Chaos Arising from Euler’s Discretization
and
to Measure Asymmetry of Figures

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1 Introduction

We can show the great distinction between differential equation and difference equation from the point of view of chaos. Let us consider Euler’s discretization of autonomous ordinary differential equation with one dimension. Some kinds of differential equations which have strongly stable equilibrium point are apt to turn into chaos in its difference equation, especially with any mesh size [1]. One of necessary conditions for chaos is the break of point symmetry in the neighborhood of stable equilibrium point [2]. If differential equation is symmetric, chaos never happens. Inspired in this asymmetry, we also tried to measure of asymmetry of figures. As the first trial, we will investigate the envelope which is given by bisectors of area.

2 Chaos arising from discretization

Let us consider one dimensional autonomous ordinary differential equation

\[
\frac{du}{dt} = f(u) \quad u \in \mathbb{R}^1 ,
\]

under the following assumptions:

\[
\begin{cases}
  f(u) \text{ is continuous in } \mathbb{R}^1 \\
  f(u) > 0 \quad (u < 0) \\
  f(0) = 0 \\
  f(u) < 0 \quad (0 < u) .
\end{cases}
\]

This simple equation has a unique stable equilibrium point \( u = 0 \). Defined \( \Delta t \) as mesh size, Euler’s discretization of (1) is

\[
\frac{x_{n+1} - x_n}{\Delta t} = f(x_n) ,
\]
\[ x_{n+1} = x_n + \Delta t \cdot f(x_n) \]

Now, define difference equation \( F_{\Delta t}(x) \) as
\[ F_{\Delta t}(x) = x + \Delta t \cdot f(x) \]
and we investigate this dynamical system \( F_{\Delta t} \).

It is well known that differential equation which has more than 2 equilibrium points and if one of these is stable, its difference equation is chaotic in the sense of Li-Yorke ([3],[4]). In this case, however, we have to assume sufficiently large mesh size. The next example is a remarkable one, because \( F_{\Delta t} \) is always chaotic with any size of mesh [1].

\[
\frac{du}{dt} = \begin{cases} 
\sqrt{-u} & (u < 0) \\
-3.411\sqrt{u} & (u \geq 0)
\end{cases}
\]

Figure 1: differential equation and its solution

We can easily solve this differential equation (see, figure 1). Equilibrium point is so strongly stable that uniqueness of solution is broken at \( u = 0 \). On the other hand, in difference equation \( F_{\Delta t} \), the fixed point \( x = 0 \) is strongly unstable (see, figure 2). This example shows the great distinction between differential equation and difference equation. We found out the following two necessary conditions for chaos in the case of sufficiently small mesh size.
Theorem 1. ([5])

(i) \( F^n_{\Delta t}(x) \) converges to the equilibrium point in its neighborhood with a sufficiently small \( \Delta t \).

\[ \lim_{u \to 0^+} \frac{f(u)}{-u} < +\infty \quad \text{or} \quad \lim_{u \to +\infty} \frac{-f(u)}{u} < +\infty. \]

(ii) \( F^n_{\Delta t}(x) \) never converges to the equilibrium point with any \( \Delta t \).

\[ \lim_{u \to 0^+} \frac{f(u)}{u} = -\infty. \]

Theorem 1 shows that non-Lipschitz continuity is one of necessary condition for chaos. The next theorem provides another necessary condition, that is to say, if \( f(u) \) is point symmetric around \( u = 0 \), chaos never occurs.

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**Figure 2**: difference equation and 3-periodic orbit

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Theorem 2. ([2])

Assume that \( f \) holds the next conditions (*).

\[ \begin{align*}
& (i) \ f(u) \ 	ext{is continuous and strictly decreasing in} \ R^1 \\
& (ii) \ f(0) = 0 \\
& (iii) \ f(u) \ 	ext{is} \ C^1 \text{-class except} \ u = 0 \\
& (iv) \ \lim_{u \to 0} \frac{f(u)}{u} = -\infty \\
& (v) \ f(-u) = -f(u) \ (f(u) \ 	ext{is an odd function}).
\]
Then,
(I) for any $\Delta t > 0$, $F_{\Delta t}$ doesn't have any periodic orbits with period $n > 2$,
(II) there exists $\Delta T > 0$, such that for any $\Delta t$ ($0 < \Delta t < \Delta T$), there is a
periodic orbit with period 2, and any 2-periodic point $x_0$ satisfies $F_{\Delta t}(x_0) = -x_0$.

3 Measurement of asymmetry of figure

We have already seen that there is some relation between chaos and asym-
metry. Now, how can we measure a distance from point symmetry of figures?
Here, we will discuss about 2-dimensional figures which have finite areas.

Let $S$ be a convex 2-dimensional figure with finite area and whose bound-
ary $\partial S$ is in piecewise $C^1$-class. \{$l_\theta$\} ($\theta \in [0, \pi]$) is a one parameter family
of lines which cut $S$ into halves, and $E$ is a envelope given by \{$l_\theta$\}.

Proposition 1.
Envelope $E$ is a geometric locus of middle point of line segment $l_\theta$. Thus,

$S$ is point symmetric $\iff E = \{G\}$,

where $G$ is a center of gravity of $S$.

The above proposition indicates that $E$ has some spread unless $S$ is point
symmetric.

Proposition 2. (figure 3)
If $S$ is asymmetric and $\partial S$ is piecewise linear, $E$ is given as a piecewise
hyperbolic closed curve.

Remark: If both ends of segment $l_\theta$ are on the same ellipse (respectively,
parabola, hyperbola), $E$ is also given as ellipse (respectively, parabola, hy-
perbola) in its neighborhood.

Proposition 3.
Unless $E$ degenerates into $\{G\}$, the number of cusps of $E$ is odd and greater
than 3. In particular, if $\partial S$ is in $C^1$-class, any $S$ has at least three $l_\theta$ such
that tangents of $\partial S$ are parallel at the cross points of $\partial S$ and $l_\theta$. 

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Figure 3: envelope of triangle

Now we define a function $N(x) : \mathbb{R}^2 \to \mathbb{N}$ as follows:

$$N(x) \overset{\text{def}}{=} \# \{ l_0 \text{ passes through } x \} .$$

With this function, we can characterize $E$.

**Proposition 4.**

$E$ is given as a set of discontinuous points of function $N(x)$.

**Proposition 5.**

$$N(x) = 2 \cdot n(E; x) + 1 \quad (x \in \mathbb{R}^2 / E) ,$$

where $n(E; x)$ denotes a winding number of $E$ around $x$. Therefore almost everywhere, the value of $N(x)$ is odd.

From the above facts, let us define a measure of asymmetry $D(S)$ as

$$D(S) \overset{\text{def}}{=} \frac{1}{|S|} \int \int n(E; x) \, dx \, dy .$$

Example: Any triangle has the same value $D = \frac{3}{4} \log 2 - \frac{1}{2}$, and of course, $D = 0$ if $S$ is point symmetric.

As for quadrangle, using invariant property of $D$ under affine transformation, without loss of generality, we can set 4 vertexes of quadrangle
\((0,0), (1,0), (0,1), (a,b)\) where \(0 \leq a \leq b \leq 1\).

\[
D(\text{triangle}) = \frac{3}{4} \log 2 - \frac{1}{2} \approx 0.01986
\]

\[
D(\text{rhombus } (b = 1)) = \frac{1}{2} \log \frac{2}{1+a} + \frac{1+a^2}{4(1-a^2)} \log \frac{2}{1+a^2} - \frac{1}{2(a+1)}
\]

\[
D(\text{kite } (b = a)) = \frac{1+a}{4(1-a)} \log \frac{1}{a} - \frac{1}{2}
\]

\[
D(\text{convex quadrangle}) = \frac{1}{2} \log \frac{2}{a+b} + \frac{2b - (a+b)(1-a)}{4(a+b)(1-a)} \log \frac{2b}{2b - (a+b)(1-a)}
\]

\[
D(\text{regular pentagon}) = \frac{1}{4 \tan^2 \theta} \log \frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} - \frac{1}{2} \approx 0.001870
\]

where \(\theta\) is \(\pi/10\).

**Theorem 3.**

*The distortion of convex quadrangle is less than that of triangle.*

**Remark:** \(D\) of concave quadrangle may be greater than that of triangle.

**Conjecture**

\[D(\forall \text{convex figure}) \leq D(\text{triangle}) = \frac{3}{4} \log 2 - \frac{1}{2}.\]

4  **Envelope of 3-dimensional solid**

In the case of 3-dimensional solid, envelope is given as a geometric locus of the center of gravity of half-cut plane domain. Regular tetrahedron is an only asymmetric solid among Platonic regular polyhedrons (figure 4). This \(E\) looks like modified octahedron but it is composed of 7 surfaces.

\(E\) of general polyhedron has self-intersection and its Eulerian number is 1. Winding number of 2-dimensional figure is correspond to number of loops which is given by points of tangency in 3-dimensional figure.
Figure 4: tetrahedron and its envelope

References


