On a certain class of cuspidal prehomogeneous vector spaces 
and its basic relative invariants

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Abstract

In this note, we give a certain class of cuspidal prehomogeneous vector spaces and determine 
explicitly two basic relative invariants of a cuspidal prehomogeneous vector space \((GL(4) \times GL(3) \times SL(2), \Lambda_1 \otimes 1 \oplus 1 + \Lambda_4 \otimes \Lambda_1 \otimes \Lambda_1^*, M(1, 4) \otimes M(4, 3) \otimes M(4, 3))\) which is a special case 
of the class. We consider everything over the complex number field \(\mathbb{C}\).

Introduction

Let \(G\) be a linear algebraic group and \(\rho\) its rational representation on a finite dimensional vector 
space \(V\), all defined over the complex number field \(\mathbb{C}\). If \(V\) has a Zariski-dense \(G\)-orbit \(\mathcal{O}\), we call 
the triplet \((G, \rho, V)\) a prehomogeneous vector space. In this case, we call \(v \in \mathcal{O}\) a generic point, 
and the isotropy subgroup \(G_v = \{ g \in G \mid \rho(g)v = v \}\) at \(v\) is called a generic isotropy subgroup. 
We call a prehomogeneous vector space \((G, \rho, V)\) a reductive prehomogeneous vector space if \(G\) is 
reductive.

Let \(\rho : G \rightarrow GL(V)\) be a rational representation of a linear algebraic group \(G\) on an \(m\)-
dimensional vector space \(V\) and let \(n\) be a positive integer with \(m > n\). A triplet \(\mathcal{C}_1 := (G \times GL(n), \rho \otimes \Lambda_1, V \otimes V(\mathcal{n}))\) is a prehomogeneous vector space if and only if a triplet \(\mathcal{C}_2 := (G \times GL(m - n), \rho^* \otimes \Lambda_1, V^* \otimes V(m - n))\) is a prehomogeneous vector space. We say that \(\mathcal{C}_1\) and \(\mathcal{C}_2\) 
are the castling transforms of each other. Two triplets are said to be castling equivalent if one is 
obtained from the other by a finite number of successive castling transformations.

Assume that \((G, \rho, V)\) is a prehomogeneous vector space with a Zariski-dense \(G\)-orbit \(\mathcal{O}\). A non-
zero rational function \(f(v)\) on \(V\) is called a relative invariant if there exists a rational character 
\(\chi : G \rightarrow GL(1)\) satisfying \(f(\rho(g)v) = \chi(g)f(v)\) for \(g \in G\). In this case, we write \(f \leftrightarrow \chi\). Let

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$S_i = \{v \in V \mid f_i(v) = 0\} \ (i = 1, \ldots, l)$ be irreducible components of $S := V \setminus \emptyset$ with codimension one. When $G$ is connected, these irreducible polynomials $f_i(v) \ (i = 1, \ldots, l)$ are algebraically independent relative invariants and any relative invariant $f(v)$ can be expressed uniquely as $f(v) = cf_1(v)^m_1 \cdots f_l(v)^m_l$ with $c \in \mathbb{C}^\times$ and $m_1, \ldots, m_l \in \mathbb{Z}$. These $f_i(v) \ (i = 1, \ldots, l)$ are called the basic relative invariants of $(G, \rho, V)$.

Prehomogeneous vector spaces $(G, \rho, V)$ with $\dim G = \dim V$ are called cuspidal prehomogeneous vector spaces (cf. [CoMc]). Cuspidal prehomogeneous vector spaces are important in the sense of contraction (cf. [Gy]) of prehomogeneous vector spaces and include arithmetical interesting examples such as the space of binary cubic forms and $(SL(5) \times GL(4), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(4))$. However it is very difficult to determine the structures of the basic relative invariants and its $b$-function. For example, the microlocal structure of $(SL(5) \times GL(4), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(4))$ is most complicated in all irreducible prehomogeneous vector spaces.

In this note, we give a certain class of cuspidal prehomogeneous vector spaces and determine explicitly two basic relative invariants of a cuspidal prehomogeneous vector space $(GL(4) \times GL(3) \times SL(2), \Lambda_1^r \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^r \otimes \Lambda_1^r, M(1, 4) \oplus M(4, 3) \oplus M(4, 3))$ which is a special case of the class. This is an interesting example of the cuspidal prehomogeneous vector spaces.

In Section 1, we give a certain class of cuspidal prehomogeneous vector spaces which was observed in [Kas, Theorem 3.22]. In Section 2, we construct two basic relative invariants of the cuspidal prehomogeneous vector space $(GL(4) \times GL(3) \times SL(2), \Lambda_1^r \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^r \otimes \Lambda_1^r, M(1, 4) \oplus M(4, 3) \oplus M(4, 3))$ which is a special case of the class in Section 1.

**Notation**

Let $V$ be an $n$-dimensional vector space spanned by $u_1, \ldots, u_n$. For $G = GL(n)$, $SL(n)$, we denote by $\Lambda_1$ a representation of $G$ on $V$ which is defined by $(u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_n)g$ for $g \in G$. Let $\bigwedge^k V \ (1 \leq k \leq n - 1)$ be a vector space spanned by exterior products $u_{i_1} \wedge \cdots \wedge u_{i_k} \ (1 \leq i_1 < \cdots < i_k \leq n)$. We denote by $\Lambda_k \ (1 \leq k \leq n - 1)$ a representation of $SL(n)$ on $\bigwedge^k V$ which is defined by $u_{i_1} \wedge \cdots \wedge u_{i_k} \mapsto \Lambda_1(g)u_{i_1} \wedge \cdots \wedge \Lambda_1(g)u_{i_k}$ for $g \in SL(n)$. Let $S^k V \ (k \geq 1)$ be a vector space spanned by symmetric tensor products $u_{i_1} \cdots u_{i_k} \ (1 \leq i_1 \leq \cdots \leq i_k \leq n)$. We denote by $k\Lambda_1 \ (k \geq 1)$ a representation of $SL(n)$ on $S^k V$ which is defined by $u_{i_1} \cdots u_{i_k} \mapsto \Lambda_1(g)u_{i_1} \cdots \Lambda_1(g)u_{i_k}$ for $g \in SL(n)$. We denote by $\rho^*$ the contragredient representation of a rational representation $\rho$. For a rational representation $\rho$, $\rho^{(\ast)}$ stands for $\rho$ or $\rho^*$. We denote by $V(n)$ an $n$-dimensional vector space. If $V(n)$ and $V(n)^*$ appear at the same time, $V(n)^*$ denotes the dual space of $V(n)$. We use $+$ instead of $\oplus$ if $\otimes$ and $\oplus$ appear at the same time.

1 **A certain class of cuspidal prehomogeneous vector spaces**

In [Kas, Theorem 3.22], a certain class of cuspidal prehomogeneous vector spaces was observed.
Theorem 1.1 (S. Kasai). Let $\rho : G \rightarrow GL(V(m))$ be an irreducible rational representation of a connected semisimple linear algebraic group $G$ with the finite kernel. Assume that a triplet $\mathcal{P} := (G \times SL(2) \times GL(l), \rho \otimes 3\Lambda \otimes \Lambda_1, V(m) \otimes V(4) \otimes V(l))$ is casting equivalent to $(SL(2) \times GL(1), 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(1))$. Then $T := (G \times GL(4l) \times GL(3l) \times SL(2), \rho \otimes \Lambda_1^{(s)} \otimes 1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(4l)^{(s)} + V(4l) \otimes V(3l) \otimes V(2))$ is a reductive cuspidal prehomogeneous vector space.

Remark 1.2. The triplet $(GL(4l) \times GL(3l) \times SL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(4l) \otimes V(3l) \otimes V(2))$ is obtained from the regular trivial prehomogeneous vector space $(GL(2l) \times GL(l) \times SL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(2l) \otimes V(l) \otimes V(2))$ by applying a casting transformation two times.

Remark 1.3. A correction to [Kas, Theorem 3.22] is given in [Ku, Correction 1.2].

Example 1.4. If $G = \{1\}$ and $l = 1$, then $\mathcal{P} = (SL(2) \times GL(1), 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(1))$. By Theorem 1.1, $T = (GL(4) \times GL(3) \times SL(2), \Lambda_1^{(s)} \otimes 1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(4)^{(s)} + V(4) \otimes V(3) \otimes V(2))$ is a reductive cuspidal prehomogeneous vector space.

Example 1.5. If $(G, \rho) = (SL(3), \Lambda_1)$ and $l = 11$, then $\mathcal{P} = (SL(3) \times SL(2) \times GL(11), \Lambda_1 \otimes 3\Lambda_1 \otimes \Lambda_1, V(3) \otimes V(4) \otimes V(11))$ is casting equivalent to $(SL(2) \times GL(1), 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(1))$. By Theorem 1.1, $T = (SL(3) \times GL(44) \times GL(33) \times SL(2), \Lambda_1 \otimes \Lambda_1^{(s)} \otimes 1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(44)^{(s)} + V(44) \otimes V(33) \otimes V(2))$ is a reductive cuspidal prehomogeneous vector space.

Example 1.6. We define a sequence $\{a_i\}_{i \geq 0}$ by $a_0 = a_1 = 1$ and $a_{i+2} = 4a_{i+1} - a_i (i \geq 0)$. Put $A_i := (SL(a_i) \times SL(2) \times GL(a_{i+1}), \Lambda_1 \otimes 3\Lambda_1 \otimes \Lambda_1, V(a_i) \otimes V(4) \otimes V(a_{i+1})) (i \geq 0)$. Then we see that $A_0 = (SL(2) \times GL(1), 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(1))$ and $A_{i+1}$ is a casting transform of $A_i$. By Theorem 1.1, $T = (SL(a_i) \times GL(4a_{i+1}) \times GL(3a_{i+1}) \times SL(2), \Lambda_1 \otimes \Lambda_1^{(s)} \otimes 1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(a_i) \otimes V(4a_{i+1})^{(s)} + V(4a_{i+1}) \otimes V(3a_{i+1}) \otimes V(2)) (i \geq 0)$ is a reductive cuspidal prehomogeneous vector space.

By Theorem 1.1, we can obtain infinitely many reductive cuspidal prehomogeneous vector spaces. From here, we shall give the preliminaries for the proof of Theorem 1.1.

Proposition 1.7 (cf. [K]). Let $\rho_i : G \rightarrow GL(V_i) (i = 1, 2)$ be a rational representation of a linear algebraic group $G$ on a finite dimensional vector space $V_i$. Assume that $(G, \rho_2, V_2)$ is a prehomogeneous vector space with a generic isotropy subgroup $H$ and $(H, \rho_1|_H, V_1)$ is a prehomogeneous vector space. Then $(G, \rho_1 \oplus \rho_2, V_1 \oplus V_2)$ is a prehomogeneous vector space.

Lemma 1.8. Let $\rho : G \rightarrow GL(V)$ be a rational representation of a linear algebraic group $G$ on an $m$-dimensional vector space $V$ and let $n$ be a positive integer with $m > n$. Assume that $Q := (G \times GL(n), \rho \otimes \Lambda_1, V \otimes V(n))$ is a prehomogeneous vector space and the $G$-part of its generic isotropy subgroup is reductive. When the representation space $V \otimes V(n)$ is identified with $\bigoplus_{i=1}^{n} V$, the
representation $\rho \otimes \Lambda_1$ is given by $(\rho \otimes \Lambda_1)(g, A)(v_1, \ldots, v_n) = (\rho(g)v_1, \ldots, \rho(g)v_n)^t A$ for $(g, A) \in G \times GL(n)$ and $v_1, \ldots, v_n \in V$. Then we have the following assertions.

1. Let $v_0 = (v_1^{(0)}, \ldots, v_n^{(0)})$ be a generic point of $Q$ and let $H$ be the $G$-part of the generic isotropy subgroup $(G \times GL(n))_{v_0}$ at $v_0$. Then $v_1^{(0)}, \ldots, v_n^{(0)}$ are linearly independent and there exists the rational representation $\phi : H \to GL(n)$ such that $(G \times GL(n))_{v_0} = \{(h, \phi(h)) \in G \times GL(n) | h \in H\}$.

2. Let $\{f_1^{(0)}, \ldots, f_{m-n}^{(0)}\}$ be a basis of $\langle v_1^{(0)}, \ldots, v_n^{(0)} \rangle^\perp := \{f \in V^* | f(v) = 0 \text{ for all } v \in \langle v_1^{(0)}, \ldots, v_n^{(0)} \rangle\}$ as vector spaces, where $\langle v_1^{(0)}, \ldots, v_n^{(0)} \rangle$ denotes the $n$-dimensional subspace of $V$ generated by $v_1^{(0)}, \ldots, v_n^{(0)}$. Then $f_0 := (f_1^{(0)}, \ldots, f_{m-n}^{(0)}) \in \bigoplus_{k=0}^{m-n} V^*$ is a generic point of $(G \times GL(m-n), \rho^* \otimes \Lambda_1, V^* \otimes V(m-n))$ which is a castling transform of $Q$. Furthermore, there exists the rational representation $\psi : H \to GL(m-n)$ such that $(G \times GL(m-n))_{f_0} = \{(h, \psi(h)) \in G \times GL(m-n) | h \in H\}$ and $\rho|_H = \phi^* \circ \psi$.

**Proof.**

1. Put $W = \{(v_1, \ldots, v_n) \in \bigoplus_{k=1}^n V | v_1, \ldots, v_n \text{ are linearly independent}\}$. Note that $W$ is a nonempty open subset in $\bigoplus_{k=1}^n V$ and $G \times GL(n)$ acts on $W$ by $\rho \otimes \Lambda_1$. Let $\varnothing$ be the open orbit of $Q$. Since $\bigoplus_{k=1}^n V$ is irreducible, we have $\varnothing \subset W$. Since $v_1^{(0)}, \ldots, v_n^{(0)}$ are linearly independent, for $h \in H$, there exists a unique $A \in GL(n)$ such that $(\rho(h)v_1^{(0)}, \ldots, \rho(h)v_n^{(0)})^t A = (v_1^{(0)}, \ldots, v_n^{(0)})$. Hence we can define a map $\phi : H \to GL(n)$ by $(\rho(h)v_1^{(0)}, \ldots, \rho(h)v_n^{(0)})^t \phi(h) = (v_1^{(0)}, \ldots, v_n^{(0)})$ for $h \in H$. Since $\rho : G \to GL(V)$ is a rational representation, we see that $\phi$ is a rational representation. Thus we obtain (1).

2. Since $H$ is reductive and $\rho|_H : H \to GL(V)$ is a rational representation, there exist $v_{n+1}^{(0)}, \ldots, v_m^{(0)} \in V$ and the rational representation $\psi : H \to GL(m-n)$ such that $\{v_1^{(0)}, \ldots, v_m^{(0)}\}$ is a basis of $V$ as vector spaces and $(\rho(h)v_1^{(0)}, \ldots, \rho(h)v_m^{(0)}) = (v_1^{(0)}, \ldots, v_m^{(0)}) \begin{pmatrix} t\phi(h)^{-1} & 0 \\ 0 & \psi(h) \end{pmatrix}$ for $h \in H$. Let $\{w_1^{(0)}, \ldots, w_m^{(0)}\}$ be the dual basis of $\{v_1^{(0)}, \ldots, v_m^{(0)}\}$. Since $\{w_1^{(0)}, \ldots, w_m^{(0)}\}$ is a basis of $\langle v_1^{(0)}, \ldots, v_n^{(0)} \rangle^\perp$, there exists an element $P \in GL(m-n)$ such that $(w_1^{(0)}, \ldots, w_m^{(0)}) = (f_1^{(0)}, \ldots, f_{m-n}^{(0)})P$. Since $(\rho^*(h)w_1^{(0)}, \ldots, \rho^*(h)w_m^{(0)}) = (w_1^{(0)}, \ldots, w_m^{(0)})^t \psi(h)^{-1}$ for $h \in H$, we have $(\rho^*(h)f_1^{(0)}, \ldots, \rho^*(h)f_{m-n}^{(0)})^t (P^{-1} \psi(h)^t P) = (f_1^{(0)}, \ldots, f_{m-n}^{(0)})$ for $h \in H$. Since $\rho(g)(\langle v_1^{(0)}, \ldots, v_n^{(0)} \rangle) = \langle v_1^{(0)}, \ldots, v_n^{(0)} \rangle$ if and only if $\rho^*(g)(\langle v_1^{(0)}, \ldots, v_n^{(0)} \rangle) = \langle v_1^{(0)}, \ldots, v_n^{(0)} \rangle$, we see that the $G$-part of the isotropy subgroup $(G \times GL(m-n))_{f_0}$ at $f_0$ coincides with $H$. Then we have $(G \times GL(m-n))_{f_0} = \{(h, tP^{-1} \psi(h)^t P) \in G \times GL(m-n) | h \in H\}$. Since $\dim(G \times GL(m-n))_{f_0} = \dim H = \dim(G \times GL(m-n)) - \dim(V^* \otimes V(m-n))$, we see that $f_0$ is a generic point. Thus we obtain (2).

**Lemma 1.9.** For irreducible rational representations $i\Lambda_1$ ($i = 1, 2$) and $\Lambda_1$ of $SL(2)$, the tensor product representation $i\Lambda_1 \otimes \Lambda_1$ decomposes to the direct sum representation of two irreducible
representations as follows:

\[
(1.1) \quad i \Lambda_1 \otimes \Lambda_1 = \begin{cases} 
2 \Lambda_1 \oplus 1 & (i = 1) \\
3 \Lambda_1 \oplus \Lambda_1 & (i = 2)
\end{cases}
\]

Here 1: \( SL(2) \to GL(\mathbb{C}) \) is the unit representation.

**Proof.** Let \( V_j = \{ F(u,v) = \sum_{m=0}^{j} x_m u^m v^m \mid x_1, \ldots, x_{j+1} \in \mathbb{C} \} \) \( (j = 1, 2, 3) \) be the vector space of all homogeneous polynomials in two variables \( u,v \) of degree \( j \). When a representation space of \( j \Lambda_1 \) \( (j = 1, 2, 3) \) is identified with \( V_j \), the representation \( j \Lambda_1 \) is given by \( j \Lambda_1(A)F(u,v) = F((u,v)A) = F(au+cv, bu+dv) \) for \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2) \) and \( F(u,v) \in V_j \).

Let \( T := \left\{ t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL(2) \mid a \in \mathbb{C}^\times \right\} \) is a maximal torus of \( SL(2) \) and its character group \( X(T) \) is given by \( X(T) = \{ e^n : T \to \mathbb{C}^\times \mid n \in \mathbb{Z} \} \), where \( e^n(t(a)) = a^n \) for \( a \in T \). Since \( j \Lambda_1(t(a))u^{j-m}v^m = (au)^{j-m}(a^{-1}v)^m = e^{j-2m}(t(a))u^{j-m}v^m \) for \( 0 \leq m \leq j \), we see that the set of all the weights of \( j \Lambda_1 \) \( (j = 1, 2, 3) \) is \( \{ e^{j-2m} \mid m \in \mathbb{Z}, 0 \leq m \leq j \} \), where the multiplicity of the weight \( e^{j-2m} \) \( (0 \leq m \leq j) \) is one. Since \( i \Lambda_1 \otimes \Lambda_1(t(a))u^{i-m}v^m \otimes u = (au)^{i-m}(a^{-1}v)^m \otimes (au) = e^{i+1-2m}(t(a))u^{i-m}v^m \otimes v \) and \( i \Lambda_1 \otimes \Lambda_1(t(a))u^{i-m}v^m \otimes v = (au)^{i-m}(a^{-1}v)^m \otimes (a^{-1}v) = e^{i+1-2m}(t(a))u^{i-m}v^m \otimes v \) for \( 0 \leq m \leq i \), we see that the set of all the weight of \( i \Lambda_1 \otimes \Lambda_1 \) \( (i = 1, 2) \) is \( \{ e^{i+1-2l} \mid l \in \mathbb{Z}, 0 \leq l \leq i + 1 \} \), where the multiplicity of the weight \( e^{i+1-2l} \) \( (l = 0, i + 1) \) (resp. \( 0 < l < i + 1 \) ) is one (resp. two). We shall show the case \( i = 1 \). Since \( 2 \Lambda_1 \oplus 1(t(a))u^{2-m}v^m, 0 = ((au)^2-m(a^{-1}v)^m, 0) = e^{2-2m}(t(a))(u^{2-m}v^m, 0) \) and \( 2 \Lambda_1 \oplus 1(t(a))(0, 1) = (0, 1) = e^0(t(a))(0, 1) \) for \( 0 \leq m \leq 2 \), we see that all the weights of \( 2 \Lambda_1 \oplus 1 \) coincide with those of \( \Lambda_1 \otimes \Lambda_1 \), including weight multiplicities. Thus we obtain \( \Lambda_1 \otimes \Lambda_1 = 2 \Lambda_1 \oplus 1 \). We shall show the case \( i = 2 \). Since \( 3 \Lambda_1 \oplus \Lambda_1(t(a))(u^{3-m}v^m, 0) = ((au)^3-m(a^{-1}v)^m, 0) = e^{3-2m}(t(a))(u^{3-m}v^m, 0) \) and \( 3 \Lambda_1 \oplus \Lambda_1(t(a))(0, u^{1-m'}v'^m) = (0, (au)^{1-m'}(a^{-1}v'^m) = e^{1-2m'}(t(a))(0, u^{1-m'}v'^m) \) for \( 0 \leq m \leq 3 \) and \( 0 \leq m' \leq 1 \), we see that all the weights of \( 3 \Lambda_1 \oplus \Lambda_1 \) coincide with those of \( 2 \Lambda_1 \otimes \Lambda_1 \), including weight multiplicities. Thus we obtain \( 2 \Lambda_1 \otimes \Lambda_1 = 3 \Lambda_1 \oplus \Lambda_1 \).

**Proposition 1.10.** \( \{ (3 \Lambda_1 \oplus \Lambda_1^*(A, B), 2 \Lambda_1 \oplus \Lambda_1(A, B), \Lambda_1(A)) \mid A \in SL(2), B \in GL(l) \} \) is a generic isotropy subgroup of \( (GL(4l) \times GL(3l) \times SL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(4l) \otimes V(3l) \otimes V(2)) \).

**Proof.** Recall Remark 1.2. We shall calculate a generic isotropy subgroup of \( \mathcal{R}_0 := (GL(2l) \times SL(2) \times GL(l), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(2l) \otimes V(2) \otimes V(l)) \). When its representation space is identified with \( M(2l) \), the representation \( \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \) is given by \( \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1(C, A, B)X = CX^t(\Lambda_1 \otimes \Lambda_1(A, B)) \) for \( (C, A, B) \in GL(2l) \times SL(2) \times GL(l), X \in M(2l) \). Then \( I_{2l} \in M(2l) \) is a generic point and the generic isotropy subgroup at \( I_{2l} \) is \( \{ (\Lambda_1^* \otimes \Lambda_1^*(A, B), \Lambda_1(A, B), \Lambda_1(B)) \mid A \in SL(2), B \in GL(l) \} \). Since two representations \( \Lambda_1 \) and \( \Lambda_1^* \) of \( SL(2) \) are equivalent, \( \{ (\Lambda_1 \otimes \Lambda_1^*(A, B), \Lambda_1(A), \Lambda_1(B)) \mid A \in SL(2), B \in GL(l) \} \) is a generic isotropy subgroup of \( \mathcal{R}_0 \). Since \( \mathcal{R}_1 := (GL(2l) \times SL(2) \times GL(3l), \Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_1^*(A, B), \Lambda_1(A), \Lambda_1(B)) \) is a generic isotropy subgroup of \( \mathcal{R}_1 \), we obtain the desired result.

**Remarks.**
\( \Lambda_1, V(2l)^* \otimes V(2)^* \otimes V(3l) \) is a castling transform of \( \mathcal{R}_0 \), by Lemmas 1.8 and 1.9, we see that 
\( \{ (\Lambda_1 \otimes \Lambda_1^*(A, B), \Lambda_1(A), 2\Lambda_1 \otimes \Lambda_1^*(A, B)) | A \in SL(2), B \in GL(l) \} \) is a generic isotropy subgroup of \( \mathcal{R}_1 \). Then \( \{ (2\Lambda_1 \otimes \Lambda_1^*(A, B), \Lambda_1(A), \Lambda_1^* \otimes \Lambda_1(A, B)) | A \in SL(2), B \in GL(l) \} \) is a generic isotropy subgroup of \( \mathcal{R}_2 := (GL(3l) \times SL(2) \times GL(2l), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3l) \otimes V(2) \otimes V(2l)) \). Since \( \mathcal{R}_3 := (GL(3l) \times SL(2) \times GL(4l), \Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_1, V(3l)^* \otimes V(2)^* \otimes V(4l)) \) is a castling transform of \( \mathcal{R}_2 \), by Lemmas 1.8 and 1.9, we see that \( \{ (2\Lambda_1 \otimes \Lambda_1^*(A, B), \Lambda_1(A), 3\Lambda_1 \otimes \Lambda_1^*(A, B)) | A \in SL(2), B \in GL(l) \} \) is a generic isotropy subgroup of \( \mathcal{R}_3 \). Since two representations \( 2\Lambda_1 \) and \( (2\Lambda_1)^* \) of \( SL(2) \) are equivalent, we obtain our assertion.

**Proof of Theorem 1.1.** Thus we can prove Theorem 1.1. By Propositions 1.7 and 1.10, we see that \( T \) is a prehomogeneous vector space. Since \( \dim G = -l^2 + 4ml - 3 \), we see that the dimension of a group of \( T \) is equal to that of a representation space of \( T \). Thus we obtain our assertion.

## 2 Basic relative invariants of a cuspidal prehomogeneous vector space \((GL(4) \times GL(3) \times SL(2), M(1, 4) \oplus M(4, 3) \oplus M(4, 3))\)

In this section, we will construct two basic relative invariants of the cuspidal prehomogeneous vector space \((G, \rho, V) = (GL(4) \times GL(3) \times SL(2), \Lambda_1^* \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1^*, M(1, 4) \oplus M(4, 3) \oplus M(4, 3))\) which is a special case of the class in \( \S 1 \) (See Example 1.4). This example is related to parabolic type (not necessarily irreducible) associated an \( sl_2 \)-triple (cf. [R]) and Dynkin-Kostant type for the exceptional groups, that is, \( E_8 \)-type (cf. [Uk]). The group action on the space is the following:

\[
\tag{2.1} M(1, 4) \oplus M(4, 3) \oplus M(4, 3) \ni (p, X, Y) \mapsto (pg^{-1}, (g_4 X g_3^{-1}, g_4 Y g_3^{-1}) g_2^{-1})
\]

for an element \( g = (g_4, g_3, g_2) \in GL(4) \times GL(3) \times SL(2) \). This space is a cuspidal prehomogeneous vector space with \( \dim G = \dim V = 28 \). Therefore a generic isotropy subgroup is finite. Here we chose an element \( (p_0, X_0, Y_0) = ((1001), ^t(I_5 | 0), ^t(0 | I_3)) \) as a generic point of the prehomogeneous vector space.

**Lemma 2.1.** For elements \( A = (a_{ij}), B = (b_{ij}), C = (c_{ij}) \in M(3) \), we define a polynomial \( \alpha(A, B, C) \) on \( M(3) \oplus M(3) \oplus M(3) \) as follows:

\[
\tag{2.2} \alpha(A, B, C) := \det(A + B + C) - \{ \det(B + C) + \det(A + C) + \det(A + B) \} + \{ \det A + \det B + \det C \}
\]

Then \( \alpha(A, B, C) \) is a symmetric trilinear form on \( M(3) \oplus M(3) \oplus M(3) \).

**Proof.** By direct calculation, we have
\[ \alpha(A, B, C) = \sum_{\sigma \in S_3} \text{sgn}(\sigma)a_{1\sigma(1)}b_{2\sigma(2)}c_{3\sigma(3)} + \sum_{\sigma \in S_3} \text{sgn}(\sigma)a_{2\sigma(1)}b_{1\sigma(2)}c_{2\sigma(3)} + \sum_{\sigma \in S_3} \text{sgn}(\sigma)b_{1\sigma(1)}a_{2\sigma(2)}c_{3\sigma(3)} + \sum_{\sigma \in S_3} \text{sgn}(\sigma)b_{2\sigma(1)}a_{1\sigma(2)}c_{3\sigma(3)} + \sum_{\sigma \in S_3} \text{sgn}(\sigma)c_{1\sigma(1)}a_{2\sigma(2)}b_{3\sigma(3)} + \sum_{\sigma \in S_3} \text{sgn}(\sigma)c_{2\sigma(1)}a_{1\sigma(2)}b_{3\sigma(3)}. \]

Thus we obtain our assertion.

For an element \( X \in M(4, 3) \), define \( X(i) \in M(3) (1 \leq i \leq 4) \) by the matrix obtained by deleting the \( i \)-th row from \( X \). Then, for \( X, Y, Z \in M(4, 3) \), we put

\[ \alpha_i(X, Y, Z) := (-1)^{i-1}\alpha(X(i), Y(i), Z(i)) \quad (1 \leq i \leq 4) \]

and \( \mathcal{A}(*, *, *) := i(\alpha_1(*, *, *) \alpha_2(*, *, *) \alpha_3(*, *, *) \alpha_4(*, *)) \in M(4, 1) \). Thus we define the mapping \( \Phi \) from \( M(4, 3) \oplus M(4, 3) \) to \( M(4) \) as follows:

\[ \Phi(X, Y) := \begin{pmatrix} \mathcal{A}(X, X, X) | \mathcal{A}(X, Y, X) | \mathcal{A}(Y, X, Y) | \mathcal{A}(Y, Y, Y) \end{pmatrix} \in M(4) \]

Here we remark the followings:

**Lemma 2.2.** For \( F_i(X, Y)(u, v) := \frac{1}{3}\alpha_i(X, X, X)u^3 + \alpha_i(X, Y, X)u^2v + \alpha_i(Y, X, Y)uv^2 + \frac{1}{3}\alpha_i(Y, Y, Y)v^3 \) \( (1 \leq i \leq 4) \), we have

\[ F_i(X, Y)g_2^{-1}(u, v) = F_i(X, Y)((u, v)^t g_2^{-1}) \text{ for } g_2 \in SL(2). \]

**Proof.** We may check the compatibility for elements \( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2). \)

\[
\begin{align*}
F_i(X, Y)^t(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix})(u, v) & = \frac{1}{3}\alpha_i(aX, aX, aX)u^3 + \alpha_i(aX, a^{-1}Y, aX)u^2v + \alpha_i(a^{-1}Y, aX, aY)uv^2 + \frac{1}{3}\alpha_i(a^{-1}Y, a^{-1}Y, a^{-1}Y)v^3 \\
& = \frac{1}{3}a^3\alpha_i(X, X, X)u^3 + a\alpha_i(X, Y, X)u^2v + a\alpha_i(Y, X, Y)uv^2 + \frac{1}{3}a^{-3}\alpha_i(Y, Y, Y)v^3 \\
& = \frac{1}{3}\alpha_i(X, X, X)(au)^3 + \alpha_i(X, Y, X)(au)^2(a^{-1}v) + \alpha_i(Y, X, Y)(au)(a^{-1}v)^2 + \frac{1}{3}\alpha_i(Y, Y, Y)(a^{-1}v)^3 \\
& = F_i(X, Y)(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^t, (u, v)) \\
F_i(X, Y)^t(\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix})(u, v) & = \frac{1}{3}\alpha_i(X + \varepsilon Y, X + \varepsilon Y, X + \varepsilon Y)u^3 + \alpha_i(X + \varepsilon Y, Y, X + \varepsilon Y)u^2v \\
& + \alpha_i(Y, X + \varepsilon Y, Y)uv^2 + \frac{1}{3}\alpha_i(Y, Y, Y)v^3 \\
& = \frac{1}{3}\{\alpha_i(X, X, X) + 3\varepsilon\alpha_i(X, Y, X) + 3\varepsilon^2\alpha_i(Y, X, Y) + \varepsilon^3\alpha_i(Y, Y, Y)\}u^3 \\
& + \{\alpha_i(X, Y, X) + 2\varepsilon\alpha_i(Y, X, Y) + \varepsilon^2\alpha_i(Y, Y, Y)\}u^2v \\
& + \{\alpha_i(Y, Y, Y) + \varepsilon\alpha_i(Y, Y, Y)\}uv^2 + \frac{1}{3}\alpha_i(Y, Y, Y)v^3 \\
\end{align*}
\]
Proposition 2.4.

Let \( A \) be an \( m \) by \( n \) matrix and \( B \) an \( n \) by \( m \) matrix. We denote by \( a_i \) \((1 \leq i \leq n)\) (resp. \( b_i \) \((1 \leq i \leq n)\)) the \( i \)-th column (resp. row) of \( A \) (resp. \( B \)). Then we have the following assertions.

1. If \( m > n \), then \( \det(AB) = 0 \).
2. If \( m = n \), then \( \det(AB) = \det A \det B \).
3. If \( m < n \), then \( \det(AB) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \det(a_{i_1}) \cdots \det(a_{i_m}) \det(t^{(b_1)} \cdots t^{(b_m)}) \).

Proposition 2.4. For \( g_2 \in SL(2) \), \( g_3 \in GL(3) \) and \( g_4 \in GL(4) \), we have

\[
\Phi((X,Y)g_2^{-1}) = \Phi(X,Y)(3\Lambda_1(g_2^{-1})),
\]

\[
\Phi(g_4Xg_3^{-1},g_4Yg_3^{-1}) = (\det g_3)^{-1}(\det g_4)^{-1}(\det g_4^{-1})\Phi(X,Y).
\]

Proof. (1) follows directly from Lemma 2.2. We will prove (2). For \( X \in M(4,3) \), we put

\[
S(X) := t^{(s(1))} s(2) s(3) s(4),
\]

where \( s(i) := (-1)^{i-1} \det X(i) \) \((1 \leq i \leq 4)\). By Lemma 2.3, we have \( S(g_4X) = (\det g_4)^{i}g_4^{-1}S(X) \) for \( g_4 \in GL(4) \). Then we see that \( \frac{1}{3} \mathfrak{A}(g_4X,g_4X,g_4X) = (\det g_4)^{i}g_4^{-1}\frac{1}{3}\mathfrak{A}(X,X,X) \) and \( \frac{1}{3} \mathfrak{A}(g_4Y,g_4Y,g_4Y) = (\det g_4)^{i}g_4^{-1}\frac{1}{3}\mathfrak{A}(Y,Y,Y) \) for \( g_4 \in GL(4) \). Note that \( \alpha_i(X,Y,X) = (-1)^{i-1}\{\det((X+Y)(i)) - 2\det((X+Y)(i)) - 2\det((X+i)) + 2\det((X+i)) \} \) \((1 \leq i \leq 4)\). Then we see that \( \mathfrak{A}(g_4X,g_4Y,g_4X) = (\det g_4)^{i}g_4^{-1}\mathfrak{A}(X,Y,X) \) and \( \mathfrak{A}(g_4Y,g_4X,g_4Y) = (\det g_4)^{i}g_4^{-1}\mathfrak{A}(Y,X,Y) \) for \( g_4 \in GL(4) \). Thus we obtain (2).

Since \( \det \Phi(X_0,Y_0) = 16 \), we see that \( \det \Phi(X,Y) \) is not identically zero. Thus we see that

\[
\det \Phi((X,Y)g_2^{-1}) = \det \Phi(X,Y)
\]

\[
\det \Phi(g_4Xg_3^{-1},g_4Yg_3^{-1}) = (\det g_3)^{-4}(\det g_4)^{3}\det \Phi(X,Y)
\]

and hence we have the following Theorem.
Theorem 2.5. \( f_1(X,Y) = \det \Phi(X,Y) \) is a basic relative invariant of the prehomogeneous vector space \( (GL(4) \times GL(3) \times SL(2), \Lambda_1^4 \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^4 \otimes \Lambda_1^4, M(1,4) \oplus M(4,3) \oplus M(4,3)) \) corresponding to the rational character \( \chi_1(g_4, g_3, g_2) = (\det g_3)^{-1}(\det g_4)^3 \).

Next we consider the construction of the second basic relative invariant of the prehomogeneous vector space \( (GL(4) \times GL(3) \times SL(2), \Lambda_1^4 \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^4 \otimes \Lambda_1^4, M(1,4) \oplus M(4,3) \oplus M(4,3)) \). For \( \Phi(X,Y) \) in (2.4), we put \( \Phi(X,Y) = (\alpha_{ij})_{1 \leq i,j \leq 4} \). Then we have the following proposition:

**Proposition 2.6.** We define polynomials \( \psi_{ij} (1 \leq i < j \leq 4) \) in 16 variables \( \alpha_{ik} (1 \leq k, l \leq 4) \) as follows:

\[
\psi_{ij} := 3\alpha_{i1}\alpha_{j4} - 3\alpha_{j1}\alpha_{i4} - \alpha_{i2}\alpha_{j3} + \alpha_{j2}\alpha_{i3}.
\]

Then we have the following assertions.

1. \( \psi_{ij} \) are \( SL(2) \)-invariant and \( GL(3) \)-relative invariant polynomials.
2. Define

\[
\Psi(X,Y) := \begin{pmatrix}
0 & \psi_{12} & \psi_{13} & \psi_{14} \\
-\psi_{12} & 0 & \psi_{23} & \psi_{24} \\
-\psi_{13} & -\psi_{23} & 0 & \psi_{34} \\
-\psi_{14} & -\psi_{24} & -\psi_{34} & 0
\end{pmatrix} \in \text{Alt}(4).
\]

Then we have

\[
\Psi(g_4 X g_3^{-1}, g_4 Y g_3^{-1}) = (\det g_3)^{-2}(\det g_4)^2(t_4 g_4^{-1})\Psi(X,Y)g_4^{-1}.
\]

We need the following Lemma 2.7 to prove Proposition 2.6

**Lemma 2.7.** For two binary cubic forms

\[
F_s(u,v) = s_1 u^3 + s_2 u^2 v + s_3 u v^2 + s_4 v^3,
F_t(u,v) = t_1 u^3 + t_2 u^2 v + t_3 u v^2 + t_4 v^3,
\]

we put \( G(s,t) := 3(s_1 t_4 - s_4 t_1) - (s_2 t_3 - s_3 t_2) \). Then \( G(s,t) \) is an \( SL(2) \)-invariant polynomial, that is, \( G(3\Lambda_1(g_2^{-1})s, 3\Lambda_1(g_2^{-1})t) = G(s,t) \) for \( g_2 \in SL(2) \).

**Proof.** Since \( SL(2) \) is generated by \( t(a) := \begin{pmatrix} a & 0 \\ a^{-1} & 1 \end{pmatrix} \), \( h := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), \( u(\varepsilon) := \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \), it is enough to show the case \( g_2 = t(a) \), \( h \), \( u(\varepsilon) \). It is obvious to show the case \( g_2 = t(a) \) (resp. \( g_2 = h \)). For \( g_2^{-1} = u(\varepsilon) \),

\[
G(3\Lambda_1(g_2^{-1})s, 3\Lambda_1(g_2^{-1})t)
\]

\[
= 3 \{(s_1 + \varepsilon s_2 + \varepsilon^2 s_3 + \varepsilon^3 s_4)t_4 - s_4(t_1 + \varepsilon t_2 + \varepsilon^2 t_3 + \varepsilon^3 t_4)\}
\]

\[
- \{(s_2 + 2\varepsilon s_3 + 3\varepsilon^2 s_4)(t_3 + 3\varepsilon t_4) - (s_3 + 3\varepsilon s_4)(t_2 + 2\varepsilon t_3 + 3\varepsilon^2 t_4)\}
\]

\[
= G(s,t)
\]

Thus we obtain our assertion.
[proof of Proposition 2.6]

By Lemma 2.7, we have (1). We shall show \( \Psi(g_4X, g_4Y) = (\det g_4)^2 (t^{g_4^{-1}}) \Psi(X, Y) g_4^{-1} \) for \( g_4 \in GL(4) \). For \( \sigma \in S_4 \), we put \( g(\sigma) := (\delta_{i \sigma(j)})_{1 \leq i, j \leq 4} \), where \( \delta_{st} = \begin{cases} 1 & (s = t) \\ 0 & (s \neq t) \end{cases} \). Since \( GL(4) \) is generated by \( \text{diag}(a, b, c, d) \), \( g(\sigma), u(\varepsilon) := \begin{pmatrix} 1 & \varepsilon & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \), it is enough to show the case \( g_4 = \text{diag}(a, b, c, d) \), \( g(\sigma), u(\varepsilon) \). It is obvious to show the case \( g_4 = \text{diag}(a, b, c, d) \) (resp. \( g_4 = g(\sigma) \)).

For \( g_4^{-1} = u(\varepsilon) \),

\[
\Psi(g_4X, g_4Y) = \begin{pmatrix} 0 & \psi_{12} & \psi_{13} & \psi_{14} \\ -\psi_{12} & 0 & \varepsilon \psi_{13} + \psi_{23} & \varepsilon \psi_{14} + \psi_{24} \\ -\psi_{13} & -\varepsilon \psi_{13} - \psi_{23} & 0 & \psi_{34} \\ -\psi_{14} & -\varepsilon \psi_{14} - \psi_{24} & -\psi_{34} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \psi_{12} & \psi_{13} & \psi_{14} \\ -\psi_{12} & 0 & \psi_{23} & \psi_{24} \\ 0 & -\psi_{13} & -\psi_{23} & 0 \\ -\psi_{14} & -\psi_{24} & -\psi_{34} & 0 \end{pmatrix} = (t^{g_4^{-1}}) \Psi(X, Y) g_4^{-1}.
\]

Thus we have (2). \( \square \)

Here we remark \( \text{Pf}(\Psi(X, Y)) = -3 \det(\Phi(X, Y)) = -3f_1(X, Y) \). We have the following lemma from Proposition 2.6.

**Lemma 2.8.** Put \( R(X, Y) := t^X \Psi(X, Y)Y \), we have

\[
(2.10) \quad R(g_4X g_3^{-1}, g_4Y g_3^{-1}) = (\det g_3)^{-2} (\det g_4)^2 (t^{g_3^{-1}}) R(X, Y) g_3^{-1}.
\]

Here if we consider a 2 \( \times \) 3 matrix \( H(p, X) := \begin{pmatrix} pX \\ pY \end{pmatrix} \), then \( H(pg_4^{-1}) = (g_4X g_3^{-1}, g_4Y g_3^{-1}) g_2^{-1} = (t^{g_3^{-1}}) \begin{pmatrix} pXg_3^{-1} \\ pYg_3^{-1} \end{pmatrix} \). We put \( H(p, X) = (\ell_1 \mid \ell_2 \mid \ell_3) \), with \( \ell_1, \ell_2, \ell_3 \in \mathbb{C}^2 \) and we make the following 3 \( \times \) 3 alternating matrix : for \( z_{ij} := \det(\ell_i \mid \ell_j) \), \( Z(p, X, Y) := \begin{pmatrix} 0 & z_{12} & z_{13} \\ -z_{12} & 0 & z_{23} \\ -z_{13} & -z_{23} & 0 \end{pmatrix} \).

Then

\[
(2.11) \quad Z(pg_4^{-1}, (g_4X g_3^{-1}, g_4Y g_3^{-1}), g_2^{-1}) = (t^{g_3^{-1}}) Z(p, X, Y) g_3^{-1}.
\]
Thus if we put $\Delta(Z(p, X, Y)) := \begin{pmatrix} (\zeta_{23})^2 & -z_{13}\zeta_{23} & z_{12}\zeta_{23} \\ -z_{13}\zeta_{23} & (\zeta_{13})^2 & -z_{12}\zeta_{23} \\ z_{12}\zeta_{23} & -z_{12}\zeta_{23} & (\zeta_{12})^2 \end{pmatrix}$, we have

(2.12) \[ \Delta(Z(pg_4^{-1}, (g_4Xg_3^{-1}, g_4Yg_3^{-1})g_2^{-1})) = (\det g_3)^{-2}g_3\Delta(Z(p, X, Y))^t g_3. \]

Thus we have

(2.13) \[ \Delta(Z(pg_4^{-1}, (g_4Xg_3^{-1}, g_4Yg_3^{-1})g_2^{-1}))R((g_4Xg_3^{-1}, g_4Yg_3^{-1})g_2^{-1}) = (\det g_3)^{-4}(\det g_4)^2g_3\Delta(Z(p, X, Y))R(X, Y)g_3^{-1}. \]

Hence we have

(2.14) \[ \tr(\Delta(Z(pg_4^{-1}, (g_4Xg_3^{-1}, g_4Yg_3^{-1})g_2^{-1}))R((g_4Xg_3^{-1}, g_4Yg_3^{-1})g_2^{-1})) = (\det g_3)^{-4}(\det g_4)^2\tr(\Delta(Z(p, X, Y))R(X, Y)). \]

We put $f_2(p, X, Y) := \tr(\Delta(Z(p, X, Y))R(X, Y))$. For the generic point $(p_0, X_0, Y_0)$, we have $f_2(p_0, X_0, Y_0) = -4 \neq 0$, that is, $f_2$ is not identically zero. Here we remark $SL(2)$-invariance of $f_2$ as follows: for $g_2^{-1} = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$,

\[ f_2(p, (X, Y)g_2^{-1}) = \tr(\Delta(Z(p, X, Y))^t X\Psi(X, Y)(\varepsilon X + Y)) = \tr(\Delta(Z(p, X, Y))^t X\Psi(X, Y)Y) + \varepsilon\tr(\Delta(Z(p, X, Y))^t X\Psi(X, Y)X) = \tr(\Delta(Z(p, X, Y))^t X\Psi(X, Y)Y). \]

For another generators $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$ of $SL(2)$, we can easily check $SL(2)$-invariance of $f_2(p, X, Y)$.

We summarize the argument above as follows:

**Theorem 2.9.** Two basic relative invariants of the prehomogeneous vector space $(GL(4) \times GL(3) \times SL(2), \Lambda_1^* \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1^*, M(1, 4) \oplus M(4, 3) \oplus M(4, 3))$ are the followings:

(2.15) \[ f_1(X, Y) = \det \Phi(X, Y) = -\frac{1}{3}\text{Pf}(\Psi(X, Y)) \leftrightarrow (\det g_3)^{-4}(\det g_4)^3, \quad \deg_{d(X, Y)} f_1(X, Y) = 12, \]

(2.16) \[ f_2(p, X, Y) = \tr(\Delta(Z(p, X, Y))^t X\Psi(X, Y)Y) \leftrightarrow (\det g_3)^{-4}(\det g_4)^2 \quad \deg_{f(X, Y)} f_2 = (4, 12). \]

where $f \leftrightarrow \chi$ means that the rational character $\chi$ of the algebraic group corresponds to the polynomial $f$ on the representation space.

**Remark 2.10.** It is the open problem to give an explicit construction of basic relative invariants of another cuspidal prehomogeneous vector spaces of the type in §1 (See Theorem 1.1).
References

[AFK] K. Amano, M. Fujigami and T. Kogiso, Construction of irreducible relative invariant of the prehomogeneous vector space \((SL_5 \times GL_4, A_2 \otimes A_1, \Lambda^2(C^5) \otimes C^4)\), Linear Algebra Appl. \textbf{355} (2002), 215–222.


