Toward a combinatorial formula for an irregular conformal block of rank one

Hajime NAGOYA and Yasuhide NUMATA

Abstract. In this note, we give a combinatorial formula for a particular three-point irregular conformal block of rank one using the Littlewood-Richardson numbers and propose a conjectural formula for the general three-point irregular conformal block of rank one.

1. Introduction

The aim of this note is to study explicit presentations for series expansions of irregular conformal blocks at irregular singular points for the Virasoro algebra. Our study is motivated by AGT correspondence [1] which gives an explicit presentation of the four-point conformal block in terms of Nekrasov partition function. This correspondence has some degenerate versions, the investigation of which was started by Gaiotto [4].

As is well-known, the ordinary conformal block is a building block of conformal field theory with Virasoro symmetry. The degeneration procedure mentioned above yields explicit presentations for series expansions of ‘degenerate’ conformal blocks at regular singular points. Such conformal blocks could have been investigated since the beginning of conformal field theories, but actually very few studies, such as [10], [16], have been done before AGT. ‘Degenerate’ conformal blocks are now called irregular conformal blocks since the Ward equations (some people call it the BPZ equations) have irregular singularities as ordinary differential equations.

This introductory section 1 will give some definitions and basic facts necessary for our study. We begin with the definition of vertex operators for irregular conformal blocks. In the second subsection irregular conformal blocks will be defined. The third subsection gives a brief explanation of the consequence of AGT correspondence in our situation.

1.1. Vertex operator

First of all, let us recall the definition of vertex operators of Virasoro algebra on irregular Verma modules and irregular conformal blocks.

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Denote the Virasoro algebra by $\text{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}c$, which is the Lie algebra with the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{n^3 - n}{12} c, \quad [L_m, c] = 0.$$ 

For $r \in \mathbb{Z}_{\geq 0}$, we define an irregular Verma module $M^{[r]}_\Lambda$ as a representation of $\text{Vir}$ generated by an element $|\Lambda\rangle$ ($\Lambda = (\Lambda_r, \ldots, \Lambda_{2r}) \in \mathbb{C}^{r+1}$) such that

$$L_n|\Lambda\rangle = \Lambda_n|\Lambda\rangle \quad (n = r, r+1, \ldots, 2r), \quad L_n|\Lambda\rangle = 0 \quad (n \geq 2r + 1).$$

Hereafter, we call $|\Lambda\rangle$ the irregular vector. The module $M^{[r]}_\Lambda$ is spanned by linearly independent vectors of the form

$$L_{i_1} \cdots L_{i_k}|\Lambda\rangle \quad (i_1 \leq \cdots \leq i_k < r).$$

When $r = 0$, $M^{[0]}_\Lambda$ is the usual Verma module. The non-negative integer $r$ corresponds to the rank of singularity, as explained below.

**Definition 1.1.** A vertex operator $\Phi^\Delta_{\Lambda', \Lambda}(z) : M^{[r]}_\Lambda \to M^{[r]}_{\Lambda'}$ is a linear operator satisfying

$$[L_n, \Phi^\Delta_{\Lambda', \Lambda}(z)] = z^n \left( z \frac{d}{dz} + (n + 1)\Delta \right) \Phi^\Delta_{\Lambda', \Lambda}(z),$$

$$\Phi^\Delta_{\Lambda', \Lambda}(z)|\Lambda\rangle = z^\alpha \exp \left( \sum_{n=1}^r \frac{\beta_n}{z^n} \right) \sum_{n=0}^\infty w_n z^n,$$

where $\alpha, \beta_n \in \mathbb{C}$, $w_n \in M^{[r]}_{\Lambda'}$ and $w_0 = |\Lambda'\rangle$.

In the case of $r = 0$, the vertex operator $\Phi^{\Delta_2}_{\Delta_3, \Delta_1}(z) : M^{[0]}_{\Delta_1} \to M^{[0]}_{\Delta_3}$ was introduced in [2] and exists if $M^{[0]}_{\Delta_1}$ is irreducible.

In the case of $r > 0$, we have the following theorem.

**Theorem 1.2 ([18]).** For a positive integer $r$, if $\Lambda_{2r} \neq 0$, then the vertex operator $\Phi^{\Delta}_{\Lambda', \Lambda}(z) : M^{[r]}_\Lambda \to M^{[r]}_{\Lambda'}$ exists and is uniquely determined by the given parameters $\Lambda, \Delta, \beta_r$ with $\alpha = -(r+1)\Delta + \tilde{\alpha}(\beta_r, \Lambda)$, $\beta_n = \beta_n(\beta_r, \Lambda)$ ($n = 1, \ldots, r-1$) and

$$\Lambda'_n = \Lambda_n - \delta_{n,r}\beta_r \quad (n = r, \ldots, 2r).$$

Moreover, the coefficients $w_n$ are polynomials of $\Lambda, \beta_r, \Delta$ and $\Lambda_{2r}^{-1}$.

We remark that $M^{[r]}_{\Lambda}$ is irreducible if and only if $\Lambda_{2r-1} \neq 0$ or $\Lambda_{2r} \neq 0$ [11],[3]. We expect the irregular Verma module $M^{[r]}_{\Lambda}$ with $\Lambda_{2r} = 0$ and $\Lambda_{2r-1} \neq 0$ can be
used to describe a system in a ramified case. Because the pairing of the Whittaker (or Gaiotto) vector is the building block of a short-distance expansion of Painlevé III \(\tau\) function \([7]\), which has only ramified singular points. Here, the Whittaker vector is the rank one irregular vector \(|\Lambda_1, 0\rangle\) embedded in a Virasoro Verma module \(M_\Delta\).

Denote a dual irregular Verma module by \(M^*_{\Lambda, [r]}\) with irregular vector \(|\Lambda|\) such that

\[\langle \Lambda | L_n = \Lambda_n \langle \Lambda | \quad (n = -r, -r - 1, \ldots, -2r),\]

and \(M^*_{\Lambda, [r]}\) is spanned by linearly independent vectors of the form

\[\langle \Lambda | L_{i_1} \cdots L_{i_k} \quad (-r < i_1 \leq \cdots \leq i_k).\]

A dual vertex operator \(\Phi^*_{\Lambda, \Delta}(z): M^*_{\Lambda, [0]} \to M^*_{\Lambda, [r]}\) is defined in the same manner and the similar statement to Theorem 1.2 holds for the dual vertex operator.

### 1.2. Conformal block

Recall the Shapovalov bilinear form on the usual Verma module. We denote it by \(\langle \cdot, \cdot \rangle: M^*_{\Delta, [0]} \times M^*_{\Delta, [0]} \to \mathbb{C}\). It satisfies

\[
\langle \Delta | \cdot | \Delta \rangle = 1, \\
\langle u | L_n \cdot | v \rangle = \langle u | \cdot L_n | v \rangle \equiv \langle u | L_n | v \rangle,
\]

where \(\langle u \rangle \in M^*_{\Delta, [0]}, \langle v \rangle \in M^*_{\Delta, [0]}\). We also use a bilinear pairing \(\langle \cdot, \cdot \rangle: M^*_{\Delta, [0]} \times M^*_{\Delta, [1]} \to \mathbb{C}\) as

\[
\langle \Delta | \cdot | \Lambda \rangle = 1, \quad \langle u | L_n \cdot | v \rangle = \langle u | \cdot L_n | v \rangle \equiv \langle u | L_n | v \rangle,
\]

where \(\langle u \rangle \in M^*_{\Delta, [0]}, \langle v \rangle \in M^*_{\Delta, [1]}\). It is well-defined because \(L_n\) acts on \(|\Delta\rangle\) diagonally for \(n > 0\) and \(L_n\) acts on \(|\Delta\rangle\) diagonally for \(n \leq 0\).

A four-point regular conformal block is defined as an expectation value of a composition of vertex operators. For example,

\[
\langle \Delta_{\infty} | \cdot \left(\Phi^*_{\Delta_{\infty}, \Delta}(1) \circ \Phi^*_{\Delta_{\infty}, \Delta_0}(z)\right) | \Delta_0 \rangle
\]

is a regular conformal block for the points 0, 1, \(z, \infty\) and a series expansion at \(z = 0\). By definition, it coincides with

\[
\langle \Delta_{\infty} | \Phi^*_{\Delta_{\infty}, \Delta}(1) \cdot \Phi^*_{\Delta_{\infty}, \Delta_0}(z) | \Delta_0 \rangle.
\]

Here, we suppose that Verma modules \(M_{\infty}, M_\Delta\) and \(M_{\Delta_0}\) are irreducible. The
following
\[ \langle \Delta_\infty | \Phi^{\Delta z}_{\Delta} (z) \cdot \Phi^{\Delta_0}_{\Delta} (1) | \Delta_0 \rangle \] (2)
is also a regular conformal block for the points 0, 1, z, \infty and a series expansion at 
\( z = \infty \). The connection problem of conformal blocks has been still an open problem.
See, for example, [20], [9]. In addition, we have not understood convergence of 
regular conformal blocks neither.

A three-point irregular conformal block is also defined as an expectation value 
of a composition of vertex operators. The following 
\[ \langle \Delta_\infty | \cdot \Phi^{\Delta_z}_{\Delta} (z) \cdot \Phi^{\Delta_0}_{\Delta} | \Lambda \rangle \] (3)
is a three-point irregular conformal block for two regular singular points z, \infty and 
one irregular singular point 0 and a series expansion at \( z = 0 \). A series expansion 
at \( z = \infty \) of the three-point irregular conformal block is given by
\[ \langle \Delta_\infty | \Phi^{\Delta_z}_{\Delta} (z) \cdot | \Lambda \rangle. \] (4)

If the parameters \( \Delta_z, \Lambda' \) and \( \Lambda \) take certain special values, then the irregular 
conformal blocks (3) and (4) are solutions to a second-order ordinary differential 
equation equivalent to the Kummer confluent hypergeometric equation. See Sub-
section III B 1 in [18]. In this case, the irregular conformal block (3) is a divergent 
series and (4) is absolutely convergent in \( \mathbb{C} \).

1.3. Combinatorial formula

By AGT correspondence [1], the four-point conformal block (2) with \( c = 1 \) is 
expressed as
\[ \langle \theta^2 \Phi^{\theta^2}_{\theta^2, \theta^2} (z) \cdot \Phi^{\theta^2}_{\theta^2, \theta^0} (1) | \theta^2 \rangle = \frac{z^{2 \Delta_{\infty} - \Delta_z^2} - \sigma^2 (1-z)^{2 \theta_z \theta_0}}{\sum_{\lambda, \mu \in Y} F_{\lambda, \mu} \left( \theta_z, \theta_1, \theta_\infty, \tilde{\sigma}, \theta_0 \right) z^{-|\lambda|-|\mu|}}, \] (5)
where
\[ F_{\lambda, \mu} \left( \theta_z, \theta_1, \theta_\infty, \tilde{\sigma}, \theta_0 \right) = \prod_{(i,j) \in \lambda} \frac{\left( \theta_z + \tilde{\sigma} + i - j \right)^2 - \theta_\infty^2 \left( \theta_z + \tilde{\sigma} + i - j \right)^2 - \theta_\infty^2}{h^2_{\lambda}(i,j) \left( \lambda_i + \mu_i - i - j + 1 + 2 \tilde{\sigma} \right)^2} \times \prod_{(i,j) \in \mu} \frac{\left( \theta_z - \tilde{\sigma} + i - j \right)^2 - \theta_\infty^2 \left( \theta_z - \tilde{\sigma} + i - j \right)^2 - \theta_\infty^2}{h^2_{\mu}(i,j) \left( \lambda_i + \lambda_j + i - j + 1 - 2 \tilde{\sigma} \right)^2}. \]

Here \( Y \) denotes the set of all partitions, \( \lambda' \) denotes the transposition of \( \lambda \) and 
h_{\lambda}(i, j) is the hook length defined by \( h_{\lambda}(i, j) = \lambda_i + \lambda_j - i - j + 1 \). It is natural to 
extpect that our irregular conformal blocks (3) and (4) have explicit combinatorial 
formulas, since degenerate Nekrasov partition functions are known [4].
If we use the explicit representation (5) of the four-point conformal block, computations of degeneration limits are easy. We can take a limit of (5) by
\[ \theta_1 + \theta_0 = \epsilon^{-1}, \quad \theta_1 - \theta_0 = \theta_s, \quad z \to \epsilon z, \quad \epsilon \to 0, \]  
which corresponds to confluence process moving the regular singular point 1 to 0, and obtain a combinatorial formula for the three-point irregular conformal block (4) with \( c = 1 \) [14], [7]. This is also true for any \( c \).

On the other hand, by the limit (6), the four-point regular conformal block (1) with \( c = 1 \) diverges although it has a similar expression to (5). It turns out that we have to also move \( \Delta = \sigma^2 \) to infinity and multiply some function such as \((1 - z)^4\) in order to obtain a limit of (1) [12]. This multiplication breaks the combinatorial structure of the four-point regular conformal block. As a result, we cannot obtain a combinatorial formula for the three-point irregular conformal block (3) by the limit.

An explanation why the four-point regular conformal block with \( c = 1 \) admits the combinatorial formula (5) is given by expanding the integral representation of it in terms of Schur functions [15]. In the following, we discuss the expansion of the integral representation of our irregular conformal block (3) to construct an explicit combinatorial formula.

2. Integral representation

The integral formula for the three-point irregular conformal block (3) with \( c = 1 \) is
\[
z^{2\alpha_0 \alpha_s} e^{-\alpha_s/z} \int_0^\infty dy_1 \cdots \int_0^\infty dy_n \int_0^t dx_1 \cdots \int_0^t dx_m \prod_{i=1}^n \prod_{j=1}^m (x_i - y_j)^2 \times \prod_{i<j} (y_i - y_j)^2 \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^m x_i^{2\alpha_0} e^{-1/x_i} (z - x_i)^{2\alpha_s} \prod_{j=1}^n y_j^{2\alpha_0} e^{-1/y_j} (y_i - z)^{2\alpha_s}.
\]
(7)

Precisely, the expansion of the integral above at \( z = 0 \) is equal to
\[
\langle (\alpha_\infty + n)^2 \cdot \Phi^{\alpha_0^2}_{(\alpha_0 + \alpha_s + m, 1/4), (\alpha_0, 1/4)}(z) \rangle (\langle \alpha_0, 1/4 \rangle)
\]
(8)
up to scalar, where \( \alpha_0 + \alpha_s + \alpha_\infty + m + n = 0 \). The integral formula above is obtained by the free field realization of vertex operators. Because the construction is a standard one, we omit the explanation.

Let us recall the Selberg integral formulas with Schur polynomials inserted in
the integrand:

\[
\int_0^1 dx_1 \cdots \int_0^1 dx_n s_\lambda(x) \prod_{i<j}(x_i - x_j)^2 \prod_{i=1}^n x_i^u(1-x_i)^v = n! f_n(\lambda) \prod_{i=1}^n \frac{\Gamma(u + (n - i) + \lambda_i) \Gamma(v + (n - i))}{\Gamma(u + v + (2n - i - 1) + \lambda_i)},
\]

where \(s_\lambda(x)\) is the Schur polynomial defined for a Young diagram \(\lambda\), \(\Gamma(u)\) is the Gamma function and

\[
f_n(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j + j - i).
\]

Thus, we expect that we obtain a combinatorial formula for the special three-point irregular conformal block (8), if we expand the integral representation (7) at \(z = 0\) in terms of Schur polynomials.

By Theorem 1.2, the coefficients of the three-point irregular conformal block (3) are polynomials of \(\Delta_0, \Lambda_1, \beta_1\) and \(\Delta_\infty\). For example, in the case of \(c = 1\) we have

\[
\langle \theta_0^2 | \Phi_{(\theta,1/4), (\theta-\beta,1/4)}^g(z) | (\theta,1/4) \rangle = z^{2\theta_0^2+2\beta(\theta-\beta)}e^{\beta/2} \left( 1 + 2 \left( 2\beta^3 - 3\beta^2\theta + \beta\theta^2 - \beta\theta_0^2 - \theta^2 + \theta_0^2 \right) z 
+ 2 \left( 4\beta^6 - 12\beta^5\theta + 13\beta^4\theta^2 - 4\beta^4\theta_0^2 - 4\beta^3\theta_0^2\theta^2 + 5\beta^4 - 6\beta^3\theta^3 + 6\beta^3\theta_0^2 \right) 
+ 10\beta^3\theta^2\theta_0^2 - 10\beta^3\theta + \beta^2\theta^4 - 2\beta^2\theta_0^2\theta^2 - 8\beta^2\theta_0^2\theta^2 + 6\beta^2\theta^2 + \beta^2\theta_0^4 
+ 2\beta^2\theta_0^2\theta_0^2 - 3\beta^2\theta_0^2 + \beta^2\theta_4 - 3\beta^2\theta_0^2 + 2\beta^3\theta_0^2 + \beta^3 
- 2\beta\theta_0^2\theta_0^2 + 2\beta\theta_0^2 + 5\beta\theta_0^2 + \theta^2\theta_0^4 - 2\theta^2\theta_0^2 + \theta_0^2\theta_0^2 \right) z^2 + \cdots \right).
\]

We remark that \(\Lambda_2\) can be taken as any non-zero complex number by scaling \(z\). Therefore, the combinatorial formula for the special three-point irregular conformal block (8) will give a combinatorial formula for the general three-point irregular conformal block (3).

By the following change of the integral variables

\[
x_i \rightarrow \frac{z}{1-zx_i}, \quad y_i \rightarrow \frac{1}{y_i},
\]

the integral becomes

\[
z^{2(\alpha_x+m)(\alpha_x+m+\alpha_0)-2\alpha_x^2}e^{-(\alpha_x+m)/z} \times \int_0^\infty dy_1 \cdots \int_0^\infty dy_n \int_0^\infty dx_1 \cdots \int_0^\infty dx_m \prod_{j=1}^n y_j^{2\alpha_0-4}e^{-y_j} \prod_{i=1}^m x_i^{2\alpha_x}e^{x_i}
\]
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\[ \times \prod_{i=1}^{m}(1-zx_i)^{-2(\alpha_0+\alpha_x+m+n)} \prod_{j=1}^{n}(1-zy_j)^{2\alpha_z} \prod_{i=1}^{m} \prod_{j=1}^{n}(1-z(x_i+y_j))^2. \]

Thus, the problem reduces to how we expand

\[ \prod_{i=1}^{m}(1-zx_i)^{-u} \prod_{j=1}^{n}(1-zy_j)^{-v} \prod_{i=1}^{m} \prod_{j=1}^{n}(1-z(x_i+y_j))^2 \]  

in terms of Schur polynomials for \( u, v \in \mathbb{Z}_{\geq 0} \).

3. Combinatorial formula

3.1. By the Littlewood-Richardson numbers

Denote by \( d_\lambda(n) \), the number of tableaux on the shape \( \lambda \) whose entries are taken from \( \{1, \ldots, n\} \). Then, we have

\[ \prod_{i=1}^{m}(1-x_i)^{-n} = \sum_{\lambda, \ell(\lambda) \leq m} d_\lambda(n)s_\lambda(x) \ (n \in \mathbb{Z}_{\geq 1}). \]  

Indeed, the left one is the sum of all products of \( x_i^{a_{ij}} \) over all \( m \) by \( n \) matrices \( A = (a_{ij}) \) with non-negative integer entries. By Robinson-Schensted-Knuth correspondence, \( A \) corresponds to pair \( (P, Q) \) of tableaux of the same shape, with \( P \) having its entries in \( \{1, \ldots, n\} \) and \( Q \) in \( \{1, \ldots, m\} \). The original product is \( x^Q \).

Recall the formula

\[ d_\lambda(u) = \prod_{(i,j) \in \lambda} \frac{u+j-i}{h_\lambda(i,j)}. \]

Set

\[ d_{a,b}^n = \det \left( \begin{array}{c} a_i + n - i \\ b_j + n - j \end{array} \right)_{1 \leq i,j \leq n}, \]

where \( n \in \mathbb{N}, \ a = (a_1, \ldots, a_n), \ b = (b_1, \ldots, b_n) \in \mathbb{Z}_{\geq 0}^n \) and \( \binom{a_i}{b_j} \) is the binomial coefficient. On the expansion of the final part of (9), the following formula

\[ \prod_{i=1}^{m} \prod_{j=1}^{n}(1+x_i+y_j) = \sum_{\mu \subset \lambda \subset (n,m)} d_{\lambda,\mu}^\nu s_\mu(x)s_{\tilde{\lambda}'}(y) \]  

is known, where \( \tilde{\lambda}' = (m-n, \ldots, m-\lambda_1') \). See p.65 in [13].

From the formulas (10) and (11), we can expand the product (9) in terms of Schur polynomials using the Littlewood-Richardson numbers \( c_{\lambda,\mu}^\nu \).
Proposition 3.1. We have

\[
\prod_{i=1}^{m} (1 - zx_i)^{-u} \prod_{j=1}^{n} (1 - zy_j)^{-v} \prod_{i=1,j=1}^{m,n} (1 - z(x_i + y_j))^2
\]

\[
= \sum (-1)^{\lvert \alpha \rvert + |\beta| + |\gamma'| + |\delta|} d_{\lambda}(u) d_{\mu}(v) d_{\alpha,\beta}^{\delta} \sum_{\gamma', \delta} c_{\beta, \delta}^{\gamma'} c_{\alpha, \delta}^{\gamma'} c_{\lambda, \kappa} s_{\mu} s_{\nu} s_{\alpha, \beta}^{\delta} s_{\gamma', \delta} s_{\alpha, \beta}^{\delta} s_{\gamma', \delta},
\]

where the sum is taken over all Young diagrams \( \alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \eta, \kappa \) and \( \tau \) such that \( \beta \subset \alpha \subset (m^n) \), \( \delta \subset \gamma \subset (n^m) \).

Hence, we have a combinatorial formula for the particular irregular conformal block (8):

\[
\{((\alpha_\infty + n)^2) \cdot \Phi^{2}_{a, \infty + m, 1/4}((\alpha_0, 1/4)) \}
\]

for \( m, n \in \mathbb{Z}_{\geq 0} \), \( \alpha_0, \alpha_\infty \in \mathbb{C} \). We want to replace \( m, n \) by complex parameters. In order to do that, we show that the determinant \( d_{\alpha, \beta}^{\delta} \) appeared in (12) is a polynomial in \( m, n \). Note that in the formula (11), the coefficient of \( s_{\mu}(x)s_{\lambda}(y) \) is

\[
d_{\lambda', \mu}^{\lambda, \mu} = \det \left( \begin{bmatrix} m - \lambda_{n+1-i} + n - i \\ \mu_j + n - j \end{bmatrix} \right)_{1 \leq i, j \leq n}.
\]

We see that the determinant \( d_{\lambda', \mu}^{\lambda, \mu} \) is a polynomial in \( m \) whose degree may depend on \( n \). Since we also have the formula

\[
\prod_{i=1}^{m} \prod_{j=1}^{n} (1 + x_i + y_j) = \sum_{\lambda \subset \mu' \subset (m^n)} d_{\mu', \lambda}^{\mu, \lambda} s_{\lambda}(y) s_{\mu}(x),
\]

the determinant \( d_{\mu', \lambda}^{\mu, \lambda} \) should be equal to \( d_{\mu', \lambda}^{\mu, \lambda} \), which is a polynomial in \( n \). We expect the degrees of the determinant \( d_{\lambda', \mu}^{\lambda, \mu} \) as a polynomial in \( m, n \) do not depend on \( n, m \), respectively.

Set

\[
d_{a,b}^{m,n} = \det \left( \begin{bmatrix} m - a_{n+1-i} + n - i \\ b_j + n - j \end{bmatrix} \right)_{1 \leq i, j \leq n}.
\]

Proposition 3.2. The determinant \( d_{a,b}^{m,n} \) is a polynomial in \( m \) with degree \( b_1 + \cdots + b_n \).

Proof. Instead of dealing with \( d_{a,b}^{m,n} \) itself, we consider the determinant \( \tilde{d}_{a,b}^{m,n} \) whose \((i, j)\) entries are obtained by extracting the part \( \sum_{k=b_j - j + 2}^{b_j + n - j} c_{ijk} m^k \) from \((i, j)\) entries \( \sum_{k=0}^{b_j + n - j} c_{ijk} m^k \) of \( d_{a,b}^{m,n} \). We show \( \tilde{d}_{a,b}^{m,n} = 0 \). Then the proposition follows by induction on \( n \).
All the coefficients $c_{ijk}$ are written as

$$
c_{ijk} = \sum_{\ell=1}^{b_j+n-j} f_i^{b_j+n-j-\ell} g_{jkl},
$$

where each $f_i = -a_{n+1-i} + n - i$ depends only on $i$ and each $g_{jkl}$ is independent of $i$. Then by subtracting the first row from the other rows, $(i, j)$ entries for $i \geq 2$ of $\tilde{d}_{a,b}^{m,n}$ are computed as

$$
\begin{align*}
&= \left( f_i - f_1 \right) \sum_{k=b_j-j+2}^{b_j+n-j} \left( \sum_{\ell=k}^{b_j+n-j} \left( f_i^{b_j+n-j-\ell} - f_1^{b_j+n-j-\ell} \right) g_{jkl} \right) m^k \\
&= \left( f_i - f_1 \right) \sum_{k=b_j-j+2}^{b_j+n-j} \left( \sum_{\ell=k}^{b_j+n-j-1} \left( f_i^{b_j+n-j-\ell} - f_1^{b_j+n-j-\ell} \right) g_{jkl} \right) m^k \\
&= \left( f_i - f_1 \right) \sum_{k=b_j-j+2}^{b_j+n-j} \sum_{\ell=k}^{b_j+n-j-1} \sum_{s=0}^{b_j+n-j-\ell-1} f_i^{b_j+n-j-\ell-1-s} f_1^{s} g_{jkl} m^k.
\end{align*}
$$

Hence, we can write $(i, j)$ entries for $i \geq 2$ of $\tilde{d}_{a,b}^{m,n} / \prod_{k=2}^{n}(f_k - f_1)$ as

$$
\sum_{k=b_j-j+2}^{b_j+n-j-1} \left( \sum_{\ell=k}^{b_j+n-j-1} f_i^{b_j+n-j-\ell-1} g_{jkl} \right) m^k,
$$

where $g'_{jkl}$ is independent of $i$. We repeat the process above, and finally when we subtract the $(n-2)$-th row from the $(n-1)$-th row and the $n$-th row, $(i, j)$ entries for $i = n - 1, n$ of $\tilde{d}_{a,b}^{m,n} / \prod_{q=1}^{n-2} \prod_{p=q+1}^{n}(f_p - f_q)$ are

$$
\sum_{k=b_j-j+2}^{b_j+n-j-(n-2)} \left( \sum_{\ell=k}^{b_j+n-j-(n-2)} f_i^{b_j+n-j-(n-2)-\ell} g'_{jkl} \right) m^k,
$$

which is independent of $i$. Therefore, the determinant $\tilde{d}_{a,b}^{m,n}$ is equal to zero. 

**Corollary 3.3.** The determinant $d_{\lambda,\mu}^{n,n}$ is a polynomial in $m,n$ with degree $|\mu|, |\lambda|$, respectively. In other words, The coefficient of $s_{\mu}(x)s_{\lambda}(y)$ in the expansion of $\prod_{i=1}^{m} \prod_{j=1}^{n}(1 + x_i + y_j)$ in terms of Schur polynomials is a polynomial in $m,n$ with degree $|\mu|, |\lambda|$, respectively.
From Corollary 3.3, we can regard $m$ and $n$ as complex numbers in principle. Unfortunately, the determinant expression of $d_{m,n}^{a,b}$ is not useful if $n$ is not a positive integer. We should know an explicit polynomial expression of the determinant $d_{m,n}^{a,b}$ if we want to compute $d_{m,n}^{a,b}$ in such a case.

### 3.2. Conjecture

In this subsection, we give another approach for an expansion of (9). Put

$$
M_{λ,µ} = \prod_{(i,j) \in λ} (2(β - θ) + i - j) \prod_{(i,j) \in µ} (-2β + i - j),
$$

$$
N_{λ,µ} = (-1)^{|µ|} \prod_{(i,j) \in λ} \frac{(β + i - j)^2 - θ^2}{h_λ(i,j)^2} \prod_{(i,j) \in µ} \frac{(θ - β + i - j)^2 - θ^2}{h_µ(i,j)^2}.
$$

**Conjecture 3.4 ([19]).** The three-point irregular conformal block with two regular singular points $z$, $∞$ and one irregular singular point $0$ of rank one admits the following combinatorial formula

$$
\langle θ^2_0 | \Phi^{θ^2_2}_{(θ,1/4),(θ-β,1/4)}(z) | (θ,1/4) \rangle = z^{-2θ^2_0-2θ(θ-β)} \sum_{λ,µ ∈ Y} c_{λ,µ}(u,v,w) s_λ(x)s_µ(y)z^{|λ|+|µ|},
$$

where $c_{λ,µ}(u,v,w) ∈ \mathbb{Z}_{≥0}$, as an expansion at the irregular singular point $0$.

Conjecture 3.4 can be obtained from the next conjecture.

**Conjecture 3.5.** For $u, v, w ∈ \mathbb{Z}_{≥0}$, we have

$$
\prod_{i=1}^m (1 - zx_i)^{-u} \prod_{j=1}^n (1 - zy_j)^{-v} \prod_{i=1, j=1}^{m,n} (1 - z(x_i + y_j))^{-w} = \sum_{λ,µ ∈ Y} c_{λ,µ}(u,v,w) s_λ(x)s_µ(y)z^{|λ|+|µ|},
$$

where the coefficients $c_{λ,µ}(u,v,w)$ admits the unique expansion

$$
c_{λ,µ}(u,v,w) = \sum_{ν⊂λ,η⊂µ, |ν|=|η|} \frac{c_{ν,η}}{h_λ h_µ} d_{ν,η}(w) P_{λ/ν,µ/η}(u,v,w,m,n),
$$

where $c_{ν,η} ∈ \mathbb{Q}$, $d_{ν,η}(w) ∈ \mathbb{Q}[w]$ is a monic polynomial of degree $|ν|$ and

$$
P_{λ/ν,µ/η}(u,v,w,m,n) = \prod_{(i,j) ∈ λ/ν} (u + nw + j - i) \prod_{(i,j) ∈ µ/η} (v + mw + j - i),
$$
$h_\lambda = \prod_{(i,j) \in \lambda} (\lambda_i - j + \lambda'_j - i + 1)$.

In other words, for any $\lambda, \mu, \nu$ and $\eta$ such that $\nu \subset \lambda$, $\eta \subset \mu$ and $|\nu| = |\eta|$, a rational number $c^r_{\lambda,\mu}^{\nu,\eta}$ and a monic polynomial $d_{\nu,\eta}(w)$ are determined uniquely by (15) and (16).

Conjectures 3.4 and 3.5 imply that the parts consisting of parameters $\theta_0, \theta, \theta_z, \beta$ are explicit and the remained problem, except for proving conjectures, is how to describe the coefficients $a^r_{\nu,\eta}^{\lambda,\mu}$ or $c^r_{\lambda,\mu}^{\nu,\eta}$ and the polynomials $d_{\nu,\eta}(w)$ combinatorially.

Proposition 3.6. For any non-negative integer $p, q, r$ such that $r \leq p, q$, there exists uniquely a rational number $c^r_{p,q}$ and a monic polynomial $d_r(w)$ such that

$$(1 - zx)^{-u}(1 - zy)^{-v}(1 - z(x + y))^{-w} = \sum_{p,q=0}^{\infty} \frac{1}{p!q!} x^p y^q z^{p+q} \sum_{0 \leq r \leq \min(p,q)} c^r_{p,q} d_r(w) P_{(p)/(r), (q)/(r)}(u, v, w, 1, 1).$$

Moreover, the rational numbers $c^r_{p,q}$ and the monic polynomials $d_r(w)$ are given as

$$c^r_{p,q} = r! \left( \begin{array}{c} p \\ r \end{array} \right) \left( \begin{array}{c} q \\ r \end{array} \right), \quad d_r(w) = w(w + 1) \cdots (w + r - 1).$$

Proof. Firstly, we show that the expansion (17) holds with (18). Let $f(p,q,u,v,w)$ be the coefficient of $x^p y^q z^{p+q}$ in the expansion, that is,

$$(1 - zx)^{-u}(1 - zy)^{-v}(1 - z(x + y))^{-w} = \sum_{p,q=0}^{\infty} f(p,q,u,v,w) x^p y^q z^{p+q}.$$ 

Denote by $R(w)$ the left hand side of (17). Then, we have a relation

$$R(w - 1) = (1 - z(x + y)) R(w),$$

which is equivalent to all of recursive relations

$$f(p,q,u,v,w) = f(p,q,u,v,w - 1) + f(p - 1,q,u,v,w) + f(p,q - 1,u,v,w)$$

for $p, q \in \mathbb{Z}_{\geq 0}$. Since $f(0,0,u,v,w) = 1$ and from (10), we have

$$f(p,q,u,v,0) = P_{(p),(q)}(u,v,0,1,1),$$
it is sufficient to verify that
\[
\sum_{0 \leq r \leq p, q} c_{r}^{p,q} d_{r}(w) \frac{P_{(p)/(r),(q)/(r)}(u, v, w, 1, 1)}{p! q!}
= \sum_{0 \leq r \leq p, q} \binom{w + r - 1}{r} \binom{u + w + p - 1}{p - r} \binom{v + w + q - 1}{q - r} \quad (p, q \in \mathbb{Z}_{\geq 0})
\tag{20}
\]
satisfy all of the recursive relations (19). Using the following relation
\[
\sum_{r=0}^{i} \left( g(p, q, r, w - 1) - g(p, q, r, w) + g(p - 1, q, r, w) + g(p, q, r - 1, w) \right)
= \binom{w + i - 1}{i} \binom{u + w + p - 2}{p - i - 1} \binom{v + w + q - 2}{q - i - 1},
\]
where
\[
g(p, q, r, w) = \binom{w + r - 1}{r} \binom{u + w + p - 1}{p - r} \binom{v + w + q - 1}{q - r},
\]
the computation for the left hand side of (20) satisfying (19) is straightforward.

Secondly, we show that the recursive relations (19) determine the rational numbers \(c_{r}^{p,q}\) and the monic polynomials \(d_{r}(w)\) uniquely. We immediately know that
\[
c_{0}^{p,q} = 1, \quad d_{0}(w) = 1
\]
for \(p, q \in \mathbb{Z}_{\geq 0}\).

Suppose \(p > q\) and \(c_{r-1,q}^{p-1,q}, c_{r,q-1}^{p,q-1}, d_{r}(w)\) for \(r = 1, \ldots, q\) satisfy (18). Then, the recursive relation (19) reads as
\[
\sum_{r=0}^{q} c_{r}^{p,q} d_{r}(w) P_{(p)/(r),(q)/(r)}(u, v, w, 1, 1)
= \sum_{r=0}^{q} c_{r}^{p,q} d_{r}(w - 1) P_{(p)/(r),(q)/(r)}(u, v, w - 1, 1, 1) + F(p, q, u, v, w),
\tag{21}
\]
where \(F(p, q, u, v, w)\) is a known polynomial in \(u, v, w\). By the relations
\[
d_{r}(w - 1) = d_{r}(w) - rd_{r-1}(w),
\]
\[
P_{(p)/(r),(q)/(r)}(u, v, w - 1, 1, 1) = P_{(p)/(r),(q)/(r)}(u, v, w, 1, 1)
+ (r - p) P_{(p-1)/(r),(q-1)/(r-1)}(u, v, w, 1, 1),
\]
the recursive relation (21) becomes

\[
\sum_{r=0}^{q} c_{p,q}^r \left( rd_{r-1}(w)P_{(p)}/(r),(q)/(r)(w) + (p-r)(d_r(w) - rd_{r-1}(w))P_{(p-1)/(r),(q-1)/(r-1)}(w) \right) = F(p, q, u, v, w).
\]

From the equation above, by comparing the coefficient of \( w^{p+q-r-1} \), \( c_{p,q}^r \) is uniquely determined as a rational number inductively.

Suppose \( p = q \) and \( c_{p-1,p}^r, c_{p,p}^r, d_r(w) \) for \( r = 1, \ldots, p-1 \) satisfy (18). Then, the recursive relation (19) reads as

\[
\sum_{r=0}^{p} c_{p,p}^r d_r(w)P_{(p)}/(r),(p)/(r)(u, v, w, 1, 1) = \sum_{r=0}^{p} c_{p,p}^r d_r(w-1)P_{(p)}/(r),(p)/(r)(u, v, w-1, 1, 1) + F(p, p, u, v, w). \tag{22}
\]

By the same way to the case of \( p > q \), \( c_{p,q}^r \) for \( r = 1, \ldots, p-1 \) are uniquely determined by (22). Put \( c_{p,p}^r d_p(w) = \sum_{i=0}^{p} a_i w^i \). Then, \( a_p, a_{p-1}, \ldots, a_0 \) are uniquely determined by (22) again, respectively.

\[\square\]

When \( m \) or \( n \) is greater than 1, the computations of \( c_{\nu,\eta}^{\lambda,\mu} \) and the polynomials \( d_{\nu,\eta}(w) \) become much difficult. We give some observations for \( c_{\lambda,\mu}^{\nu,\eta} \) and the polynomials \( d_{\nu,\eta}(w) \) below:

\[
c_{(1),1}^{(1),1} = |\lambda||\mu|,
\]

\[
c_{(m,n),(p)}^{(p),(p)} = \frac{p!}{p!} \left( \begin{array}{c} m \\ p \end{array} \right) + \sum_{i=0}^{p-1} \sum_{j=1}^{p-i} (-1)^{p+i+j} j(p-j)! \prod_{k=1}^{j-1} (p-k-i) \left( \begin{array}{c} m \\ i \end{array} \right) \left( \begin{array}{c} n \\ j \end{array} \right) \left( \begin{array}{c} r \\ p \end{array} \right),
\]

\[
c_{(p,1),(p+1)}^{(p,1),(p+1)} = \frac{pp!}{p!} \left( \begin{array}{c} m \\ p \end{array} \right) \left( \begin{array}{c} n \\ 1 \end{array} \right) + \sum_{i=0}^{p-1} \sum_{j=1}^{p+1-i} (-1)^{p+i+j} pj(p-j)! \prod_{k=1}^{j-1} (p-k-(1+\delta_{k,j-1}(j-1))i) \left( \begin{array}{c} m \\ i \end{array} \right) \left( \begin{array}{c} n \\ j \end{array} \right) \left( \begin{array}{c} r \\ p+1 \end{array} \right),
\]

\[
d_{\nu,(p)}(w) = \prod_{(i,j) \in \nu} (w+j-i).
\]
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References

Hajime Nagoya
School of Mathematics and Physics, Kanazawa University
Ishikawa, Japan
nagoya@se.kanazawa-u.ac.jp

Yasuhide Numata
Department of Mathematical Sciences, Shinshu University
Nagano, Japan
nu@math.shinshu-u.ac.jp