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On Oshima's middle convolution

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Abstract. Oshima's middle convolution operation for scalar higher order differential equations is a powerful tool in the study of linear ordinary differential equations in the complex domain. We rephrase its definition so that the reader can directly apply it. We also study the middle convolution from analytic viewpoint. An application to Appell's hypergeometric function F_4 is given.

1. Introduction

After Nicolas M. Katz [5], Toshio Oshima [7] defined the middle convolution operation for scalar higher order linear ordinary differential equations, and obtained a lot of remarkable basic results. In this note, we focus on a theorem ([7, Theorem 5.2]) that describes the change of the Riemann scheme by the middle convolution. The theorem is shown by calculus on the Weyl algebra, and we shall study it from analytic viewpoint.

Analytic realization of the middle convolution is the Riemann-Liouville transform. The Riemann-Liouville transform is an integral transformation, and determined by a complex parameter together with an end point of the integral. We will explain how to choose the end point, which will give another proof of Oshima's theorem.

In the last section, we apply the middle convolution to Appell's hypergeometric function F_4. It is expected that F_4 gives an algebraic solution to Painlevé VI equation, however, we have not yet obtained it. We explain where the difficulty is.

2. Oshima's middle convolution

The book [7] is full of beautiful results, however, it may be somewhat difficult to take a particular notion or result in a plenty of notions and results. Then, in

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this section, we will rephrase the definition of the middle convolution, which is the central notion, so that the reader can immediately apply it. Also we discuss its analytic aspect.

Let $W[x]$ be the Weyl algebra in one variable x. Then $W[x]$ is the set of polynomials in non-commutative variables x and ∂ with coefficients in $\mathbb{C}$, where the commutation relation between x and ∂ is given by

$$[\partial, x] = 1. \tag{1}$$

By using (1), any $P \in W[x]$ can be expressed in the *normal* form

$$P = \sum_{j=0}^{n} a_j(x)\partial^j \quad (a_j(x) \in \mathbb{C}[x]).$$

Also we can express P in the form

$$P = \sum_{j=0}^{n} \partial^j b_j(x) \quad (b_j(x) \in \mathbb{C}[x]),$$

which we call the *transposed* form.

We define the weight w of x and ∂ by

$$w(x) = 1, \; w(\partial) = -1,$$

and set

$$w(c) = 0 \quad (c \in \mathbb{C} \setminus \{0\}).$$

Since the relation (1) is compatible with this weight, we can define the weight of every monomial in $W[x]$. Then, for any $P \in W[x]$, we can define the weight $w(P)$ by the maximum of the weights of the monomials in P.

Example 2.1.

$$\begin{aligned} &w(x\partial) = w(x) + w(\partial) = 1 - 1 = 0, \\ &w(\partial x - 1) = \max\{w(\partial x), w(-1)\} = \max\{0, 0\} = 0, \\ &w(x\partial^2 + x^4\partial + x) = \max\{1-2, 4-1, 1\} = 3. \end{aligned}$$

Let $\mu \in \mathbb{C}$ be a parameter, and $c \in \mathbb{C} \cup \{\infty\}$ a point. The Riemann-Liouville transform (the Euler transform) I_c^μ is defined by

$$(I_c^\mu u)(x) = \frac{1}{\Gamma(\mu)} \int_c^x u(t)(x-t)^{\mu-1}\,dt \tag{2}$$

for a function $u(x)$ holomorphic in a neighborhood of $L \setminus \{c\}$, where L is a compact one chain from c to x:

$$L : [0,1] \to \mathbb{C} \quad (L(0) = c, L(1) = x).$$

We understand that the integral in (2) is regularized at the end points if necessary. We take small positive numbers ϵ_1 and ϵ_2, and set

$$L_1 = L|_{[0,\epsilon_1]},\ L_2 = L|_{[\epsilon_1,1-\epsilon_2]},\ L_3 = L|_{[1-\epsilon_2,1]}.$$

The regularization at x is defined as follows. Put $a = L(1-\epsilon_2)$, and consider the integral

$$J = \int_c^a u(t)(x-t)^{\mu-1}\,dt - \frac{1}{e^{2\pi i\mu}-1}\int_C u(t)(x-t)^{\mu-1}\,dt,$$

where C is the circle with center x of radius $|a-x|$. The integral over C is determined so that the branch of the integrand at the starting point a of C coincides with the branch at a on L_2. Symbolically we denote the right hand side by

$$L_1 \cdot L_2 \cdot \left(-\frac{1}{e^{2\pi i\mu}-1}\right) C.$$

Namely, for one chains A, B such that the starting point of B coincides with the end point of A and for scalars α, β, we denote by $\alpha A \cdot \beta B$ the integral

$$\alpha \int_A f(t)\,dt + \beta \int_B f(t)\,dt,$$

where the branch of $f(t)$ at the starting point of B coincides with that at the end point of A. It is shown that, if $\Re\mu > 0$ and if $\mu \notin \mathbb{Z}$, the integral J coincides with the integral from c to x:

$$J = \int_c^x u(t)(x-t)^{\mu-1}\,dt.$$

The integral of the right hand side is defined for $\Re\mu > 0$, and the left hand side is defined for $\mu \notin \mathbb{Z}$. Thus the equality shows that the integrals are defined for

$\mu \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. This analytic continuation with respect to μ is the regularization at x. Moreover, also for $\mu \in \mathbb{Z}_{\leq 0}$, the transform (2) is defined thanks to the gamma factor. Namely, by the formula

$$\frac{1}{\Gamma(\mu)} \cdot \frac{1}{1 - e^{2\pi i \mu}} \to \frac{(-1)^{n+1} n!}{2\pi i} \quad (\mu \to -n \in \mathbb{Z}_{\leq 0}),$$

the right hand side of (2) with the integral replaced by J gives n-th derivative of $u(x)$ when $\mu = -n \in \mathbb{Z}_{\leq 0}$. (Note that the limit becomes the Cauchy integral for $u^{(n)}(x)$.) Hence we obtain

$$(I_c^{-n} u)(x) = \partial^n u(x) \quad (n \in \mathbb{Z}_{\geq 0}). \tag{3}$$

The regularization at c is similar. We assume that $u(x)$ takes the form

$$u(x) = (x - c)^\lambda \varphi(x)$$

in a neighborhood of c, where $\lambda \in \mathbb{C}$ and $\varphi(x)$ is holomorphic at $x = c$. Put $a = L(\epsilon_1)$. Then, if $\Re\lambda > 0$ and $\lambda \notin \mathbb{Z}_{<0}$, we have

$$\frac{1}{e^{2\pi i \lambda} - 1} \int_C (t - c)^\lambda \varphi(t)(x - t)^{\mu - 1}\, dt = \int_c^a (t - c)^\lambda \varphi(t)(x - t)^{\mu - 1}\, dt,$$

where C is the circle with center c of radius $|a - c|$. By using this identity, we can extend the domain of λ to $\mathbb{C} \setminus \mathbb{Z}_{<0}$. Note that, if we regularize the integral in (2) at both c and x, we get an integral over a compact chain.

The Riemann-Liouville transform I_c^μ satisfies the identities

$$I_c^0 = \mathrm{id.}, \tag{4}$$

$$I_c^\lambda \circ I_c^\mu = I_c^{\lambda + \mu}. \tag{5}$$

(Remark that these identities hold under some generic condition.) These identities together with (3) may allow us to regard the pseudo-differential operator $\partial^{-\mu}$ as a formal Riemann-Liouville transform. Note that the Riemann-Liouville transform I_c^μ depends on the point c, while $\partial^{-\mu}$ does not.

A function $f(x)$ is called *regularizable* at $x = a$ if there exist a function $\varphi(x)$ holomorphic at $x = a$ and $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ such that

$$f(x) = (x - a)^\lambda \varphi(x).$$

We give fundamental properties of regularized integrals.

PROPOSITION 2.2. (i) *Let γ be a path from a to b, and D a neighborhood of γ. Suppose that $f(t)$ (resp. $g(t)$) is holomorphic in $D \setminus \{a\}$ (resp. $D \setminus \{b\}$), and is regularizable at $t = a$ (resp. $t = b$). Then we have*

$$\int_a^b f(t)g'(t)\,dt = -\int_a^b f'(t)g(t)\,dt.$$

(ii-i) *Let D be a simply connected domain, and $c \in D$. Suppose that $u(t)$ is holomorphic in D. If $\mu \notin \mathbb{Z}$, we have*

$$\frac{d}{dx}\int_c^x u(t)(x-t)^{\mu-1}\,dt = u(c)(x-c)^{\mu-1} + \int_c^x u'(t)(x-t)^{\mu-1}\,dt.$$

(ii-ii) *Suppose that $u(t)$ is holomorphic in $D \setminus \{c\}$ and is regularizable at $t = c$. If $\mu \notin \mathbb{Z}$, we have*

$$\frac{d}{dx}\int_c^x u(t)(x-t)^{\mu-1}\,dt = \int_c^x u'(t)(x-t)^{\mu-1}\,dt.$$

PROOF. (i) Let γ_a, γ_b be circles centered at a and b, respectively, with small radii so that these are contained in D. By the assumption, there exist $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}$ such that

$$(\gamma_a)_* f(t) = f(t)\alpha, \ (\gamma_b)_* g(t) = g(t)\beta,$$

where $(\gamma_a)_*$ denotes the analytic continuation along γ_a. Note that

$$(\gamma_b)_* g'(t) = g'(t)\beta$$

also holds. Take a point a_1 (resp. b_1) on γ_a (resp. γ_b), and set $a_1 = a+p$, $b_1 = b+q$. We regularize the integral of $f(t)g'(t)$ on γ as

$$\int_a^b f(t)g'(t)\,dt = \frac{1}{\alpha-1}\int_{\gamma_a} f(t)g'(t)\,dt + \int_{a_1}^{b_1} f(t)g'(t)\,dt - \frac{1}{\beta-1}\int_{\gamma_b} f(t)g'(t)\,dt$$

By the parametrization $t = a + pe^{i\theta}$ $(\theta \in [0, 2\pi])$ of γ_a, we have

$$\begin{aligned}\int_{\gamma_a} f(t)g'(t)\,dt &= \int_0^{2\pi} f(a+pe^{i\theta})g'(a+pe^{i\theta})ipe^{i\theta}\,d\theta \\ &= \int_0^{2\pi} f(a+pe^{i\theta})\frac{d}{d\theta}g(a+pe^{i\theta})\,d\theta\end{aligned}$$

$$
\begin{aligned}
&= \left[f(a+pe^{i\theta})g(a+pe^{i\theta})\right]_0^{2\pi} - \int_0^{2\pi} \frac{d}{d\theta} f(a+pe^{i\theta}) \cdot g(a+pe^{i\theta})\, d\theta \\
&= [f(t)g(t)]_{a+p}^{a+pe^{2\pi i}} - \int_0^{2\pi} f'(a+pe^{i\theta})g(a+pe^{i\theta}) ipe^{i\theta}\, d\theta \\
&= [f(t)g(t)]_{a+p}^{a+pe^{2\pi i}} - \int_{\gamma_a} f'(t)g(t)\, dt \\
&= (\alpha-1)f(a_1)g(a_1) - \int_{\gamma_a} f'(t)g(t)\, dt.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\int_{a_1}^{b_1} f(t)g'(t)\, dt &= [f(t)g(t)]_{a_1}^{b_1} - \int_{a_1}^{b_1} f'(t)g(t)\, dt, \\
\int_{\gamma_b} f(t)g'(t)\, dt &= (\beta-1)f(b_1)g(b_1) - \int_{\gamma_b} f'(t)g(t)\, dt.
\end{aligned}
$$

Combining these equalities, we get

$$
\begin{aligned}
\int_a^b f(t)g'(t)\, dt &= f(a_1)g(a_1) - \frac{1}{\alpha-1}\int_{\gamma_a} f'(t)g(t)\, dt \\
&\quad + f(b_1)g(b_1) - f(a_1)g(a_1) - \int_{a_1}^{b_1} f'(t)g(t)\, dt \\
&\quad - f(b_1)g(b_1) + \frac{1}{\beta-1}\int_{\gamma_b} f'(t)g(t)\, dt \\
&= -\biggl(\frac{1}{\alpha-1}\int_{\gamma_a} f'(t)g(t)\, dt + \int_{a_1}^{b_1} f'(t)g(t)\, dt \\
&\qquad - \frac{1}{\beta-1}\int_{\gamma_b} f'(t)g(t)\, dt\biggr) \\
&= -\int_a^b f'(t)g(t)\, dt.
\end{aligned}
$$

(ii-i) By the assumption $\mu \notin \mathbb{Z}$, we can regularize the integral as

$$
\int_c^x u(t)(x-t)^{\mu-1}\, dt = \int_c^b u(t)(x-t)^{\mu-1}\, dt - \frac{1}{\beta-1}\int_S u(t)(x-t)^{\mu-1}\, dt,
$$

where S is a circle surrounding $t = x$, b a point on S and $\beta = e^{2\pi i\mu}$. We see that there exists a constant $M > 0$ such that

$$\left| \frac{1}{h} \int_S u(t) \left\{ ((x+h)-t)^{\mu-1} - (x-t)^{\mu-1} - h(\mu-1)(x-t)^{\mu-2} \right\} dt \right| \leq M|h|$$

holds for sufficiently small h. Then we have

$$\frac{d}{dx} \int_S u(t)(x-t)^{\mu-1}\, dt = \int_S u(t)(\mu-1)(x-t)^{\mu-2}\, dt.$$

Therefore we get

$$\begin{aligned}
\frac{d}{dx} \int_c^x u(t)(x-t)^{\mu-1}\, dt &= \int_c^b u(t)(\mu-1)(x-t)^{\mu-2}\, dt \\
&\quad - \frac{1}{\beta-1} \int_S u(t)(\mu-1)(x-t)^{\mu-2}\, dt \\
&= - \int_c^b u(t) \frac{\partial}{\partial t}(x-t)^{\mu-1}\, dt \\
&\quad + \frac{1}{\beta-1} \int_S u(t) \frac{\partial}{\partial t}(x-t)^{\mu-1}\, dt \\
&= - \left[u(t)(x-t)^{\mu-1} \right]_c^b + \int_c^b u'(t)(x-t)^{\mu-1}\, dt \\
&\quad + \frac{1}{\beta-1} \left[u(t)(x-t)^{\mu-1} \right]_b^{\tilde{b}} \\
&\quad - \frac{1}{\beta-1} \int_S u'(t)(x-t)^{\mu-1}\, dt \\
&= -u(b)(x-b)^{\mu-1} + u(c)(x-c)^{\mu-1} \\
&\quad + \int_c^b u'(t)(x-t)^{\mu-1}\, dt \\
&\quad + u(b)(x-b)^{\mu-1} - \frac{1}{\beta-1} \int_S u'(t)(x-t)^{\mu-1}\, dt \\
&= u(c)(x-c)^{\mu-1} + \int_c^x u'(t)(x-t)^{\mu-1}\, dt,
\end{aligned}$$

where $\tilde{b}$ is the point on S with $\arg(\tilde{b} - x) = \arg(b-x) + 2\pi$.

We can show (ii-ii) in a similar argument. □

The middle convolution with parameter μ is an operation that sends a differential operator $P \in W[x]$ to a differential operator $Q \in W[x]$ satisfied by $I_c^\mu(u)$

for any u with $Pu = 0$. This operation can be obtained by taking the adjoint of the formal Riemann-Liouville transform $\partial^{-\mu}$. Namely, for u with $Pu = 0$, we set $v = \partial^{-\mu}u$, and then have

$$P\partial^{\mu}v = 0 \Leftrightarrow \partial^{-\mu}P\partial^{\mu}v = 0 \Leftrightarrow \mathrm{Ad}(\partial^{-\mu})(P)v = 0$$

as long as $P\partial^{\mu}v \notin \mathrm{Ker}\,\partial^{-\mu}$. However, the result $\mathrm{Ad}(\partial^{-\mu})(P)$ does not necessarily belong to $W[x]$. We have

$$\mathrm{Ad}(\partial^{-\mu})x^m = \sum_{\nu=0}^{m}(-1)^{\nu}(\mu)_{\nu}\binom{m}{\nu}x^{m-\nu}\partial^{-\nu}, \tag{6}$$

where

$$(\mu)_{\nu} = \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)} = \begin{cases} 1 & (\nu = 0), \\ \mu(\mu+1)\cdots(\mu+\nu-1) & (\nu \geq 1), \end{cases} \tag{7}$$

and hence a negative power of ∂ appears if P contains a monomial of positive weight. In order to avoid this, we replace P by $\partial^k P$ before operating the adjoint of $\partial^{-\mu}$, if the weight k of P is positive. Then, if we set

$$k_0 = \max\{w(P), 0\}, \tag{8}$$

the result $Q = \mathrm{Ad}(\partial^{-\mu})(\partial^{k_0}P)$ belongs to $W[x]$. Now we consider the algebra $W(x) = \mathbb{C}(x)[\partial]$. We decompose Q into irreducible elements in $W(x)$:

$$Q = Q_1Q_2\dots Q_l.$$

It would be natural to define that the rightmost factor Q_l were the result of the middle convolution. If $Q_l \in W(x) \setminus W[x]$, we multiply a polynomial $\psi(x) \in \mathbb{C}[x]$ so that $\psi(x)Q_l \in W[x]$. This operation is denoted by $\mathrm{R}(Q_l)$ ([7, Definition 1.1]). Thus we would have the operation on $W[x]$

$$P \mapsto \mathrm{R}(Q_l)(= \psi(x)Q_l).$$

However, there are several problems. First, the irreducible decomposition in $W(x)$ is not unique. Second, the definition of the middle convolution (for local systems) by Katz is independent of the irreducibility. The assertion that the result of the middle convolution is irreducible if the source is irreducible is one of the main results in the Katz theory. Then we want to get an effective definition of the middle convolution not using the irreducibility.

In [7], the middle convolution mc_μ is first defined in Chapter 1, (1.36) as a composition of R and $\mathrm{Ad}(\partial^{-\mu})$, and algorithmically defined for Fuchsian case in Chapter 5, Theorem 5.2. We rephrase the latter definition.

Let μ be a complex number. Let $P \in W[x]$ be a Fuchsian operator with Riemann scheme

$$\left\{\begin{matrix} x=c_1 & x=c_2 & \cdots & x=c_p & x=\infty \\ [\lambda_{1,1}]_{(m_{1,1})} & [\lambda_{2,1}]_{(m_{2,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} & [\lambda_{0,1}]_{(m_{0,1})} \\ [\lambda_{1,2}]_{(m_{1,2})} & [\lambda_{2,2}]_{(m_{2,2})} & \cdots & [\lambda_{p,2}]_{(m_{p,2})} & [\lambda_{0,2}]_{(m_{0,2})} \\ \vdots & \vdots & & \vdots & \vdots \\ [\lambda_{1,n_1}]_{(m_{1,n_1})} & [\lambda_{2,n_2}]_{(m_{2,n_2})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} & [\lambda_{0,n_0}]_{(m_{0,n_0})} \end{matrix}\right\}, \tag{9}$$

where we set

$$[\lambda]_{(m)} = (\lambda, \lambda+1, \dots, \lambda+m-1)^T.$$

Also we set $c_0 = \infty$. Let n be the order of P. Then we have

$$\sum_{\nu=1}^{n_j} m_{j,\nu} = n \quad (0 \le j \le p).$$

Although it is not assumed in [7, Theorem 5.2], we assume the followings in order to make the argument simple:

(A1) there is no integral difference among $\{\lambda_{j_1}, \lambda_{j,2}, \dots, \lambda_{j,n_j}\}$ for each $j = 0, 1, \dots, p$,
(A2) for each $j = 1, \dots, p$, if some of $\lambda_{j,\nu}$ are integers, then one of them is 0.

If there is $\lambda_{j,\nu} \in \mathbb{Z} \setminus \{0\}$, we operate the addition

$$\mathrm{Ad}((x-c_j)^{-\lambda_{j,\nu}})(P)$$

to reduce $\lambda_{j,\nu}$ to 0, so that the assumption (A2) is satisfied. Later, we will discuss the cases where the assumptions are not satisfied. We may assume that, for every $1 \le j \le p$,

$$\lambda_{j,1} = 0;$$

if there is ν such that $\lambda_{j,\nu} = 0$, we change the indices ν and 1, and if there is no such ν, we set $\lambda_{j,1} = 0$ with $m_{j,1} = 0$. Similarly, we may assume

$$\lambda_{0,1} = \mu + 1,$$

where μ is the parameter of the middle convolution. We rewrite the Riemann scheme of P under this convention:

$$\left\{ \begin{array}{ccccc} x = c_1 & x = c_2 & \cdots & x = c_p & x = \infty \\ [0]_{(m_{1,1})} & [0]_{(m_{2,1})} & \cdots & [0]_{(m_{p,1})} & [\mu+1]_{(m_{0,1})} \\ [\lambda_{1,2}]_{(m_{1,2})} & [\lambda_{2,2}]_{(m_{2,2})} & \cdots & [\lambda_{p,2}]_{(m_{p,2})} & [\lambda_{0,2}]_{(m_{0,2})} \\ \vdots & \vdots & & \vdots & \vdots \\ [\lambda_{1,n_1}]_{(m_{1,n_1})} & [\lambda_{2,n_2}]_{(m_{2,n_2})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} & [\lambda_{0,n_0}]_{(m_{0,n_0})} \end{array} \right\}. \tag{10}$$

Then P can be written in the normal form

$$P = \prod_{j=1}^{p} (x - c_j)^{n - m_{j,1}} \partial^n + a_1(x)\partial^{n-1} + \cdots + a_{n-1}(x)\partial + a_n(x), \tag{11}$$

where

$$\deg a_k(x) \leq \sum_{j=1}^{p} (n - m_{j,1}) - k \quad (1 \leq k \leq p).$$

The last inequalities come from Fuchs's criterion for regular singularity at $x = \infty$. Therefore we have

$$w(P) = (p-1)n - \sum_{j=1}^{p} m_{1,j}.$$

Define k_0 by (8). Now we give an algorithmic definition of the middle convolution.

Definition 2.3. Retain the notation above. Namely we consider a Fuchsian differential operator P given by (11) with Riemann scheme (10). Since the weight of $\partial^{k_0} P$ is non-positive, we can write it as a polynomial in ∂ and the Euler operator $\vartheta = x\partial$:

$$\partial^{k_0} P = F(\partial, \vartheta)$$

with $F(X, Y) \in \mathbb{C}[X, Y]$. Then replace ϑ in $F(\partial, \vartheta)$ by $\vartheta - \mu$:

$$Q = F(\partial, \vartheta - \mu).$$

Write Q in transposed form, and divide it by ∂ from the left as many times as possible:

$$Q = \partial^q Q_1.$$

The operation

$$P \mapsto Q_1$$

is the *middle convolution* with parameter μ, and we denote

$$mc_\mu(P) = Q_1.$$

Here we introduce the result of [7, Theorem 5.2]. We set

$$\delta = \sum_{j=0}^{p} m_{j,1} - (p-1)n.$$

THEOREM 2.4. ([7, Theorem 5.2]) *Retain the notation above. We do not assume (A1) nor (A2), but instead assume*

(B1) $$m_{j,1} \geq \delta \quad (0 \leq j \leq p),$$

for $j = 0$ and $\nu \geq 1$, in the case $m_{1,1} m_{2,1} \cdots m_{p,1} \neq 0$,

(B2) $$\lambda_{0,\nu} \notin \{0, -1, \ldots, 2 - (m_{0,\nu} - m_{0,1} + \delta)\} \text{ if } m_{0,\nu} - m_{0,1} + \delta \geq 2,$$

and for $1 \leq j \leq p$ and $\nu \geq 2$, in the case $m_{j,1} \neq 0$,

(B3) $$\lambda_{j,\nu} \notin \{0, -1, \ldots, 2 - (m_{j,\nu} - m_{j,1} + \delta)\} \text{ if } m_{j,\nu} - m_{j,1} + \delta \geq 2.$$

Then the order of $mc_\mu(P) = Q_1$ becomes $n - \delta$, and the Riemann scheme of Q_1 is *given by*

(12)
$$\left\{\begin{array}{ccccc} x=c_1 & x=c_2 & \cdots & x=c_p & x=\infty \\ [0]_{(m_{1,1}-\delta)} & [0]_{(m_{2,1}-\delta)} & \cdots & [0]_{(m_{p,1}-\delta)} & [1-\mu]_{(m_{0,1}-\delta)} \\ [\lambda_{1,2}+\mu]_{(m_{1,2})} & [\lambda_{2,2}+\mu]_{(m_{2,2})} & \cdots & [\lambda_{p,2}+\mu]_{(m_{p,2})} & [\lambda_{0,2}-\mu]_{(m_{0,2})} \\ \vdots & \vdots & & \vdots & \vdots \\ [\lambda_{1,n_1}+\mu]_{(m_{1,n_1})} & [\lambda_{2,n_2}+\mu]_{(m_{2,n_2})} & \cdots & [\lambda_{p,n_p}+\mu]_{(m_{p,n_p})} & [\lambda_{0,n_0}-\mu]_{(m_{0,n_0})} \end{array}\right\}.$$

Moreover,

$$mc_\mu(P) = \partial^{-\delta}\mathrm{Ad}(\partial^{-\mu})(P)$$

holds.

We explain the reason why, in Definition 2.3, we replace ϑ by $\vartheta-\mu$. By using the relation (6) with $m=1$, we have

$$\mathrm{Ad}(\partial^{-\mu})(\vartheta) = \mathrm{Ad}(\partial^{-\mu})(x)\partial = \vartheta-\mu, \tag{13}$$

and of course $\mathrm{Ad}(\partial^{-\mu})(\partial^m)=\partial^m$. Thus we get

$$\mathrm{Ad}(\partial^{-\mu})F(\partial,\vartheta) = F(\partial,\vartheta-\mu).$$

Note that we may use any integer k greater than k_0 instead of k_0, because ∂ is commutative with $\mathrm{Ad}(\partial^{-\mu})$.

The result $mc_\mu(P)=Q_1$ is expected to be irreducible in $W(x)$ when P is irreducible in $W(x)$. For this problem, an answer is given in [7, Theorem 10.10]. Also a good criterion is obtained in [4, Proposition 3.3]. For rigid P, an exact result for irreducibility is obtained in [7, Theorem 10.13] and [6].

2.1. On assumptions (A1) and (A2)

Let $P\in W[x]$ be a Fuchsian operator with Riemann scheme (9). First we consider what happens if (A1) is not satisfied.

Suppose that $\lambda_{j,\nu}-\lambda_{j,\nu'}\in\mathbb{Z}$ for some $\nu\neq\nu'$. Without loss of generality, we may set $\lambda_{j,\nu'}-\lambda_{j,\nu}=k\in\mathbb{Z}_{\geq 0}$. Then, in general, solutions at $x=c_j$ of exponents $\lambda_{j,\nu},\lambda_{j,\nu}+1,\dots,\lambda_{j,\nu}+\min\{m_{j,\nu}-1,k\}$ have logarithmic terms. When there is no logarithmic term in these solutions, we call the case apparent. Notice that, in some literatures, the term apparent may be used only when the exponent is a non-negative integer.

We assume $\lambda_{j,1}=0$ with $m_{j,1}>0$, $\lambda_{j,\nu}\in\mathbb{Z}_{\geq 0}$ for some $\nu>1$, and there is no other exponent in $\mathbb{Z}$ at c_j. Theorem 2.4 can be applied in this case, and then in the result of a middle convolution with parameter $\mu \notin \mathbb{Z}$, the integral difference between $\lambda_{j,1}$ and $\lambda_{j,\nu}$ disappears. This implies that, if there is a logarithmic term in the solutions of exponent $\lambda_{j,1}$, it disappears by the middle convolution. On the other hand, if it is apparent, the solutions of exponent $\lambda_{j,1} = 0$ and of exponent $\lambda_{j,\nu}$ are both holomorphic at $x = c_j$. However, by a middle convolution with parameter $\mu \notin \mathbb{Z}$, it seems that the former remains holomorphic while the latter is sent to a singular solution. We shall explain the reason of this curious phenomenon in the next section.

Second, we shall see, by two examples, what occurs if the assumption (A2) is not satisfied.

Example 2.5. Let

$$P = x(1-x)\partial^2 + (c - (a+b+1)x)\partial - ab \tag{14}$$

be the differential operator for the Gauss hypergeometric differential equation. The Riemann scheme is given by

$$\begin{Bmatrix} x=0 & x=1 & x=\infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{Bmatrix}.$$

In order to increase the exponent 0 at $x = 0$ by 1, we operate the addition $\mathrm{Ad}(x)$ to get

$$P_1 = x^2(1-x)\partial^2 + ((c-2) - (a+b-1)x)x\partial - (a-1)(b-1)x + 2 - c.$$

The Riemann scheme for P_1 is given by

$$\begin{Bmatrix} x=0 & x=1 & x=\infty \\ 1 & 0 & a-1 \\ 2-c & c-a-b & b-1 \end{Bmatrix}.$$

We compute the middle convolution $mc_\mu(P_1)$. Noting that $w(P_1) = 1$, we have

$$\begin{aligned} Q_1 &= mc_\mu(P_1) \\ &= \mathrm{Ad}(\partial^{-\mu})(\partial P_1) \\ &= x^2(1-x)\partial^3 + x((c-2\mu) - (a+b+2-3\mu)x)\partial^2 \\ &\quad + ((1-a-b-ab+(1+2a+2b)\mu - 3\mu^2)x + \mu(1-c+\mu))\partial \end{aligned}$$

$$- (a-1-\mu)(b-1-\mu)(1-\mu).$$

The Riemann scheme for Q_1 is

$$\begin{Bmatrix} x=0 & x=1 & x=\infty \\ 0 & [0]_{(2)} & 1-\mu \\ 1+\mu & & a-1-\mu \\ 2-c+\mu & c-a-b+\mu & b-1-\mu \end{Bmatrix},$$

whose spectral type is $(111, 21, 111)$. Then Q_1 is the operator for the generalized hypergeometric series ${}_3F_2$, and we have the irreducibility condition [7, (10.58)]. Since $(1+\mu)+(1-\mu) \in \mathbb{Z}$, Q_1 is reducible. We can decompose Q_1 in $W(x)$:

$$Q_1 = K_1 L_1$$

with

$$\begin{aligned} K_1 &= \frac{x}{x-y}\partial - \frac{\mu}{x-y}, \\ L_1 &= x(1-x)(x-y)\partial^2 + f_1(x)\partial + f_2(x), \end{aligned}$$

where

$$\begin{aligned} y &= \frac{(\gamma-2)\mu}{(\alpha-1)(\beta-1)}, \\ f_1(x) &= (2\mu-\alpha-\beta+1)x^2 \\ &\quad - \frac{2(\gamma-2)\mu^2 + ((1+\alpha)(1+\beta)-\gamma(\alpha+\beta))\mu - (\alpha-1)(\beta-1)(\gamma-2)}{(\alpha-1)(\beta-1)}x \\ &\quad - \frac{(\gamma-2)(\gamma-\mu-1)\mu}{(\alpha-1)(\beta-1)}, \\ f_2(x) &= -(\alpha-\mu-1)(\beta-\mu-1)\left[x - \frac{(\gamma-2)(\mu-1)}{(\alpha-1)(\beta-1)}\right]. \end{aligned}$$

According to Definition 2.3, the operator Q_1 is the result of the middle convolution. However, as we have discussed, it may be natural to define that L_1 is the result of the middle convolution.

The Riemann scheme for L_1 is

$$\begin{Bmatrix} x=0 & x=1 & x=y & x=\infty \\ 0 & 0 & 0 & a-\mu-1 \\ 2-c+\mu & c-a-b+\mu & 2 & b-\mu-1 \end{Bmatrix},$$

and $x = y$ is an apparent singular point.

Example 2.6. We take the same P as Example 2.5, and operate the addition $\mathrm{Ad}(x^{-1})$. Then we get

$$P_2 = x^2(1-x)\partial^2 + ((c+2) - (a+b+3)x)x\partial - (a+1)(b+1)x + c,$$

whose Riemann scheme is

$$\begin{Bmatrix} x=0 & x=1 & x=\infty \\ -1 & 0 & a+1 \\ -c & c-a-b & b+1 \end{Bmatrix}.$$

We operate the middle convolution mc_μ to get

$$\begin{aligned} Q_2 &= mc_\mu(P_2) \\ &= \mathrm{Ad}(\partial^{-\mu})(\partial P_2) \\ &= x^2(1-x)\partial^3 + (q_{10} + q_{11}x)x\partial + (q_{20} + q_{21}x)\partial + q_3, \end{aligned}$$

where

$$\begin{aligned} q_{10} &= c + 4 - 2\mu, \\ q_{11} &= 3\mu - a - b - 6, \\ q_{20} &= 2c + 2 - (c+3)\mu + \mu^2, \\ q_{21} &= -3a - 3b - ab - 7 + (2a + 2b + 9)\mu - 3\mu^2, \\ q_3 &= (\mu - 1)(\mu - a - 1)(\mu - b - 1). \end{aligned}$$

The Riemann scheme of Q_2 is

$$\begin{Bmatrix} x=0 & x=1 & x=\infty \\ 0 & [0]_{(2)} & 1-\mu \\ -1+\mu & & a+1-\mu \\ -c+\mu & c-a-b+\mu & b+1-\mu \end{Bmatrix},$$

and we see that Q_2 is reducible. In fact, Q_2 is decomposed as

$$Q_2 = K_2 L_2$$

with

$$K_2 = x(1-x)\partial^2 + f(x)\partial - (\mu - a - 1)(\mu - b - 1),$$
$$L_2 = x\partial - (\mu - 1),$$

where

$$f(x) = (2\mu - a - b - 3)x + c + 1 - \mu.$$

According to Definition 2.3, Q_2 is the result of the middle convolution, but is reducible. In this case, the right factor is of order one.

As the above examples shows, if we use Definition 2.3, the result of a middle convolution of an irreducible equation is not necessarily irreducible if the assumption (A2) is not satisfied.

3. Analytic aspects of middle convolution

Oshima's theorem (Theorem 2.4) tells the change of the Riemann scheme by a middle convolution. We are interested in whether the Riemann-Liouville transform actually sends a solution to the equation before middle convolution to a solution to the equation after middle convolution. Oshima [7, Proposition 3.1] already showed that this holds for a class of solutions. In this section, we review Oshima's result, and then study the Riemann-Liouville transform in a little more detail. Throughout this section, we assume $\mu \notin \mathbb{Z}$.

Let P be a Fuchsian operator (11) with the Riemann scheme (10). We assume (A2). We consider a local solution at a finite singular point $x = c$ of exponent λ. (Namely c is one of c_j with $j \neq 0$ and $\lambda \in [\lambda_{j,\nu}]_{(m_{j,\nu})}$ with some $\lambda_{j,\nu}$.) We set

$$\vartheta_c = (x - c)\partial.$$

Let k_0 be given by (8). Then we can write

$$\partial^{k_0} P = \sum_{j,k \geq 0} a_{jk} \partial^j \vartheta_c^k \tag{15}$$

with $a_{jk} \in \mathbb{C}$. Then, since $\vartheta_c = \vartheta - c\partial$, the result of the middle convolution $Q = mc_\mu(P)$ is obtained from

$$\partial^q Q = \sum_{j,k \geq 0} a_{jk} \partial^j (\vartheta_c - \mu)^k.$$

We study three cases.

(i) $\lambda \notin \mathbb{Z}$ and $\lambda + \mu \notin \mathbb{Z}$.
This case is studied in [7, Proposition 3.1].
First we consider a solution to P of the form

$$u(x) = (x-c)^{\lambda}\varphi(x),$$

where

$$\varphi(x) = \sum_{l=0}^{\infty} \varphi_l (x-c)^l, \ \varphi_0 = 1.$$

We take a Riemann-Liouville transform with parameter μ and with end point c

$$\begin{aligned} v(x) &= I_c^{\mu}(u)(x) \\ &= \frac{1}{\Gamma(\mu)} \int_c^x u(t)(x-t)^{\mu-1}\,dt \\ &= \frac{1}{\Gamma(\mu)} \int_c^x (t-c)^{\lambda}\varphi(t)(x-t)^{\mu-1}\,dt. \end{aligned}$$

Since the exponent of the integrand at $t = c$ is not an integer, we can regularize the integral at $t = c$. Then by Proposition 2.2 (ii-ii), we have

$$\partial(I_c^{\mu}(u)(x)) = I_c^{\mu}(\partial u)(x).$$

Moreover, by the help of Proposition 2.2 (i), we further have

$$\begin{aligned} \vartheta_c(I_c^{\mu}(u)(x)) &= (x-c)\frac{1}{\Gamma(\mu)} \int_c^x u'(t)(x-t)^{\mu-1}\,dt \\ &= \frac{1}{\Gamma(\mu)} \int_c^x (x-t+t-c)u'(t)(x-t)^{\mu-1}\,dt \\ &= \frac{1}{\Gamma(\mu)} \left(\int_c^x u'(t)(x-t)^{\mu}\,dt + \int_c^x (t-c)u'(t)(x-t)^{\mu-1}\,dt \right) \\ &= \frac{1}{\Gamma(\mu)} \int_c^x u(t)\mu(x-t)^{\mu-1}\,dt + I_c^{\mu}(\vartheta_c u)(x) \\ &= \mu I_c^{\mu}(u)(x) + I_c^{\mu}(\vartheta_c u)(x), \end{aligned}$$

and then

$$(\vartheta_c - \mu)(I_c^{\mu}(u)(x)) = I_c^{\mu}(\vartheta_c u)(x). \tag{16}$$

These relations yields

$$\partial^j(\vartheta_c-\mu)^k(I_c^\mu(u)(x)) = I_c^\mu((\partial^j\vartheta_c^k u)(x)),$$

from which we obtain

$$\partial^q Qv = 0.$$

By putting the Taylor expansion of $\varphi(t)$ into the integral, we get

$$\begin{aligned}
v(x) &= \frac{1}{\Gamma(\mu)}\int_c^x \sum_{l=0}^\infty \varphi_l (t-c)^{\lambda+l}(x-t)^{\mu-1}\,dt \\
&= \frac{1}{\Gamma(\mu)}\sum_{l=0}^\infty \varphi_l \int_c^x (t-c)^{\lambda+l}(x-t)^{\mu-1}\,dt \quad (t-c=(x-c)s) \\
&= \frac{1}{\Gamma(\mu)}\sum_{l=0}^\infty \varphi_l \int_0^1 ((x-c)s)^{\lambda+l}((x-c)(1-s))^{\mu-1}(x-c)\,ds \\
&= (x-c)^{\lambda+\mu}\frac{1}{\Gamma(\mu)}\sum_{l=0}^\infty \varphi_l B(\lambda+l+1,\mu)(x-c)^l \\
&= (x-c)^{\lambda+\mu}\sum_{l=0}^\infty \frac{\Gamma(\lambda+l+1)}{\Gamma(\lambda+\mu+l+1)}\varphi_l(x-c)^l.
\end{aligned} \tag{17}$$

Thus $v(x)$ has the exponent $\lambda+\mu$. By the assumption $\lambda+\mu\notin\mathbb{Z}$, we get $Qv=0$ ([7, Proposition 3.1]). Hence in this case, the Riemann-Liouville transform sends the solution u to P of exponent λ at $x=c$ to the solution to Q of exponent $\lambda+\mu$ at $x=c$.

Next we consider a solution of exponent λ with logarithmic terms

$$u(x) = (x-c)^\lambda \sum_{j=0}^r \varphi_j(x)(\log(x-c))^j, \tag{18}$$

where $\varphi_j(x)$ are holomorphic at $x=c$. Such solution appears only when there is another exponent $\lambda+k$ at $x=c$ with $k\in\mathbb{Z}_{\geq 0}$. We shall study the Riemann-Liouville transform of this $u(x)$.

Note that the method of the regularization we have used so far does not work for logarithmic case. Nevertheless, we see that such integral can be regularized in the sense of Hadamard's finite part of divergent integral.

Let $\varphi(x)$ be holomorphic in a neighborhood U of c, and b a point in $U\setminus\{c\}$.

We consider the integral

$$L(\lambda) = \int_c^b \varphi(t)(t-c)^\lambda \log(t-c)\, dt$$

for $\lambda \in \mathbb{C}$. It is seen that the integral $L(\lambda)$ converges uniformly on any compact subset in $\{\lambda \in \mathbb{C} \mid \Re\lambda > 0\}$. By integration by parts, we obtain

$$\begin{aligned}
L(\lambda) &= \frac{1}{\lambda+1}(b-c)^{\lambda+1}\varphi(b)\log(b-c) \\
&\quad - \frac{1}{\lambda+1}\int_c^b (t-c)^{\lambda+1}\left\{\varphi'(t)\log(t-c) + \varphi(t)\cdot\frac{1}{t-c}\right\} dt \\
&= \frac{1}{\lambda+1}(b-c)^{\lambda+1}\varphi(b)\log(b-c) \\
&\quad - \frac{1}{\lambda+1}\int_c^b (t-c)^{\lambda+1}\varphi'(t)\log(t-c) - \frac{1}{\lambda+1}\int_c^b (t-c)^{\lambda}\varphi(t)\, dt \\
&= \frac{1}{\lambda+1}(b-c)^{\lambda+1}\varphi(b)\log(b-c) - \frac{1}{(\lambda+1)^2}(b-c)^{\lambda+1}\varphi(b) \\
&\quad - \frac{1}{\lambda+1}\int_c^b (t-c)^{\lambda+1}\varphi'(t)\log(t-c) + \frac{1}{(\lambda+1)^2}\int_c^b (t-c)^{\lambda+1}\varphi'(t)\, dt.
\end{aligned}$$

The integrals in the last side converge in $\Re(\lambda+1) > 0$. Thus $L(\lambda)$ is analytically continued to $\{\Re\lambda > -1\}$. Continuing this process, we see that $L(\lambda)$ can be analytically continued to $\mathbb{C} \setminus \mathbb{Z}_{<0}$. For $r \geq 1$, we set

$$L_r(\lambda) = \int_c^b \varphi(t)(t-c)^\lambda (\log(t-c))^r\, dt.$$

This integral is also holomorphic in $\{\Re\lambda > 0\}$. Again by integration by parts, we get

$$\begin{aligned}
L_r(\lambda) &= \frac{1}{\lambda+1}(b-c)^{\lambda+1}\varphi(b)(\log(b-c))^r p \\
&\quad - \frac{1}{\lambda+1}\int_c^b \varphi'(t)(t-c)^{\lambda+1}(\log(t-c))^r p\, dt - \frac{r}{\lambda+1}L_{r-1}(\lambda).
\end{aligned}$$

Thus $L_r(\lambda)$ can be analytically continued to $\{\Re\lambda > -1\}$ if $L_{r-1}(\lambda)$ can be. Then the problem is reduced to $L_1(\lambda) = L(\lambda)$, and hence $L_r(\lambda)$ is also analytically continued to $\mathbb{C} \setminus \mathbb{Z}_{<0}$.

Therefore, when we consider the Riemann-Liouville transform $I_c^\mu(u)$ for u in

(18), we may assume that the real part of λ is sufficiently large. Then we have similar assertions as Proposition 2.2 for the integral of $u(x)$, and hence (16) also holds in this case. Moreover, the formula

$$I_c^\mu((x-c)^\lambda\varphi(x)(\log(x-c))^r) = \frac{\partial^r}{\partial\lambda^r} I_c^\mu(\varphi(x)(x-c)^\lambda)$$

holds for $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, since it holds for $\Re\lambda > 0$. Note that, as a function in (λ, μ), this formula holds on $(\mathbb{C} \setminus \mathbb{Z}_{<0}) \times \mathbb{C}$. Then we have

$$\begin{aligned} v(x) &= I_c^\mu(u)(x) \\ &= \sum_{j=0}^r I_c^\mu((x-c)^\lambda\varphi_j(x)(\log(x-c))^j) \\ &= \sum_{j=0}^r \frac{\partial^j}{\partial\lambda^j} I_c^\mu((x-c)^\lambda\varphi_j(x)) \\ &= (x-c)^{\lambda+\mu}\sum_{j=0}^r \hat{\varphi}_j(x)(\log(x-c))^j, \end{aligned}$$

where $\hat{\varphi}_j(x)$ is a power series in $x-c$ obtained from $\varphi_j(x)$ in a similar way as in (17). Thus we have the solution to Q of exponent $\lambda+\mu$ with logarithmic terms.

In conclusion, the solution to P of exponent $[\lambda_{j,\nu}]_{(m_{j,\nu})}$ with $\lambda_{j,\nu} \notin \mathbb{Z}$ and $\lambda_{j,\nu}+\mu \notin \mathbb{Z}$ in the Riemann scheme (10) is sent by the Riemann-Liouville transform $I_{c_j}^\mu$ to the solution to Q of exponent $[\lambda_{j,\nu}+\mu]_{(m_{j,\nu})}$ in the Riemann scheme (12).

(ii) $\lambda \in [0]_{(m)}$.

We first consider a holomorphic solution of exponent $\lambda \in [0]_{(m)}$. If there is no other $[k]_{(m')}$ with $k \in \mathbb{Z}_{\geq 0}$ at $x=c$, the solution of exponent $\lambda \in [0]_{(m)}$ is always holomorphic at $x=c$. We can apply Proposition 2.2 (ii-i) to get

$$\partial(I_c^\mu(u)(x)) = \frac{1}{\Gamma(\mu)}u(c)(x-c)^{\mu-1} + I_c^\mu(\partial u)(x). \tag{19}$$

By a repeated use of this formula, we have

$$\partial^j(I_c^\mu(u)(x)) = \frac{1}{\Gamma(\mu)}\sum_{l=0}^{j-1}(\partial^l u)(c)\partial^{j-1-l}\left((x-c)^{\mu-1}\right) + I_c^\mu(\partial^j u)(x) \tag{20}$$

for $j \geq 1$. Multiplying $(x-c)$ to (19), we get

$$\begin{aligned}
\vartheta_c(I_c^\mu(u)(x)) &= \frac{1}{\Gamma(\mu)}\left(u(c)(x-c)^\mu + (x-c)\int_c^x (\partial u)(t)(x-t)^{\mu-1}\,dt\right)\\
&= \frac{1}{\Gamma(\mu)}\left(u(c)(x-c)^\mu + \int_c^x (x-t+t-c)(\partial u)(t)(x-t)^{\mu-1}\,dt\right)\\
&= \frac{1}{\Gamma(\mu)}\left(u(c)(x-c)^\mu + \int_c^x u'(t)(x-t)^\mu\,dt + \int_c^x (\vartheta_c u)(t)(x-t)^{\mu-1}\,dt\right)\\
&= \frac{1}{\Gamma(\mu)}\left(u(c)(x-c)^\mu - u(c)(x-c)^\mu + \mu\int_c^x u(t)(x-t)^{\mu-1}\,dt\right.\\
&\qquad\left. + \int_c^x (\vartheta_c u)(t)(x-t)^{\mu-1}\,dt\right)\\
&= \mu I_c^\mu(u)(x) + I_c^\mu(\vartheta_c u)(x).
\end{aligned}$$

Thus we obtain

$$(\vartheta_c - \mu)(I_c^\mu(u)(x)) = I_c^\mu(\vartheta_c u)(x).$$

Therefore

$$(\vartheta_c - \mu)^k(I_c^\mu(u)(x)) = I_c^\mu(\vartheta_c^k u)(x) \tag{21}$$

holds for $k \geq 1$.

Lemma 3.1. *Let $(x-c)$ denote the right ideal in $W[x]$ generated by $x-c$. Then, for $j,k \geq 0$, we have*

$$\partial^j \vartheta_c^k \equiv j^k \partial^j \qquad \mathrm{mod}\ (x-c).$$

Proof. We write

$$\partial^j \vartheta_c^k = \partial^j (x-c)\partial(x-c)\partial\cdots(x-c)\partial.$$

The left most $x-c$ is killed by one of j ∂'s in the left of the factor. The second left $x-c$ is killed by one of ∂'s in the left of the factor, whose number is $(j+1)-1=j$ when one ∂ has been used to kill the left most $x-c$. In a similar way, we see that there are j possibilities to kill each $x-c$. Thus the total number is $j\times j\times\cdots\times j = j^k$. □

Since the elements in the ideal $(x-c)$ vanish when we put $x=c$, this lemma shows

$$(\partial^j \vartheta_c^k u)(c) = j^k (\partial^j u)(c).$$

Combining this with (19) and (21), we obtain

$$\partial^j (\vartheta_c - \mu)^k (I_c^\mu (u)(x)) = \sum_{l=1}^{j-1} l^k (\partial^l u)(c) \partial^{j-1-l} \left((x-c)^{\mu-1} \right) + I_c^\mu (\partial^j \vartheta_c^k u)(x). \tag{22}$$

Then we get

$$\partial^q Q(I_c^\mu (u)(x)) = I_c^\mu (\partial^{k_0} P u)(x) + R, \tag{23}$$

where

$$\begin{aligned} R = \frac{1}{\Gamma(\mu)} \Bigg(& \sum_{j \geq 1} a_{j0} \sum_{l=0}^{j-1} (\partial^l u)(c) \partial^{j-1-l} \left((x-c)^{\mu-1} \right) \\ & + \sum_{k \geq 1} \sum_{j \geq 2} a_{jk} \sum_{l=1}^{j-1} l^k (\partial^l u)(c) \partial^{j-1-l} \left((x-c)^{\mu-1} \right) \Bigg). \end{aligned} \tag{24}$$

Because of the existence of this remainder R, $I_c^\mu (u)(x) = v(x)$ does not become a solution to Q.

In order to obtain a solution to Q from a solution to P of exponent $\lambda \in [0]_{(m)}$, we may take another end point of the Riemann-Liouville transform. Suppose that there is an l such that all solutions at $x = c_l$ can be sent to solutions to Q by the Riemann-Liouville transform $I_{c_l}^\mu$ with the end point c_l. For example, if $\lambda_{l,\nu} \notin \mathbb{Z}$ for some $l \neq 0$ and for all ν, we can take this l.

The solution $u(x)$ of exponent $\lambda \in [0]_{(m)}$ at $x = c$ is written as a linear combination of the solutions to P of exponents $[\lambda_{l,2}]_{(m_{l,2})}, \dots, [\lambda_{l,n_l}]_{(m_{l,n_l})}$ in a neighborhood of $x = c_l$. Thanks to the result in the case (i), we see that each solution in the linear combination is sent to a solution to Q by the Riemann-Liouville transform $I_{c_l}^\mu$ with end point c_l. Thus

$$v(x) = I_{c_j}^\mu (u)(x) = \frac{1}{\Gamma(\mu)} \int_{c_j}^x u(t)(x-t)^{\mu-1}\, dt$$

is a solution to Q. We may deform the path of integration so that it does not pass through $t = c$. Then, by noting the expansion

$$(x-t)^{\mu-1} = ((x-c)+(c-t))^{\mu-1} = (c-t)^{\mu-1}\sum_{l=0}^{\infty}\binom{\mu-1}{l}\left(\frac{x-c}{c-t}\right)^{l},$$

we find that $v(x)$ is holomorphic at $x = c$. Hence the solution to P of exponent $\lambda \in [0]_{(m)}$ at $x = c$ is sent to a holomorphic solution to Q.

Next we consider the logarithmic case. If there is another $[k]_{(m')}$ with $k \in \mathbb{Z}_{\geq 0}$ at $x = c$, the solution of exponent $\lambda \in [0]_{(m)}$ may contain logarithmic terms. Take $\lambda \in [0]_{(m)}$, and assume that the solution $u(x)$ is of the form (18) with this integer λ. We take $h \in \mathbb{C} \setminus \{0\}$ with $|h|$ sufficiently small. Then $u(x)$ is holomorphic at $x = c + h$, and hence the above results can be applied to this $u(x)$ with the Riemann-Liouville transform with the end point $c + h$. In particular, we get

$$\partial^q Q(I^{\mu}_{c+h}(u)(x)) = I^{\mu}_{c+h}(\partial^{k_0} Pu)(x) + R_h,$$

where R_h is obtained from R in (24) by replacing c by $c + h$. Since

$$\lim_{h\to 0}(\partial^{\lambda}u)(c+h) = \infty$$

by the existence of the logarithmic terms $(\log(x-c))^j$, and since $I^{\mu}_{c+h}(\partial^{k_0}Pu)(x)$ converges as $h \to 0$, we see that $\partial^q Q(I^{\mu}_{c+h}(u)(x))$ diverges. Then the Riemann-Liouville transform I^{μ}_{c} does not send $u(x)$ to a solution to Q.

Also in this case, we may take another Riemann-Liouville transform $I^{\mu}_{c_l}$ to get a solution to Q by a similar reason.

(iii) $m > 0$ and $\lambda \in \mathbb{Z}$ with $\lambda > m$.

Namely we assume that $[0]_{(m)}$ exists in the Riemann scheme. This follows from the assumption (A2) when such λ exists.

First we consider the non-logarithmic case. Since λ is a positive integer, the solution $u(x)$ of exponent λ is holomorphic at $x = c$. Then we can apply the results in (ii), and hence (23) holds. We shall show that the remainder term R vanishes in this case.

For a moment, we assume that $w(P)$ is non-positive, and hence $k_0 = 0$. We can write P as

$$P = \sum_{j=0}^{n-m-1}(x-c)^{n-m-j}\psi_j(x)\partial^{n-j} + \sum_{j=n-m}^{n}\psi_j(x)\partial^{n-j},$$

where $\psi_j(x)$ are polynomials. Now we rewrite this P as in the form (15) with $k_0 = 0$. From the term $(x-c)^{n-m-j}\psi_j(x)\partial^{n-j}$, we obtain terms $\partial^r \vartheta_c^s$ in (15), and the degree r of ∂ is at most

$$(n-j)-(n-m-j)=m.$$

From the term $\psi_j(x)\partial^{n-j}$ with $j \geq n-m$, evidently we obtain terms $\partial^r \vartheta_c^s$ with $r \leq m$. Therefore the degree of ∂ in (15) is at most m. If $w(P) > 0$, we operate ∂^{k_0}, which is used to transform $(x-c)$ to ϑ_c. Then also in this case, we see that the degree of ∂ in (15) is at most m.

Hence the remainder R given by (24) is a linear combination of $u(c), (\partial u)(c), \ldots, (\partial^{m-1}u)(c)$. Since the order of $u(x)$ at $x = c$ is greater than m, all these vanish. Then we have $R = 0$, which implies that $I_c^\mu(u)(x)$ is a solution to Q. Moreover, the exponent of the solution $I_c^\mu(u)(x)$ is $\lambda+\mu$, since the argument in (i) can be applied to this case.

The logarithmic case is similar. As in the logarithmic case in (ii), we consider the Riemann-Liouville transform I_{c+h}^μ with end point $c+h$, and take the limit $h \to 0$. Then the remainder R_h goes to 0 because $\lambda > m$. Thus, also in this logarithmic case, the Riemann-Liouville transform I_c^μ sends the solution to P of exponent λ to the solution to Q of exponent $\lambda+\mu$.

Summing up, we obtain the following assertion.

Theorem 3.2. *Let P be a Fuchsian operator (11) with the Riemann scheme (10) satisfying the assumption (A2). Let μ be in $\mathbb{C}\setminus\mathbb{Z}$, and $Q = mc_\mu(P)$ the result of the middle convolution with parameter μ. We consider solutions at a singular point $x = c_j$ with $1 \leq j \leq p$. Let λ be one of the exponents at $x = c_j$.*

(i) *If $\lambda \notin \mathbb{Z}$ and $\lambda+\mu \notin \mathbb{Z}$, the solution to P of exponent λ is sent to the solution to Q of exponent $\lambda+\mu$ by the Riemann-Liouville transform $I_{c_j}^\mu$ with the end point c_j.*

(ii) *If $\lambda \in [0]_{(m)}$, the Riemann-Liouville transform I_c^μ with the end point c does not send the solution to P of exponent λ to a solution to Q. If there is another singular point c_l such that the solution of any exponent in $[\lambda_{l,\nu}]_{(m_{l,\nu})}$ for any ν is sent to a solution to Q by $I_{c_l}^\mu$, the Riemann-Liouville transform $I_{c_l}^\mu$ with the end point c_l sends the solution to P of exponent $\lambda \in [0]_{(m)}$ at $x = c$ to a solution to Q holomorphic at $x = c$.*

(iii) *If there is $[0]_{(m)}$ at $x = c_j$ in the Riemann scheme (10) and if $\lambda \in \mathbb{Z}$ with $\lambda > m$, the solution to P of exponent λ is sent to the solution to Q of exponent $\lambda+\mu$ by the Riemann-Liouville transform $I_{c_j}^\mu$ with the end point c_j.*

Example 3.3. The generalized hypergeometric equation of order 3 is obtained from the Gauss hypergeometric equation by an addition and an middle convolution. We start from the operator P given by (14) in Example 2.5. P is the operator for the Gauss hypergeometric equation. First we operate $\mathrm{Ad}(x^d)$, and then operate mc_μ to obtain the operator Q for the generalized hypergeometric equation of order 3. The change of the Riemann scheme is given by

$$\begin{Bmatrix} x=0 & x=1 & x=\infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{Bmatrix} \to \begin{Bmatrix} x=0 & x=1 & x=\infty \\ d & 0 & a-d \\ 1-c+d & c-a-b & b-d \end{Bmatrix}$$
$$\to \begin{Bmatrix} x=0 & x=1 & x=\infty \\ 0 & [0]_{(2)} & 1-\mu \\ d+\mu & & a-d-\mu \\ 1-c+d+\mu & c-a-b+\mu & b-d-\mu \end{Bmatrix}$$

We assume that the parameters a, b, c, d and μ are generic. We shall show that the Riemann-Liouville transform I_0^μ with the end point 0 sends the solution to $\mathrm{Ad}(x^d)P$ of exponent 0 at $x = 1$ to a solution to Q holomorphic at $x = 1$.

For simplicity, we assume $x \in \mathbb{R}$ and $0 < x < 1$. Any solution to P can be expressed by an integral

$$\int_C s^{a-c}(s-1)^{c-b-1}(x-s)^{-a}\, ds$$

with a twisted cycle C. In particular the solution to $\mathrm{Ad}(x^d)P$ of exponent 0 at $x = 1$ is given by

$$u(x) = x^d \int_{-\infty}^{0} s^{a-c}(s-1)^{c-b-1}(x-s)^{-a}\, ds.$$

As we have seen, the Riemann-Liouville transform I_1^μ with the end point 1 does not send this $u(x)$ to a solution to Q. Since the exponents of $\mathrm{Ad}(x^d)P$ at $x = 0$ are not integers, we can take the Riemann-Liouville transform I_0^μ with the end point 0. Then the result $v(x) = I_0^\mu(u)(x)$ becomes

$$\begin{aligned} v(x) &= \frac{1}{\Gamma(\mu)} \int_0^x dt\, t^d \int_{-\infty}^{0} s^{a-c}(s-1)^{c-b-1}(t-s)^{-a}\, ds\, (x-t)^{\mu-1} \\ &= \frac{1}{\Gamma(\mu)} \iint_\Delta s^{a-c}(s-1)^{c-b-1}(t-s)^{-a} t^d (x-t)^{\mu-1}\, ds\, dt, \end{aligned}$$

where

$$\Delta = \{(s,t) \in \mathbb{R}^2 \mid -\infty < s < 0,\, 0 < t < x\}.$$

By a standard argument in analysis, we can see that this integral over Δ gives a holomorphic function at $x = 1$. Thus the Riemann-Liouville transform with end point 0 sends the holomorphic solution to $\mathrm{Ad}(x^d)P$ at $x = 1$ to a holomorphic solution to Q at $x = 1$.

In Theorem 3.2, we have not studied the case $\lambda \in \mathbb{Z}_{<0}$. In order to study this case, it is useful to consider the Pochhammer cycle. We use the notation L for the chain from c to x as in Section 2. Put $L(\epsilon_1) = a$ and $L(1-\epsilon_2) = b$. Let C_1 (resp. C_2) be the circle with center c (resp. x) of radius $|a-c|$ (resp. $|b-x|$). The Pochhammer cycle Δ is the connected chain

$$\Delta = C_1 \cdot L_2 \cdot C_2 \cdot L_2^{-1} \cdot C_1^{-1} \cdot L_2 \cdot C_2^{-1} \cdot L_2^{-1}, \tag{25}$$

where we have set $L_2 = L|_{[\epsilon_1, 1-\epsilon_2]}$. As the integrand, we consider a linear combination of

$$f(t) = \varphi(t)(t-c)^{\lambda}(x-t)^{\mu-1}$$

and

$$g_j(t) = \varphi_j(t)(t-c)^{\lambda}(\log(t-c))^j(x-t)^{\mu-1},$$

where $\varphi(t), \varphi_j(t)$ are holomorphic in a neighborhood of L, $\lambda, \mu \in \mathbb{C}$ and $j \in \mathbb{Z}_{>0}$. Then the branch of $f(t)$ or $g_j(t)$ at the end point a of the right most L_2^{-1} coincides with that at the starting point a of the left most component C_1. If we take $f(t)$ as the integrand, and if we assume that $\lambda, \mu \notin \mathbb{Z}$ and $\Re\lambda, \Re\mu$ are sufficiently large, we have

$$\Delta = (e^{2\pi i\lambda} - 1)(1 - e^{2\pi i\mu})L$$

as integrals. Since the integral over L is related to the Riemann-Liouville transform, under the same assumption, we also have

$$I_c^{\mu} = \frac{1}{\Gamma(\mu)(e^{2\pi i\lambda} - 1)(1 - e^{2\pi i\mu})}\Delta$$

as integral operators for $f(t)$. Thus, up to scalar multiple, we may regard Δ as an extension of I_c^μ.

Let $u(t)$ be a linear combination

$$u(t) = \varphi(t)(t-c)^\lambda + \sum_{j=1}^{r} \varphi_j(t)(t-c)^\lambda (\log(t-c))^j.$$

Then the branches of $u(t)(x-t)^{\mu-1}$ at the starting point of Δ and at the end point of Δ coincide, from which we obtain

$$\begin{aligned} \partial \int_\Delta u(t)(x-t)^{\mu-1}\,dt &= \int_\Delta (\partial u)(t)(x-t)^{\mu-1}\,dt, \\ (\vartheta_c - \mu) \int_\Delta u(t)(x-t)^{\mu-1}\,dt &= \int_\Delta (\vartheta u)(t)(x-t)^{\mu-1}\,dt. \end{aligned}$$

Hence we get

$$\partial^q Q \int_\Delta u(t)(x-t)^{\mu-1}\,dt = \int_\Delta (\partial^{k_0} P u)(t)(x-t)^{\mu-1}\,dt,$$

which means that the integral transformation

$$u(x) \mapsto \int_\Delta u(t)(x-t)^{\mu-1}\,dt$$

sends a solution $u(t)$ to P to a solution to $\partial^q Q$. This assertion holds even if $\lambda \in \mathbb{Z}_{<0}$.

4. ODE for Appell's F_4

Appell's hypergeometric series F_4 is defined by

$$F_4(a,b,c,d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m (d)_n m! n!} x^m y^n,$$

where $(a)_n$ is defined in (7). From this definition, we can derive the system of partial differential equations satisfied by F_4:

(26)

$$\begin{aligned} &\Big[x(1-x)\partial_x^2 - 2xy\partial_x\partial_y - y^2\partial_y^2 + (c-(a+b+1)x)\partial_x - (a+b+1)y\partial_y - ab\Big] z = 0, \\ &\Big[y(1-y)\partial_y^2 - 2xy\partial_x\partial_y - x^2\partial_x^2 + (d-(a+b+1)y)\partial_y - (a+b+1)x\partial_x - ab\Big] z = 0. \end{aligned}$$

We can further derive a Pfaffian system

$$dZ = \Big(A(x,y)\,dx + B(x,y)\,dy\Big)Z \tag{27}$$

from (26), and then notice that the rank of the system (26) is four and the singular locus is given by

$$\{x=0\}\cup\{y=0\}\cup\{(1-x-y)^2-4xy=0\}\cup\{x=\infty\}\cup\{y=\infty\}. \tag{28}$$

We look at the x-equation of (27):

$$\frac{\partial Z}{\partial x} = A(x,y)Z. \tag{29}$$

As an ordinary differential equation, (29) has the Riemann scheme (as a system)

$$\left\{\begin{matrix} x=0 & x=(\sqrt{y}-1)^2 & x=(\sqrt{y}+1)^2 & x=\infty \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ -c & 0 & 0 & a-d+1 \\ -c & c+d-a-b-\frac{5}{2} & c+d-a-b-\frac{5}{2} & b-d+1 \end{matrix}\right\},$$

and the spectral type is $(22, 31, 31, 1111)$, which means that the index of rigidity is 0. Moreover, this spectral type can be connected to the spectral type $(11, 11, 11, 11)$ by an iteration of middle convolutions and an addition. This can be seen by the chase of the spectral types

$$\begin{array}{ll} (\underline{2}2, \underline{3}1, \underline{3}1, \underline{1}111) & \delta = 2+3+3+1-2\times 4 = 1 \\ \quad\downarrow & \\ (12, 21, 21, 111) & \\ \quad\downarrow & \\ (\underline{2}1, \underline{2}1, \underline{2}1, \underline{1}11) & \delta = 2+2+2+1-2\times 3 = 1 \\ \quad\downarrow & \\ (11, 11, 11, 11). & \end{array}$$

The last spectral type $(11, 11, 11, 11)$ is the spectral type of the ordinary differential equation whose deformation equation yields Painlevé VI. Since the deformation equations are invariant under middle convolutions and additions [3], the deformation of (29) will give also Painlevé VI. On the other hand, the ordinary differential equation (29) is already deformed, since it is a section of the completely integrable system (27). Thus we expected that the entries of the coefficient matrix $A(x, y)$ of

(29) give an algebraic solution of Painlevé VI. However, this story does not work: As far as we know, the equation (29) cannot be transformed to a normal Fuchsian system of non-resonant type, so that we cannot apply Katz-Dettweiler-Reiter algorithm [1, 5].

Therefore we expect that Oshima's middle convolution will work. Then we shall compute the ordinary differential equation for F_4 with respect to x.

By a simple but long calculation, we derive from (26) the ordinary differential equation

$$\Big[p_0(x)\partial_x^4 + p_1(x)\partial_x^3 + p_2(x)\partial_x^2 + p_3(x)\partial_x + p_4(x)\Big]z = 0 \tag{30}$$

in x satisfied by F_4, where $p_j(x)$ are polynomials in x depending polynomially on y. The explicit form of $p_0(x)$ is given by

$$p_0(x) = x^2((1-x-y)^2 - 4xy)(v_0x - v_1),$$

where

$$\begin{aligned} v_0 &= (a-b-d+1)(a-b+d-1), \\ v_1 &= (a+b-d+1)(a+b-2c-d+3)(y-1), \end{aligned}$$

and those of the other $p_j(x)$ are given in Appendix. We see that the point $v = v_1/v_0$ is a singular point that does not come from the singular locus (28). Hence it is an apparent singular point. The Riemann scheme of (30) is

$$\begin{aligned} &\left\{\begin{matrix} x=0 & x=(\sqrt{y}-1)^2 & x=(\sqrt{y}+1)^2 & x=v & x=\infty \\ 0 & 0 & 0 & 0 & a \\ 1 & 1 & 1 & 1 & b \\ 1-c & 2 & 2 & 2 & a-d+1 \\ 2-c & -a-b+c+d-\frac{1}{2} & -a-b+c+d-\frac{1}{2} & 4 & b-d+1 \end{matrix}\right\} \\ &= \left\{\begin{matrix} x=0 & x=(\sqrt{y}-1)^2 & x=(\sqrt{y}+1)^2 & x=v & x=\infty \\ [0]_{(2)} & [0]_{(3)} & [0]_{(3)} & [0]_{(3)} & a \\ & & & & b \\ [1-c]_{(2)} & & & & a-d+1 \\ & -a-b+c+d-\frac{1}{2} & -a-b+c+d-\frac{1}{2} & 4 & b-d+1 \end{matrix}\right\} \end{aligned} \tag{31}$$

Since $x = v$ is apparent, any solution is single valued at $x = v$. Then the spectral type of the monodromy for (30) is $(22, 31, 31, 1111)$. According to [7, Definition 4.8], the spectral type of (30) is determined by the Riemann scheme (31), and then is $(22, 31, 31, 31, 1111)$. Thus there arises a difference between the spectral types of the monodromy and of the equation. For the spectral type $(22, 31, 31, 31, 1111)$ of (30), the maximum of δ is

$$\delta = 2 + 3 + 3 + 3 + 1 - 3 \times 4 = 0,$$

which implies that we cannot reduce the order by additions and middle convolutions. (Therefore we have not succeeded to obtain an algebraic solution to Painlevé VI even by Oshima's middle convolution.) We examine this.

We operate the middle convolution with parameter $\mu = a - 1$ to (30). Then the order of the result is four, and the Riemann scheme is given by

$$\left\{ \begin{array}{ccccc} x = 0 & x = (\sqrt{y} - 1)^2 & x = (\sqrt{y} + 1)^2 & x = v & x = \infty \\ [0]_{(2)} & [0]_{(3)} & [0]_{(3)} & [0]_{(3)} & 2 - a \\ & & & & 1 - a + b \\ [a - c]_{(2)} & & & & 2 - d \\ & -b + c + d - \frac{3}{2} & -b + c + d - \frac{3}{2} & 3 + a & 2 - a + b - d \end{array} \right\}.$$

We see that the singular point $x = v$ changes from an apparent singular point to a non-apparent one. Then the spectral type of the monodromy becomes $(22, 31, 31, 31, 1111)$, which is different from the one before middle convolution. Thus, owing to the existence of an apparent singular point, the compatibility of the middle convolutions for the monodromy and for the equation does not hold.

5. Appendix

The coefficients of the ordinary differential equation (30) satisfied by F_4 are given as follows:

$$\begin{aligned} p_1(x) &= 2x(p_{13}x^3 + p_{12}x^2 + p_{11}x + p_{10}), \\ p_2(x) &= p_{23}x^3 + p_{22}x^2 + p_{21}x + p_{20}, \\ p_3(x) &= p_{32}x^2 + p_{31}x + p_{30}, \\ p_4(x) &= ab(a - d + 1)(b - d + 1)(p_{41}x + p_{40}), \end{aligned}$$

where

$$\begin{aligned} p_{13} &= (a - b - d + 1)(a - b + d - 1)(a + b - d + 4), \\ p_{12} &= \big[-2(a^3 + b^3) - 2ab(a + b) + (c + 4d - 13)(a^2 + b^2) + (6c + 4d - 8)ab \\ &\quad + (-4cd - 2d^2 + 11c + 15d - 20)(a + b) + (d - 1)(3cd - 10c - 2d + 9)\big]y \\ &\quad + 4ab(a + b) + (-3c - 2d + 4)(a^2 + b^2) + (-2c - 8d + 26)ab \end{aligned}$$

$$
\begin{aligned}
&+ (c+d-2)(4d-11)(a+b) - (d-1)(cd+2d^2-8c-13d+18),\\
p_{11} &= (y-1)\Big[\big\{(a+b)^3 + (10-3d)(a^2+b^2) + (18-4c-6d)ab\\
&+ (-2c^2+2cd+3d^2-9c-19d+25)(a+b)\\
&+ (d-1)(2c^2-2cd-d^2+9c+8d-16)\big\}y\\
&+ (a^3+b^3) + 3ab(a+b) + (9-2c-3d)(a^2+b^2) + (20-6d)ab\\
&+ (-2c^2+2cd+3d^2-9c-19d+25)(a+b)\\
&+ (d-1)(2c^2-d^2+7c+9d-17)\Big],\\
p_{10} &= -(a+b-d+1)(a+b-2c-d+3)(c+1)(y-1)^3,
\end{aligned}
$$

$$
\begin{aligned}
p_{23} &= (a-b-d+1)(a-b+d-1)(a^2+4ab+b^2+(9-3d)(a+b)+d^2-8d+14),\\
p_{22} &= \big[-2(a^4+b^4) - 12a^2b^2 - 4ab(a^2+b^2) + 6(d-3)(a^3+b^3)\\
&+ 2(6c+9d-24)ab(a+b) + (-6cd-7d^2+18c+47d-68)(a^2+b^2)\\
&+ 2(-12cd-9d^2+30c+28d-61)ab\\
&+ 2(d-3)(5cd+2d^2-10c-14d+17)(a+b)\\
&- (d-1)(4cd^2+d^3-26cd-10d^2+42c+39d-50)\big]y\\
&+ (a^4+b^4) + 4ab(a^2+b^2) + 14a^2b^2 + (11-4c-4d)(a^3+b^3)\\
&+ (55-8c-20d)ab(a+b) + (10cd+7d^2-28c-42d+57)(a^2+b^2)\\
&+ (16cd+22d^2-40c-114d+148)ab\\
&+ (-6cd^2-6d^3+42cd+51d^2-56c-134d+109)(a+b)\\
&+ 2(d-1)(d^3-6cd-9d^2+16c+29d-31),\\
p_{21} &= (y-1)\Big[\big\{(a^4+b^4) + 2ab(a^2+b^2) + 2a^2b^2 + 3(3-d)(a^3+b^3)\\
&+ (17+4c-5d)ab(a+b) + (-3c^2-2cd+3d^2+3c-21d+31)(a^2+b^2)\\
&+ (-10c^2-8cd+4d^2+14c-32d+54)ab\\
&+ (8c^2d+4cd^2-d^3-18c^2-12cd+15d^2+6c-53d+51)(a+b)\\
&- (d-1)(5c^2d+2cd^2-15c^2-7cd+3d^2+3c-19d+28)\big\}y\\
&+ 2ab(a^2+b^2) + 4a^2b^2 + (4+2c-d)(a^3+b^3)\\
&+ (22+2c-7d)ab(a+b) + (-5c^2-4cd+3d^2+7c-18d+28)(a^2+b^2)\\
&+ (-6c^2-4cd+8d^2+6c-46d+64)ab
\end{aligned}
$$

$$
\begin{aligned}
&+ (8c^2d + 2cd^2 - 3d^3 - 18c^2 - 8cd + 24d^2 + 4c - 65d + 56)(a + b) \\
&- (d - 1)(3c^2d - d^3 - 13c^2 - cd + 9d^2 - c - 28d + 32)\Big], \\
p_{20} &= c(c + 1)(a + b - d + 1)(-a - b + 2c + d - 3)(y - 1)^3,
\end{aligned}
$$

$$
\begin{aligned}
p_{32} &= (a - b - d + 1)(a + b - d + 2)(a - b + d - 1)(2ab + (2 - d)(a + b + 1)), \\
p_{31} &= \big[-6a^2b^2(a + b) - 2ab(a^3 + b^3) + (d - c - 2)(a^4 + b^4) \\
&\quad + (6c + 10d - 24)ab(a^2 + b^2) + (6c + 18d - 40)a^2b^2 \\
&\quad - (d - 3)(c + 3d - 6)(a^3 + b^3) + (-15cd - 15d^2 + 31c + 73d - 84)ab(a + b) \\
&\quad + (5cd^2 + 3d^3 - 21cd - 25d^2 + 23d + 63d - 50)(a^2 + b^2) \\
&\quad + (12cd^2 + 8d^3 - 56cd - 62d^2 + 58c + 152d - 116)ab \\
&\quad - (d - 3)(3cd^2 + d^3 - 12cd - 10d^2 + 11c + 25d - 18)(a + b) \\
&\quad - (d - 1)(d - 2)(3cd + d^2 - 7c - 7d + 10)\big]y \\
&\quad + 6a^2b^2(a + b) + 2ab(a^3 + b^3) + (2 - d)(a^4 + b^4) + (40 - 4c - 18d)a^2b^2 \\
&\quad + (24 - 6c - 10d)ab(a^2 + b^2) + 3(d - 2)(c + d - 3)(a^3 + b^3) \\
&\quad + (13cd + 15d^2 - 28c - 73d + 84)ab(a + b) \\
&\quad + (d - 2)(5cd + 3d^2 - 12c - 19d + 25)(a^2 + b^2) \\
&\quad + 2(4cd^2 + 4d^3 - 23cd - 31d^2 + 26c + 76d - 58)ab \\
&\quad + (d - 2)(cd^2 + d^3 - 12cd - 11d^2 + 15c + 33d - 27)(a + b) \\
&\quad + (d - 1)(d - 2)^2(cd + 3c + d - 5), \\
p_{30} &= (y - 1)c(a + b - d + 2)(a + b - 2c - d + 3) \\
&\quad \times \big[(a^2 + b^2 + (2 - d)(a + b) - d + 1)y + 2ab - (d - 2)(a + b - d + 1)\big].
\end{aligned}
$$

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