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Unfolding of spectral types

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Dedicated to Professor Toshio Oshima on the occasion of his 70th birthday

Abstract. A conjecture on unfolding of spectral types of differential equations is presented by Toshio Oshima. We claim that a part of this conjecture is true for first order systems of linear ordinary differential equations and give a sketch of proof of it.

Introduction

This paper is an announcement of one of the results in [6].

Let us consider the differential equation of Gauss hypergeometric function,

$$z(1-z)\frac{d^2}{dz^2}y + (c-(a+b+1)z)\frac{d}{dz}y - aby = 0.$$

This is one of the most famous Fuchsian differential equation and enjoys many interesting properties. Let us put

$$z = \epsilon\xi, \quad b = \frac{1}{\epsilon}$$

and rewrite the equation as follows

$$\xi(1-\epsilon\xi)\frac{d^2}{d\xi^2}y + (c-(a+\frac{1}{\epsilon}+1)\epsilon\xi)\frac{d}{d\xi}y - ay = 0.$$

Then taking the limit $\epsilon \mapsto 0$, we can see that the regular singular point $\xi = \frac{1}{\epsilon}$ goes to infinity and the resulting differential equation is

$$\xi\frac{d^2}{d\xi^2}y + (c-\xi)\frac{d}{d\xi}y - ay = 0$$

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which is called Kummer's confluent hypergeometric equation and has an irregular singular point at $\xi = \infty$. This process called confluence of singular points and is a classical technique to study analytic properties of differential equations with irregular singular points. It might be a natural expectation that every differential equation with irregular singular points can be obtained from a Fuchsian equation, differential equation only with regular singular points, through the confluence process. This is always true in a naive sense. However, known families of differential equations obtained by confluence of singular points usually share the same numbers of accessory parameters, for example Gauss hypergeometric family: Gauss, Kummer, Hermite, and Heun's family: Heun, confluent Heun, doubly confluent Heun, biconfluent Heun, triconfluent Heun, and so on.

Thus, we may ask if we can obtain every differential equation with irregular singular points from a Fuchsian equation with the same number of accessory parameter by confluence of singular points.

To compute a number of accessory parameters of a differential equation, a key ingredient is the spectral type which consists of collections of positive integers determined from local isomorphic classes of the differential equation. And it is known that numbers of accessory parameters can be computed from indices of rigidity of spectral types. In [11] and [12], Oshima introduced a notion called unfolding of spectral type which connects different spectral types preserving indices of rigidity. Further, Oshima formulated the above question on confluence of singular points in terms of the unfolding of spectral types and presented this question as a conjecture in [12].

The purpose of this paper is to introduce this conjecture by Oshima in the case of first order systems of differential equations, as a variant of the original conjecture which is for higher order scalar differential equations. Further, we claim that a part of this conjecture for first order systems is true. The contents of this paper is as follows. We shall review the Hukuhara-Turrittin-Levelt theory of local differential equations and introduce the notion of spectral types in Section 1 and 2. In Section 3, Oshima's idea of unfolding spectral types is explained. The conjecture presented by Oshima in [12] appears in Section 4. Section 5 is the final section in which we claim that a part of the conjecture is true and give a sketch of the proof of this claim. A full-length proof of this claim will be found in the upcoming paper [6].

1. HTL normal forms and compositions of integers

From the Hukuhara-Turrittin-Levelt theory, it is known that local differential equations defined on a neighborhood of a singular point are classified by Hukuhara-

Turrittin-Levelt normal forms under gauge transformations whose coefficients are formal power series. We shall associate collections of integers, called spectral types, to Hukuhara-Turrittin-Levelt normal forms.

1.1. Compositions of integers

A *composition* $\beta = \{\beta_1, \beta_2, \dots, \beta_k\}$ of integer n, denoted by $\beta \vDash n$, is an ordered set of positive integers $\beta_1, \beta_2, \dots, \beta_k$ satisfying the equation $\beta_1 + \beta_2 + \cdots + \beta_k = n$. We call each integer β_i *component* of β and say that β has *length* $l(\beta) := k$ and *size* $|\beta| := n$.

Let us define a partial order on compositions of n as follows. For $\beta = \{\beta_1, \dots, \beta_k\}, \gamma = \{\gamma_1, \dots, \gamma_l\} \vDash n$, we say β is a *coarsening* of γ, denoted by $\beta \geq \gamma$ or equivalently γ is a *refinement* of β when there exists indices $i_0 := 0 < i_1 < i_2 < \cdots < i_k := l$ such that $\{\gamma_{i_{j-1}+1}, \gamma_{i_{j-1}+2}, \dots, \gamma_{i_j}\} \vDash \beta_j$ for $j = 1, \dots, k$. In this case we can define a surjective map from the index set of γ to that of β in the following way. Define $\phi\colon \{1, 2, \dots, l\} \to \{1, 2, \dots, k\}$ so that $\phi(i) = j$ for $i_{j-1} < i \leq i_j$, $j = 1, \dots, k$ and call this ϕ the *coarsening map* of $\beta \geq \gamma$.

1.2. HTL normal forms and compositions of integers

The following is a fundamental fact of the local formal theory of differential equations with irregular singularity.

Theorem 1.1 (Hukuhara-Turrittin-Levelt, see [13] for instance). *For any $A \in M(n, \mathbb{C}((z)))$, there exist a finite filed extension $\mathbb{C}((t))$ of $\mathbb{C}((z))$ and $X \in \mathrm{GL}(n, \mathbb{C}((t)))$ such that $t^r = z$ for some $r \in \mathbb{Z}_{\geq 1}$ and $X[A]$ is written by*

$$\mathrm{diag}\left(q_1(t^{-1}) I_{n_1} + R_1, \dots, q_m(t^{-1}) I_{n_m} + R_m\right) z^{-1}$$

where $q_i(s) \in s\mathbb{C}[s]$ satisfying $q_i \neq q_j$ if $i \neq j$, and $R_i \in M(n_i, \mathbb{C})$. We call this $X[A]$ the Hukuhara-Turrittin-Levelt normal form *of A or* HTL-normal form *for short.*

Here $X[A] := XAX^{-1} + \left(z\frac{d}{dz}X\right) X^{-1}$ is the gauge transform which is the transformation of the differential equation

$$z\frac{d}{dz}Y = AY,$$

and $\frac{d}{dz}f(t)$ for $f(t) \in \mathbb{C}((t))$ is defined by the equation $t^r = z$, namely, $\frac{d}{dz}f(t) = \frac{1}{r}t^{1-r}\frac{d}{dt}f(t)$. If $A \in M(n, \mathbb{C}((z)))$ has the normal form in $M(n, \mathbb{C}((z)))$, namely $r = 1$ in the above, then we say that the differential equation $z\frac{d}{dz}Y = AY$ has an *unramified* irregular singularity at $z = 0$.

Let us consider an HTL normal form in $M(n, \mathbb{C}(\!(z)\!))$,

$$B = \operatorname{diag}\left(q_1(z^{-1})I_{n_1} + R_1, \ldots, q_m(z^{-1})I_{n_m} + R_m\right) z^{-1}$$

and set

$$k := \max_{i=1,\ldots,m} \{\deg_{\mathbb{C}[z^{-1}]} q_i(z^{-1})\} + 1.$$

Let us see that an HTL normal form $B = \sum_{i=1}^{k} B_i z^{-i+1}$ gives an increasing sequence of compositions of n. Let $\bigoplus_{j=1}^{m(s)} V_{\langle s,j\rangle}$ be the decomposition of $\mathbb{C}^n$ as simultaneous invariant spaces of $\{B_{s+1}, B_{s+2}, \ldots, B_k\}$ for $s = 1, \ldots, k-1$. Then we have compositions

$$\mathbf{m}_s := \{\dim V_{\langle s,1\rangle}, \dim V_{\langle s,2\rangle}, \ldots, \dim V_{\langle s,m(s)\rangle}\} \vDash n$$

for $s = 1, 2, \ldots, k-1$, satisfying

$$\mathbf{m}_1 \leq \mathbf{m}_2 \leq \cdots \leq \mathbf{m}_{k-1}.$$

Here we note that $m(1) = m$ and $\mathbf{m}_1 = \{n_1, n_2, \ldots, n_m\}$.

Furthermore, the residue term $\operatorname{Res}(Bz^{-1}) = \operatorname{diag}(R_1, \ldots, R_m)$ defines a composition as follows. Let $f_{R_i}(x) = \prod_{j=1}^{l_i}(x - \xi_j^{(i)})$ be the minimal polynomial of R_i. Then we have a composition

$$\mathbf{m}_0 = \{n_{[1,1]}, n_{[1,2]}, \ldots, n_{[1,e_1]}, n_{[2,1]}, n_{[2,2]}, \ldots, n_{[m,e_m]}\}$$

where

$$\begin{aligned} n_{[i,j]} &:= \operatorname{rank} \prod_{s=1}^{j-1}(R_i - \xi_s^{(i)} I_{n_i}) - \operatorname{rank} \prod_{s=1}^{j}(R_i - \xi_s^{(i)} I_{n_i}) \qquad \text{for } j = 2, \ldots, e_i, \\ n_{[i,1]} &:= n_i - \operatorname{rank}(R_i - \xi_1^{(i)}). \end{aligned}$$

Then we can check $\mathbf{m}_1 \geq \mathbf{m}_0$ from the construction.

Definition 1.2 (spectral type of HTL normal form). The sequences of compositions of n defined above,

$$\mathbf{m}_0 \leq \mathbf{m}_1 \leq \cdots \leq \mathbf{m}_{k-1},$$

is called the *spectral type* of the HTL normal form $B \in \mathfrak{g}_k$. We call the integer k

length of the sequence.

2. Realization of spectral types

We have defined spectral types of HTL normal forms of local differential equations. In this section, we consider the existence of global differential equations whose spectral types of singular points coincide with given collection of increasing sequences of compositions of n.

2.1. Moduli spaces of differential equations

Following the papers [3], [9] and [5], we introduce isomorphic classes of global differential equations whose local HTL normal forms are fixed.

Let us define $G_k := \mathrm{GL}(n, \mathbb{C}[\![z]\!]/z^k\mathbb{C}[\![z]\!])$ which can be identified with

$$\left\{ A_0 + A_1 z + \cdots + A_{k-1}z^{k-1} \in \sum_{i=0}^{k-1} M(n,\mathbb{C})z^i \;\middle|\; A_0 \in \mathrm{GL}(n,\mathbb{C}) \right\}.$$

Also define

$$\begin{aligned} \mathfrak{g}_k &:= M(n, \mathbb{C}[\![z]\!]/z^k\mathbb{C}[\![z]\!]) \\ &\cong \left\{ A_0 + A_1 z + \cdots + A_{k-1}z^{k-1} \;\middle|\; A_i \in M(n,\mathbb{C}),\, i = 0, 1, \ldots, k-1 \right\}. \end{aligned}$$

The group G_k acts on $\mathfrak{g}_k$ by the *adjoint action* $\mathrm{Ad}(g)X := gXg^{-1}$ for $g \in G_k, X \in \mathfrak{g}_k$. The dual vector space $\mathfrak{g}_k^*$ is identified with

$$M(n, z^{-k}\mathbb{C}[\![z]\!]/\mathbb{C}[\![z]\!]) \cong \left\{ \frac{A_k}{z^k} + \cdots + \frac{A_1}{z} \;\middle|\; A_i \in M(n,\mathbb{C}),\, i = 1, \ldots, k \right\}$$

by the bilinear form

$$\mathfrak{g}_k \times \mathfrak{g}_k^* \ni (A, B) \mapsto \mathrm{Res}(\mathrm{tr}(AB)) \in \mathbb{C}$$

where $\mathrm{Res}(\sum_{i=r}^{\infty} a_i z^i) := a_{-1}$ for $\sum_{i=r}^{\infty} a_i z^i \in \mathbb{C}(\!(z)\!)$. Let us note that the *coadjoint action* of G_k on $\mathfrak{g}_k^*$ is defined by $(\mathrm{Ad}^*(g)f)(X) := f(\mathrm{Ad}(g^{-1})X)$ for $g \in G_k, f \in \mathfrak{g}_k^*, X \in \mathfrak{g}_k$.

Definition 2.1 (truncated orbit). Let us regard $B \in \mathfrak{g}_k^*$. Then the coadjoint orbit

$$\mathcal{O}_B := \{\mathrm{Ad}^*(g)B \mid g \in G_k\}$$

is called the *truncated orbit* of B.

We say that a collection of matrices $(A_1, \dots, A_s) \in M(n, \mathbb{C})^s$ is *irreducible* if $(A_1, \dots, A_s)$ has no nontrivial invariant subspace of $\mathbb{C}^n$, i.e., if a subspace $W \subset \mathbb{C}^n$ satisfies that $A_i W \subset W$ for all $i = 1, \dots, s$, then $W = \{0\}$ or $\mathbb{C}^n$. Let us consider a differential equation

$$\frac{d}{dz} Y = \sum_{i=0}^{p} \sum_{\nu=1}^{k_i} \frac{A_\nu^{(i)}}{(z-a_i)^\nu} Y,$$

with

$$\sum_{i=0}^{p} A_1^{(i)} = 0.$$

The *principal term* at the singular point a_i is

$$A_i(z_i) := \sum_{\nu=1}^{k_i} A_\nu^{(i)} z_i^{-\nu}$$

for each $i = 0, \dots, p$. Here we set $z_i := z - a_i$, $i = 0, \dots, p$. This differential equation is said to be *irreducible* if the collection of the matrices $(A_{i,j})_{\substack{0 \le i \le p, \\ 1 \le j \le k_i}}$ is irreducible.

Let us take $k_i \in \mathbb{Z}_{\ge 1}$ and HTL normal forms $B_i \in \mathfrak{g}_{k_i}^*$ for $i = 0, 1, \dots, p$. Then we define a moduli space of differential equations on $\mathbb{P}^1$,

$$\mathfrak{M}(\mathbf{B}) := \left\{ \frac{d}{dz} Y = \sum_{i=0}^{p} \sum_{\nu=1}^{k_i} \frac{A_\nu^{(i)}}{(z-a_i)^\nu} Y \;\middle|\; \begin{array}{l} \text{irreducible}, \\ \sum_{i=0}^{p} A_1^{(i)} = 0, \\ \sum_{\nu=1}^{k_i} \frac{A_\nu^{(i)}}{z^\nu} \in \mathcal{O}_{B_i}, i = 0, \dots, p \end{array} \right\} \Big/ \mathrm{GL}(n, \mathbb{C}).$$

Here the action of $\mathrm{GL}(n, \mathbb{C})$ is given by

$$\frac{d}{dz} Y = A(z) Y \mapsto \frac{d}{dz} Y = g A(x) g^{-1} Y,$$

for $g \in \mathrm{GL}(n, \mathbb{C})$, $A(z) \in M(n, \mathbb{C}(z))$.

2.2. Irreducibly realizable

Since differential equations have HTL normal forms at their singular points, they have collections of increasing sequences of compositions of n as spectral types. Conversely, we would like to find differential equations with given spectral types. In this paper, we only consider differential equations with unramified singularities. Thus, we adopt the following notion of realizability of spectral types.

Definition 2.2 (irreducibly realizable). Let us take a collection of increasing sequences of compositions of n,

$$\mathbf{m} = (\mathbf{m}_0^{(i)} \leq \mathbf{m}_1^{(i)} \leq \cdots \leq \mathbf{m}_{k_i-1}^{(i)})_{i=0,1,\ldots,p}.$$

We say $\mathbf{m}$ is *irreducibly realizable* if the following conditions are satisfied.

1. There exists a collection of HTL normal forms $\mathbf{B} = (B_0, B_1, \ldots, B_p)$ such that $B_i \in \mathfrak{g}_{k_i}$ and its spectral type is $\mathbf{m}_0^{(i)} \leq \mathbf{m}_1^{(i)} \leq \cdots \leq \mathbf{m}_{k_i-1}^{(i)}$ for each $i = 0, 1, \ldots, p$.

2. For the above $\mathbf{B}$, we have

$$\mathfrak{M}(\mathbf{B}) \neq \emptyset.$$

A necessary and sufficient condition of non-emptiness of $\mathfrak{M}(\mathbf{B})$ is already known (see [4], [1], [2], [9], [5]). The condition is given in terms of Kac-Moody root systems. Thus, we can rephrase the realizability of spectral types as a condition on Kac-Moody root systems as follows. We refer the detail to the section 4 in [5] and just give a quick review here.

For a collection of increasing sequences of compositions of n, $\mathbf{m} = (\mathbf{m}_0^{(i)} \leq \mathbf{m}_1^{(i)} \leq \cdots \leq \mathbf{m}_{k_i-1}^{(i)})_{i=0,1,\ldots,p}$, let us write

$$\mathbf{m}_s^{(i)} = (n_1^{([i,s]}, n_2^{[i,s]}, \ldots, n_{m^{(i)}(s)}^{[i,s]}) \qquad \text{for } s = 1, 2, \ldots, k_i - 1,$$
$$\mathbf{m}_0^{(i)} = (n_{[i,1,1]}, n_{[i,1,2]}, \ldots, n_{[i,1,e_{[i,1]}]}, n_{[i,2,1]}, n_{[i,2,2]}, \ldots, n_{[i,m^{(i)}(1),e_{[i,m^{(i)}(1)]}]}).$$

Let $\phi_s^i \colon \{1, 2, \ldots, m^{(i)}(s)\} \to \{1, 2, \ldots, m^{(i)}(s+1)\}$ be coarsening maps of indices of $\mathbf{m}_s^{(i)} \leq \mathbf{m}_{s+1}^{(i)}$ for $s = 1, 2, \ldots, k-2$ and $\phi_{k-1}^{(i)} \colon \{1, 2, \ldots, m^{(i)}(k-1)\} \to \{1\}$. Then we define a distance on the index set $\{1, 2, \ldots, m^{(i)}(1)\}$ of $\mathbf{m}_1^{(i)}$ as follows,

$$d_i(j,k) := \min\{l \mid \phi_l^{(i)} \circ \cdots \circ \phi_2^{(i)} \circ \phi_1^{(i)}(j) = \phi_l^{(i)} \circ \cdots \circ \phi_2^{(i)} \circ \phi_1^{(i)}(k)\} - 1$$

Set $I_{\mathrm{irr}} := \{i \in \{0,1,\dots,p\} \mid k_i \geq 2\} \cup \{0\}$ and $I_{\mathrm{reg}} := \{0,1,\dots,p\} \backslash I_{\mathrm{irr}}$. Then let us define a symmetric Dynkin diagram $D_{\mathbf{m}} = (V, E)$ with E, the set of edges, and V, the set of vertices, as follows. Set

$$V^{\mathrm{irr}} := \left\{ [i,j] \,\middle|\, \begin{array}{l} i \in I_{\mathrm{irr}}, \\ j = 1,\dots,m^{(i)} \end{array} \right\}, \qquad V^{\mathrm{leg}} := \left\{ [i,j,k] \,\middle|\, \begin{array}{l} i = 0,\dots,p, \\ j = 1,\dots,m^{(i)}, \\ k = 1,\dots,e_{[i,j]} - 1 \end{array} \right\}.$$

Here we set $m^{(i)} := m^{(i)}(1)$. Then the set of vertices V is the disjoint union

$$V := V^{\mathrm{irr}} \sqcup V^{\mathrm{leg}}.$$

Also define sets of edges between vertices,

$$\begin{aligned}
E^{0 \leftrightarrow I_{\mathrm{irr}}} &:= \left\{ \rho^{[0,j]}_{[i,j']} \colon [0,j] \longleftrightarrow [i,j'] \,\middle|\, \begin{array}{l} j = 1,\dots,m^{(0)}, \\ i \in I_{\mathrm{irr}} \backslash \{0\}, \\ j' = 1,\dots,m^{(i)} \end{array} \right\}, \\
E^{\mathbf{m}^{(i)}_1} &:= \left\{ \rho^{[k]}_{[i,j],[i,j']} \colon [i,j] \longleftrightarrow [i,j'] \,\middle|\, \begin{array}{l} 1 \leq j < j' \leq m^{(i)}, \\ 1 \leq k \leq d_i(j,j') \end{array} \right\}, \\
E^{\mathrm{leg}^{(i)}} &:= \left\{ \rho_{[i,j,k]} \colon [i,j,k] \longleftrightarrow [i,j,k-1] \,\middle|\, \begin{array}{l} j = 1,\dots,m^{(i)}, \\ k = 2,\dots,e_{[i,j]} - 1 \end{array} \right\}, \\
E^{\mathrm{leg}^{(i)} \leftrightarrow \mathbf{m}^{(i)}_1} &:= \left\{ \rho_{[i,j,1]} \colon [i,j,1] \longleftrightarrow [i,j] \mid j = 1,\dots,m^{(i)} \right\}, \\
E^{\mathrm{leg}^{(i)} \leftrightarrow 0}_1 &:= \left\{ \rho^{[i,1,1]}_{[0,j]} \colon [i,1,1] \longleftrightarrow [0,j] \mid i \in I_{\mathrm{reg}},\, j = 1,\dots,m^{(0)} \right\}.
\end{aligned}$$

Then the set of edges E is the disjoint union

$$E := E^{0 \leftrightarrow I_{\mathrm{irr}}} \sqcup \bigsqcup_{i \in I_{\mathrm{irr}}} \left(E^{\mathbf{m}^{(i)}_1} \sqcup E^{\mathrm{leg}^{(i)} \leftrightarrow \mathbf{m}^{(i)}_1} \sqcup E^{\mathrm{leg}^{(i)}} \right) \sqcup \bigsqcup_{i \in I_{\mathrm{reg}}} \left(E^{\mathrm{leg}^{(i)} \leftrightarrow 0} \sqcup E^{\mathrm{leg}^{(i)}} \right).$$

Then the relation

$$\begin{aligned}
\langle v_i, v_j \rangle &:= -\sharp\{\text{edges } v_i \leftrightarrow v_j\} \qquad \text{for } v_i \neq v_j \in V, \\
\langle v, v \rangle &:= 2 \qquad \text{for } v \in V
\end{aligned}$$

defines a bilinear form on $\mathbb{Z}^V := \bigoplus_{v \in V} \mathbb{Z}v$. Then we can define the root system on $\mathbb{Z}^V$ in the usual way (see Kac [10]).

Let us define a vector $\alpha_{\mathbf{m}} = (\alpha_v)_{v \in V} \in \mathbb{Z}^V$ from the collection of increasing sequences of compositions $\mathbf{m}$ as follows,

$$\alpha_{[i,j]} := n_j^{[i,1]} \quad \text{and} \quad \alpha_{[i,j,k]} := n_j^{[i,1]} - \sum_{l=1}^{k} n_{[i,j,l]}.$$

Then the results in [4], [1], [2], [9], [5] tell us that we can determine the condition which assures that $\mathbf{m}$ is irreducibly realizable.

THEOREM 2.3. *A collection of increasing sequences of compositions of* n, $\mathbf{m} = (\mathbf{m}_0^{(i)} \leq \mathbf{m}_1^{(i)} \leq \cdots \leq \mathbf{m}_{k_i-1}^{(i)})_{i=0,1,\ldots,p}$ *is irreducibly realizable if and only if the vector* $\alpha_{\mathbf{m}}$ *is a positive root in the root system associated with the Dynkin diagram* $D_{\mathbf{m}}$.

3. Unfolding of spectral types

In [11] and [12], Oshima introduced a process for constructing spectral types of Fuchsian equations from that of differential equations with irregular singularities, this is called *unfolding* of spectral types, which will be explained in this section.

For an HTL normal form

$$B = \operatorname{diag}\left(q_1(z^{-1})I_{n_1} + R_1 z^{-1}, \ldots, q_m(z^{-1})I_{n_m} + R_m z^{-1}\right),$$

let us construct a family of elements in $M(n, \mathbb{C}(z))$ containing this HTL normal form B as follows. Recalling equalities,

$$\frac{1}{\prod_{1\leq\nu\leq i}(z-c_\nu)} = \sum_{j=1}^{i} \frac{\left(\prod_{1\leq\nu\leq i,\,\nu\neq j}(c_j-c_\nu)\right)^{-1}}{z-c_j},$$

we define a function of c_i for $i = 1, \ldots, k$ as follows,

$$\begin{aligned} B(c_1, \ldots, c_k) &:= \sum_{i=1}^{k} \frac{B_i}{\prod_{1\leq\nu\leq i}(z-c_\nu)} \\ &= \frac{B_1}{z-c_1} + \sum_{i=2}^{k}\sum_{j=1}^{i} \left(\frac{B_i}{\prod_{1\leq\nu\leq i,\,\nu\neq j}(c_j-c_\nu)}\right)\frac{1}{z-c_j} \\ &= \sum_{j=1}^{k}\left(\sum_{i=j}^{k} \frac{B_i}{\prod_{1\leq\nu\leq i,\,\nu\neq j}(c_j-c_\nu)}\right)\frac{1}{z-c_j}. \end{aligned}$$

Here we formally set $\prod_{1\le\nu\le 1,\nu\neq 1}(c_1-c_\nu)=1$. We regard this $B(c_1,\ldots,c_k)$ as a function on the polydisc $\mathbb{D}_\epsilon^k := \{(c_1,\ldots,c_k)\in\mathbb{C}^k \mid |c_i|<\epsilon,\ i=1,\ldots,k\}$ for a sufficiently small $0<\epsilon\ll 1$. Then the graph of this function

$$\mathcal{B} := \{(B(c_1,\ldots,c_k),(c_1,\ldots,c_k))\in M(n,\mathbb{C}(z))\times\mathbb{D}_\epsilon^k \mid (c_1,\ldots,c_k)\in\mathbb{D}_\epsilon^k\}$$

and the natural projection $\pi\colon\mathcal{B}\longrightarrow\mathbb{D}_\epsilon^k$ define a deformation of B which is obtained as the special fiber $\pi^{-1}((0,\ldots,0))$.

Let us look at a generic fiber of the deformation $\pi\colon\mathcal{B}\longrightarrow\mathbb{D}_\epsilon^k$. Namely, take a generic element $\mathbf{c}=(c_1,\ldots,c_k)\in\mathbb{D}_\epsilon^k$ so that $c_i\neq c_j$ for $1\le i\neq j\le k$. Then we have

$$B(\mathbf{c})=\sum_{j=1}^{k}\frac{A_j(\mathbf{c})}{z-c_j},$$

where

$$A_j(\mathbf{c}) := \sum_{i=j}^{k}\frac{B_i}{\prod_{1\le\nu\le i,\,\nu\neq j}(c_j-c_\nu)}\in M(n,\mathbb{C}) \tag{1}$$

for $j=1,\ldots,k$. Namely, the deformation $\pi\colon\mathcal{B}\longrightarrow\mathbb{D}_\epsilon^k$ has the irregular singular HTL normal form B as a special fiber and collections of regular singular normal forms as general fibers.

Let us see that this deformation of normal forms induces the deformation of spectral types as follows. Let

$$\mathbf{m}_0\le\mathbf{m}_1\le\cdots\le\mathbf{m}_k$$

be the spectral type of B. Then we can easily deduce the following by the definition of $A_j(\mathbf{c})$ in the equation (1).

Lemma 3.1. *Let us take a generic $\mathbf{c}\in\mathbb{D}_\epsilon^k$ for a sufficiently small $\epsilon>0$. Then the spectral type of $A_j(\mathbf{c})$ is $\mathbf{m}_{j-1}$ for each $j=1,\ldots,k$.*

This lemma leads to the following notion, unfolding of spectral types.

Definition 3.2. Let

$$\mathbf{m}=(\mathbf{m}_0^{(i)}\le\mathbf{m}_1^{(i)}\le\cdots\le\mathbf{m}_{k_i-1}^{(i)})_{i=0,1,\ldots,p}$$

be a collection of increasing sequences of compositions of n. Then forgetting all

"$\leq$"s, we can consider the following collection of compositions of n

$$\mathbf{m}^{\mathrm{unf}} := (\mathbf{m}_j^{(i)})_{\substack{i=0,1,\ldots,p\\ j=1,2,\ldots,k_i-1}}$$

which can be seen as a collection of increasing sequences of length 1.

Remark 3.3. According to the construction explained in Section 2.2, we can define the Dynkin diagram $D_{\mathbf{m}^{\mathrm{unf}}}$ for $\mathbf{m}^{\mathrm{unf}}$ as well. Note that $D_{\mathbf{m}}$ and $D_{\mathbf{m}^{\mathrm{unf}}}$ are different in general.

4. Oshima's conjecture

In the previous section, we constructed a spectral type $\mathbf{m}^{\mathrm{unf}}$ of length 1, namely that of a Fuchsian differential equation, from a spectral type of an irregular singular differential equation $\mathbf{m}$. In the paper [12], Oshima gave a conjecture which concerns the irreducibly realizability of $\mathbf{m}$ and $\mathbf{m}^{\mathrm{unf}}$. Let us explain the conjecture in this section.

Let

$$\mathbf{m} = (\mathbf{m}_0^{(i)} \leq \mathbf{m}_1^{(i)} \leq \cdots \leq \mathbf{m}_{k_i-1}^{(i)})_{i=0,1,\ldots,p}$$

be a collection of increasing sequences of compositions of n.

Definition 4.1. We say that $\mathbf{m}$ is *versally realizable* if the following conditions are satisfied.

- There exists an HTL normal form $B^{(i)} \in \mathfrak{g}_{k_i}^*$ with the spectral type

  $$\mathbf{m}^{(i)} := \mathbf{m}_0^{(i)} \leq \mathbf{m}_1^{(i)} \leq \cdots \leq \mathbf{m}_{k_i-1}^{(i)}$$

  for each $i = 0, 1, \ldots, p$.

- Let $\pi^{(i)} \colon \mathcal{B}^{(i)} \to \mathbb{D}_{\epsilon_i}^{k_i}$ be the deformation of $B^{(i)}$ defined as before. Let us define $A_j^{(i)}(\mathbf{c}^{(i)}) \in M(n, \mathbb{C})$ by the equation (1) for $B^{(i)}$ with a generic $\mathbf{c}^{(i)} = (c_1^{(i)}, \ldots, c_{k_i}^{(i)}) \in \mathbb{D}_{\epsilon_i}^{k_i}$. Then there exists a differential equation

  $$\frac{d}{dz}Y = \sum_{i=0}^{p}\sum_{j=1}^{k_i} \frac{C_j^{(i)}(\mathbf{c})}{z - (a_i + c_j^{(i)})} Y$$

  which depends holomorphically on $\mathbf{c} = (\mathbf{c}^{(i)})_{i=0,\ldots,p} \in \prod_{i=0}^{p} \mathbb{D}_{\epsilon_i}^{k_i}$ and satisfies that

1. for generic $\mathbf{c}$, the differential equation is irreducible Fuchsian equation with regular singular points at $a_i + c_j^{(i)}$, and satisfies that $\sum_{i=0}^{p}\sum_{j=1}^{k_i} C_j^{(i)}(\mathbf{c}) = 0$ and $C_j^{(i)}(\mathbf{c})$ are in the conjugacy class of $A_j^{(i)}(\mathbf{c}^{(i)})$, for $i = 0, \ldots, p$, $j = 1, \ldots, k_i$,
2. the differential equation is a generic element of $\mathfrak{M}(\mathbf{B})$ when $\mathbf{c} = \mathbf{0}$.

In the paper [12], Oshima gave a conjecture for the realizability of spectral types of higher order scalar differential equations, see the section 4 in [12]. The following is a variant of this conjecture for first order systems of differential equations.

Conjecture 4.2 (Oshima [12]). *Let* $\mathbf{m}$ *be a collection of increasing sequences of compositions of* n. *Then the following are equivalent.*

1. $\mathbf{m}$ *is irreducibly realizable.*
2. $\mathbf{m}$ *is versally realizable.*
3. $\mathbf{m}^{unf}$ *is irreducibly realizable.*

It immediately follows from the definition of versally realizability that the condition 3 implies 1 and 2.

Remark 4.3. Under the assumption $\langle \alpha_{\mathbf{m}}, \alpha_{\mathbf{m}} \rangle \geq -2$, it is known that the conjecture is true (see [7] and [12]).

5. Main theorem and sketch of proof

Let us give the statement of our main theorem of this paper.

Theorem 5.1. *The direction from* 3 *to* 1 *is true in Conjecture 4.2*

In the remaining of this section, we shall give a sketch of a proof of this theorem. Let

$$\mathbf{m} = (\mathbf{m}_0^{(i)} \leq \mathbf{m}_1^{(i)} \leq \cdots \leq \mathbf{m}_{k_i-1}^{(i)})_{i=0,1,\ldots,p}$$

be a collection of increasing sequences of compositions of n. Define a sublattice $\mathcal{L}$ of $\mathbb{Z}^V$ as follows,

$$\mathcal{L} := \left\{ \beta = (\beta_{[i,j]}) \in \mathbb{Z}^V \;\middle|\; \sum_{j=1}^{m^{(0)}} \beta_{[0,j]} = \sum_{j=1}^{m^{(i)}} \beta_{[i,j]} \text{ for all } i \in I_{\mathrm{irr}} \backslash \{0\} \right\}.$$

Set $\mathcal{L}^+ := \mathcal{L} \cap \bigoplus_{v\in V} \mathbb{Z}_{\geq 0} v$. We can see that the vector $\alpha_{\mathbf{m}}$ defined before Theorem 2.3 satisfies $\alpha_{\mathbf{m}} \in \mathcal{L}^+$. Here we note that this correspondence is one to one and onto. Namely, let us fix positive integers k_i, $i = 0, 1, \ldots, p$ as above. Let us denote the set of all collections of increasing sequences of compositions of n of length k_i $(i = 0, 1, \ldots, p)$ by C_n. Then we have a bijection σ from $\bigcup_{n>0} C_n$ to $\mathcal{L}^+$,

$$\begin{array}{rcl} \sigma\colon \bigcup_{n>0} C_n & \longrightarrow & \mathcal{L}^+ \\ \Psi & & \Psi \\ \mathbf{m} = (\mathbf{m}_0^{(i)} \leq \mathbf{m}_1^{(i)} \leq \cdots \leq \mathbf{m}_{k_i-1}^{(i)})_{i=0,1,\ldots,p} & \longmapsto & \alpha_{\mathbf{m}} \end{array} . \tag{2}$$

Setting

$$\mathcal{J} := \{ \sum_{i\in I_{\mathrm{irr}}}^{p} [i, j_i] \in \mathbb{Z}^V \mid 1 \leq j_i \leq m^{(i)} \},$$

we can see that $\mathcal{L}$ is generated by $\mathcal{J} \cup V^{\mathrm{leg}}$ and $\mathcal{L}$ is closed under the action of

$$W_{\mathcal{L}} := \left\langle s_{\mathbf{j}},\, s_{[i,j,k]} \;\middle|\; \mathbf{j} \in \mathcal{J},\, [i, j, k] \in V^{\mathrm{leg}} \right\rangle$$

which is a subgroup of the Weyl group of $\mathbb{Z}^V$. Here s_a, $(a \in \mathbb{Z}^V)$, is the reflection, namely,

$$s_a(\beta) := \beta - \langle \beta, a\rangle a, \quad \text{for } \beta \in \mathbb{Z}^V.$$

The following lemmas play key roles in the proof of the theorem. We note that these lemmas can be obtained from a slight modification of Theorem 3.14 in [8] and Lemma 3.3 in [12].

Lemma 5.2. *Let $D_{\mathbf{m}^{unf}} = (V^{unf}, E^{unf})$ be the Dynkin diagram associated with $\mathbf{m}^{unf}$. Then*

$$\langle \alpha_{\mathbf{m}}, \alpha_{\mathbf{m}} \rangle = \langle \alpha_{\mathbf{m}^{unf}}, \alpha_{\mathbf{m}^{unf}} \rangle$$

Let us note that the unfolding of spectral types induces the bijection, $\mu\colon \mathcal{L}^+ \to \mathbb{Z}_{\geq 0}^{V^{\mathrm{unf}}}$ such that $\mu(\beta) = \sigma^{\mathrm{unf}}(\sigma^{-1}(\beta)^{\mathrm{unf}})$. Here σ and σ^{unf} are maps defined by (2) for $D_{\mathbf{m}}$ and $D_{\mathbf{m}^{\mathrm{unf}}}$ respectively. We can show that this map is compatible with the actions of both Weyl groups.

Lemma 5.3. *Let W^{unf} be the Weyl group associated with $D_{\mathbf{m}^{unf}}$. Then for* *each $w \in W_{\mathcal{L}}$, there exists $w' \in W^{unf}$ such that*

$$w(\alpha_{\mathbf{m}}) = \mu^{-1}(w'(\alpha_{\mathbf{m}^{unf}})).$$

Under these preparations, the proof of the theorem goes on in the following way. Suppose that $\mathbf{m}^{\mathrm{unf}}$ is irreducibly realizable. Then Theorem 2.3 tells us that $\alpha_{\mathbf{m}^{\mathrm{unf}}}$ is a positive root of $\mathbb{Z}^{V^{\mathrm{unf}}}$. Thus, the Weyl group orbit of $\alpha_{\mathbf{m}^{\mathrm{unf}}}$ is contained in the set of positive roots or $\alpha_{\mathbf{m}^{\mathrm{unf}}}$ is a real root. Then Lemma 5.2 and 5.3 show that there exists $w \in W_{\mathcal{L}}$ such that $w(\alpha_{\mathbf{m}}) \in \tilde{F}$ or $w(\alpha_{\mathbf{m}})$ is a simple root, where

$$\tilde{F} := \left\{ \beta \in \mathcal{L}^+ \backslash \{0\} \;\middle|\; \begin{array}{c} \langle \beta, a \rangle \leq 0 \text{ for all } a \in \mathcal{J} \cup V^{\mathrm{leg}} \\ \text{support of } \beta \text{ is connected} \end{array} \right\}.$$

It is known that elements in $\tilde{F}$ are positive roots (see Theorem 6.20 in [5]). Thus, in any case, $\alpha_{\mathbf{m}}$ is a positive root. Then Theorem 2.3 shows that $\mathbf{m}$ is irreducibly realizable.

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