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GG system and its application to the connection problem of GKZ hypergeometric functions

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Abstract. This is an announcement of a result on connection formula of GKZ hypergeometric functions between “nearby toric infinities”. The key is the use of contiguity relation of GKZ hypergeometric system which is known as GG system.

1. Introduction

In 80’s and 90’s, the general study of hypergeometric functions made a huge progress in a series of papers by I.M.Gelfand, M.M.Kapranov, and A.V.Zelevinsky ([GZK89], [GKZ90], [GKZ94]). One of the non-trivial consequences of their study is that there is a combinatorial structure of convex polytopes behind hypergeoemetric systems. The aim of this paper is to give a new method of deriving connection formulae of GKZ hypergeometric functions in the language of combinatorics of regular triangulations.

GKZ system is determined by two inputs: an $n\times N$ $(n<N)$ integer matrix $A=(\mathbf{a}(1)|\cdots|\mathbf{a}(N))$ and a parameter vector $c\in\mathbb{C}^{n\times 1}$. GKZ system $M_A(c)$ is defined by

$$M_A(c):\begin{cases} E_i\cdot f(z)=0 & (i=1,\dots,n)\\ \square_u\cdot f(z)=0 & \left(u={}^t(u_1,\dots,u_N)\in L_A=\mathrm{Ker}(A\times:\mathbb{Z}^{N\times 1}\to\mathbb{Z}^{n\times 1})\right),\end{cases}$$

where E_i and $\square_u$ are differential operators defined by

$$E_i=\sum_{j=1}^N a_{ij}z_j\frac{\partial}{\partial z_j}+c_i,\quad \square_u=\prod_{u_j>0}\left(\frac{\partial}{\partial z_j}\right)^{u_j}-\prod_{u_j<0}\left(\frac{\partial}{\partial z_j}\right)^{-u_j}. \tag{1.2}$$

Throughout this paper, we assume an additional condition $\mathbb{Z}A\overset{def}{=}\mathbb{Z}\mathbf{a}(1)+\cdots+$

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$\mathbb{Z}\mathbf{a}(N) = \mathbb{Z}^{n\times 1}$. The GKZ system $M_A(c)$ is holonomic ([Ado94, THEOREM 3.9]). Moreover, $M_A(c)$ is regular holonomic if and only if the primitive hyperplane condition is satisfied, i.e., there exists a rational row vector $\mathbf{l} \in \mathbb{Q}^{1\times n}$ such that $\mathbf{l}A = (1, \dots, 1)$ holds. See [Hot], [SW08, Corollary 3.16]. See also [FF10, Theorem 5.9].

If we set $\operatorname{cone}(A) = \sum_{j=1}^{N} \mathbb{R}_{\geq 0}\mathbf{a}(j)$, the notion of regular triangulation T of $\operatorname{cone}(A)$ comes into play. To each regular triangulation T, we can associate a basis Φ_T of series solutions of $M_A(c)$. Therefore, we can expect combinatorics of regular triangulations controls the solution space. The totality (or "moduli") of regular triangulations has the structure of a convex polyhedral fan, which is called the secondary fan F_A. Thus, we can deal with the notion of distance among regular triangulations. Let us denote by C_T the cone corresponding to a regular triangulation T. We say T is adjacent to T' if $C_T \cap C_{T'}$ is a common facet of C_T and $C_{T'}$. The analytic interpretation of adjacency is an analytic continuation by Mellin-Barnes integral. This crucial observation is due to the paper [ST94] by Mutsumi Saito and N.Takayama. Later, P.Horja discussed a similar problem for special configurations A with the special parameter $c = 0$ in [Hor99], and L.Borisov and P.Horja studied the general configuration with the special parameter $c = 0$ in [BH06]. In [Beu16], F.Beukers proposed a conjectural method of computing a set of generators of the monodromy group of GKZ system with a generic parameter c based on the multidimensional Mellin-Barnes integral representation under a special assumption of the configuration. Our approach can be regarded as a complement to his approach. A related problem is also discussed by S.Tanabé in [Tan17]. In this paper, we announce an explicit connection formula of GKZ hypergeoemtric functions for general regular holonomic configuration A with a generic parameter c. Namely, when T is adjacent to T', we give an analytic continuation between Φ_T and $\Phi_{T'}$. As was indicated in [ST94], the key is the method of boundary value problems ([Hec87], [KO77]). However, we can realize this structure in a more explicit way employing the viewpoint of GG system ([GG99]).

Throughout this paper, we use the following notation: for any vectors $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$, we set $e^{2\pi\sqrt{-1}\mathbf{a}} = (e^{2\pi\sqrt{-1}a_1}, \dots, e^{2\pi\sqrt{-1}a_n})$, $\mathbf{ab} = (a_1b_1, \dots, a_nb_n)$, and $\mathbf{a}^{\mathbf{b}} = a_1^{b_1} \cdots a_n^{b_n}$. For any univariate function F, we set $F(\mathbf{a}) = F(a_1) \cdots F(a_n)$. For any $1 \times n$ row vector z and any $n \times m$ matrix $B = (\mathbf{b}(1)|\cdots|\mathbf{b}(m))$, we set $z^B = (z^{\mathbf{b}(1)}, \dots, z^{\mathbf{b}(m)})$.

2. GG system and GKZ system

In [GG97], Gelfand and Graev introduced a system of difference differential equations called GG system. Here, "GG" stands for "Gauß and Graßmann". A very similar system was discussed in [AI99] and [AI01] where it is called quasi-hypergeometric system. In this section, we briefly recall the definition of GG system and discuss its relation to Γ-series. Suppose an $n \times N$ $(n < N)$ integer matrix $A = (\mathbf{a}(1)|\cdots|\mathbf{a}(N))$ is given. In this section, we do not need to assume that the configuration vectors generate the ambient lattice, i.e., we have, in general, the inclusion $\mathbb{Z}\mathbf{a}(1) + \cdots + \mathbb{Z}\mathbf{a}(N) \subset \mathbb{Z}^{n\times 1}$ but not the equality. We assume the primitive hyperplane condition. We define GG system $\mathrm{GG}(A)$ as a system of linear partial difference-differential equations on $\mathbb{C}^N \times \mathbb{C}^n$ ([GG99]):

$$\mathrm{GG}(A):\begin{cases} E_i \cdot f(z;c) = 0 & (i = 1, \cdots, n) \quad \text{(2.1a)}\\ \dfrac{\partial}{\partial z_j} f(z;c) = f(z; c + \mathbf{a}(j)) & (j = 1, \ldots, N). \quad \text{(2.1b)} \end{cases}$$

As was remarked in [GG99], solutions of GG system is automatically a solution of GKZ hypergeometric system. Indeed, if $u \in L_A$, we can decompose u as $u = u_+ - u_-$ where u_+ and u_- are integer vectors with non-negative entries and u_+ and u_- do not have common support. Then, we have an equality $\partial_z^{u_+} f(z;c) = f(z; c + Au_+) = f(z; c + Au_-) = \partial_z^{u_-} f(z;c)$.

We briefly discuss the method of constructing a basis of series solutions of a given GKZ system following the exposition of M.-C. Fernández-Fernández ([FF10]). This actually gives a method of constructing a basis of GG system $\mathrm{GG}(A)$. For any vector $v \in \mathbb{C}^{N\times 1}$ such that $Av = -c$, we put

$$\varphi_v(z) = \sum_{u \in L_A} \frac{z^{u+v}}{\Gamma(1 + u + v)} \tag{2.2}$$

and call it a Γ-series. It can readily be seen that $\varphi_v(z)$ is a formal solution of $M_A(c)$ ([GZK89]). For any subset $\tau \subset \{1, \ldots, N\}$, we denote A_τ the matrix given by the columns of A indexed by τ. In the following, we take $\sigma \subset \{1, \ldots, N\}$ such that $|\sigma| = n$ and $\det A_\sigma \neq 0$. Taking a vector $\mathbf{k} \in \mathbb{Z}^{\bar{\sigma}\times 1}$, we put

$$v_\sigma^{\mathbf{k}} = \begin{pmatrix} -A_\sigma^{-1}(c + A_{\bar{\sigma}}\mathbf{k}) \\ \mathbf{k} \end{pmatrix}, \tag{2.3}$$

where $\overline{\sigma}$ denotes the complement $\{1, \ldots, N\} \setminus \sigma$. Then, by a direct computation, we have

(2.4)

$$\varphi_{\sigma,\mathbf{k}}(z;c)\overset{\mathrm{def}}{=}\varphi_{v_\sigma^{\mathbf{k}}}(z)=z_\sigma^{-A_\sigma^{-1}\mathbf{c}}\sum_{\mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k}}}\frac{(z_\sigma^{-A_\sigma^{-1}A_{\bar\sigma}}z_{\bar\sigma})^{\mathbf{k}+\mathbf{m}}}{\Gamma(\mathbf{1}_\sigma-A_\sigma^{-1}(\mathbf{c}+A_{\bar\sigma}(\mathbf{k}+\mathbf{m})))(\mathbf{k}+\mathbf{m})!},$$

where $\Lambda_{\mathbf{k}}$ is given by

$$\Lambda_{\mathbf{k}}=\Big\{\mathbf{k}+\mathbf{m}\in\mathbb{Z}_{\geq 0}^{\bar\sigma}\mid A_{\bar\sigma}\mathbf{m}\in\mathbb{Z}A_\sigma\Big\}. \tag{2.5}$$

The following lemma is easily confirmed ([FF10]).

LEMMA 2.1. *For any $\mathbf{k},\mathbf{k}'\in\mathbb{Z}^{\bar\sigma}$, the following statements are equivalent*

1. $v_\sigma^{\mathbf{k}}-v_\sigma^{\mathbf{k}'}\in\mathbb{Z}^{N\times 1}$
2. $[A_{\bar\sigma}\mathbf{k}]=[A_{\bar\sigma}\mathbf{k}']$ *in* $\mathbb{Z}^{n\times 1}/\mathbb{Z}A_\sigma$
3. $\Lambda_{\mathbf{k}}=\Lambda_{\mathbf{k}'}$.

We claim that this function $\varphi_{\sigma,\mathbf{k}}(z;c)$ viewed as a function of space variable z and parameter variable c is actually a solution of GG(A). For later use, we formulate it in a more general form. For any $\tilde{\mathbf{k}}\in\mathbb{Z}^\sigma$ and for any partition $\sigma=\sigma^u\sqcup\sigma^d$, we put

(2.6)

$$\psi^{\sigma^u}_{\sigma^d,\tilde{\mathbf{k}}}(z;c)=\sum_{\mathbf{m}\in\mathbb{Z}^{\overline{\sigma}}_{\geq 0}}\frac{\prod_{i\in\sigma^u}\Gamma({}^t\mathbf{e}_iA_\sigma^{-1}(c+A_{\overline{\sigma}}\mathbf{m}))}{\prod_{i\in\sigma^d}\Gamma(1-{}^t\mathbf{e}_iA_\sigma^{-1}(c+A_{\bar\sigma}\mathbf{m}))\mathbf{m}!}(e^{2\pi\sqrt{-1}\tilde{\mathbf{k}}}e^{\pi\sqrt{-1}\mathbf{1}_{\sigma^u}}z_\sigma)^{-A_\sigma^{-1}(c+A_{\bar\sigma}\mathbf{m})}z_{\bar\sigma}^{\mathbf{m}}.$$

Here, $\mathbf{e}_i\in\mathbb{Z}^{\sigma\times 1}$ is the vector whose entries are all zero but i-th one is 1, and $\mathbf{1}_{\sigma^u}=\sum_{i\in\sigma^u}{}^t\mathbf{e}_i$. Note that $\psi^{\sigma^u}_{\sigma^d,\tilde{\mathbf{k}}}(z;c)$ is well-defined if the numerator does not have any pole. It is important that series $\psi^{\sigma^u}_{\sigma^d,\tilde{\mathbf{k}}}(z;c)$ is defined as a sum over the positive lattice $\mathbb{Z}^{\overline{\sigma}}_{\geq 0}$ which does not depend on the choice of $\tilde{\mathbf{k}}$. This property plays an important role when we consider restriction-extension structure of GG systems. The following statement is easy to check.

PROPOSITION 2.2. *$\psi^{\sigma^u}_{\sigma^d,\tilde{\mathbf{k}}}(z;c)$ is a formal solution of GG(A).*

Let us review the definition of a regular triangulation. In general, for any subset σ of $\{1,\dots,N\}$, we denote by cone(σ) the positive span of $\{\mathbf{a}(1),\dots,\mathbf{a}(N)\}$

i.e., $\operatorname{cone}(\sigma) = \sum_{i\in\sigma} \mathbb{R}_{\geq 0}\mathbf{a}(i)$. We often identify a subset $\sigma \subset \{1,\dots,N\}$ with the corresponding set of vectors $\{\mathbf{a}(i)\}_{i\in\sigma}$ or with the set $\operatorname{cone}(\sigma)$. A collection T of subsets of $\{1,\dots,N\}$ is called a triangulation if $\{\operatorname{cone}(\sigma) \mid \sigma \in T\}$ is the set of cones in a simplicial fan whose support equals $\operatorname{cone}(A)$. For any generic choice of a vector $\omega \in \mathbb{R}^{1\times N}$, we can define a triangulation $T(\omega)$ as follows: A subset $\sigma \subset \{1,\dots,N\}$ belongs to $T(\omega)$ if there exists a vector $\mathbf{n} \in \mathbb{R}^{1\times n}$ such that

$$\mathbf{n}\cdot\mathbf{a}(i) = \omega_i \text{ if } i \in \sigma \tag{2.7}$$

$$\mathbf{n}\cdot\mathbf{a}(j) < \omega_j \text{ if } j \in \overline{\sigma}. \tag{2.8}$$

A triangulation T is called a regular triangulation if $T = T(\omega)$ for some $\omega \in \mathbb{R}^{1\times N}$.

Let us put $H_\sigma = \{j \in \{1,\dots,N\} \mid |A_\sigma^{-1}\mathbf{a}(j)| = 1\}$. Here, $|A_\sigma^{-1}\mathbf{a}(j)|$ denotes the sum of all entries of the vector $A_\sigma^{-1}\mathbf{a}(j)$. We set

$$U_\sigma = \left\{ z \in (\mathbb{C}^*)^N \mid \operatorname{abs}\left(z_\sigma^{-A_\sigma^{-1}\mathbf{a}(j)} z_j \right) < R, \text{for all } j \in H_\sigma \setminus \sigma \right\}, \tag{2.9}$$

where $R > 0$ is a small positive real number and abs stands for the absolute value.

We define an $N \times \overline{\sigma}$ matrix B_σ by

$$B_\sigma = \begin{pmatrix} -A_\sigma^{-1}A_{\overline{\sigma}} \\ \mathbf{I}_{\overline{\sigma}} \end{pmatrix} \tag{2.10}$$

and a cone C_σ by

$$C_\sigma = \left\{ \omega \in \mathbb{R}^{N\times 1} \mid \omega \cdot B_\sigma > 0 \right\}. \tag{2.11}$$

Then, T is a regular triangulation if and only if $C_T \overset{def}{=} \bigcap_{\sigma\in T} C_\sigma$ is a non-empty open cone. In this case, the cone C_T is characterized by the formula

$$C_T = \left\{ \omega \in \mathbb{R}^{1\times N} \mid T(\omega) = T \right\}. \tag{2.12}$$

From the definition of U_σ, we can confirm that z belongs to $U_T \overset{def}{=} \bigcap_{\sigma\in T} U_\sigma$ if $(-\log|z_1|,\dots,-\log|z_N|)$ belongs to a sufficiently far translation of C_T inside itself, which implies $U_T \neq \varnothing$.

According to [FF10], the parameter vector c is said to be very generic if $A_\sigma^{-1}(c + A_{\bar{\sigma}}\mathbf{m})$ does not contain any integer entry for any $\mathbf{m} \in \mathbb{Z}^{\bar{\sigma}}$. The following proposition is well-known ([FF10]).

Proposition 2.3. *Let T be a regular triangulation. For each n-simplex σ, we decompose σ as $\sigma=\sigma^u\sqcup\sigma^d$. Assume the condition $\mathbb{Z}A=\mathbb{Z}^{n\times 1}$. Then, for a very generic parameter c, $\bigcup_{\sigma\in T}\{\psi^{\sigma^u}_{\sigma^d,\tilde{\mathbf{k}}}(z;c)\}_{[\tilde{\mathbf{k}}]\in\mathbb{Z}^{\sigma\times 1}/\mathbb{Z}^tA_\sigma}$ is a basis of solutions of $M_A(c)$ on an non-empty open subset U_T of $\mathbb{C}^N$.*

We quote a result of Gelfand, Kapranov, and Zelevinsky [GKZ94, Chapter 7, Proposition 1.5, Theorem 1.7].

Theorem 2.4 ([GKZ94]). *There exists a complete polyhedral fan F_A in $\mathbb{R}^{1\times N}$ whose maximal cones are exactly $\{C_T\}_{T:\text{regular triangulation}}$. The fan F_A is called the secondary fan. Moreover, there exists a convex polytope* $\mathrm{Sec}(A)$ *whose normal fan is equal to the secondary fan F_A.* $\mathrm{Sec}(A)$ *is called the secondary polytope.*

The important observation is that GG system has a restriction-extension structure. Namely, we can construct an explicit Green kernel for a particular class of boundary value problem of GG(A). Let $\mathbf{a}(N+1)\in\mathbb{Z}^{n\times 1}$ be a lattice vector. Let us put $\tilde{A}=(A|\mathbf{a}(N+1))$ and consider a formal solution $f(z,z_{N+1};c)$ of GG$(\tilde{A})$ formally holomorphic in z_{N+1}, i.e., a formal solution of the form $f(z,z_{N+1};c)=\sum_{m=0}^{\infty}f_m(z;c)z_{N+1}^m$. Then, we can easily see that its restriction $rest(f)(z;c)=f(z,0;c)$ along $\{z_{N+1}=0\}$ gives rise to a solution of GG(A). Conversely, if we have a solution $f(z;c)$ of GG(A), we can create a solution of GG$(\tilde{A})$. Let $\sigma_c^{\mathbf{a}(N+1)}$ be a difference operator defined by $(\sigma_c^{\mathbf{a}(N+1)}g)(z;c)=g(z;c+\mathbf{a}(N+1))$. Then, we have a

Proposition 2.5. $F(z,z_{N+1};c)\overset{def}{=}\exp\left(z_{N+1}\sigma_c^{\mathbf{a}(N+1)}\right)f(z;c)\overset{def}{=}\sum_{n=0}^{\infty}\frac{f(z;c+n\mathbf{a}(N+1))}{n!}z_{N+1}^n$ *is a formal solution of $GG(\tilde{A})$ such that $rest(F)(z;c)=f(z;c)$. The restriction operator rest gives an isomorphism between formal solutions of $GG(\tilde{A})$ formally holomorphic in z_{N+1} and formal solutions of $GG(A)$ whose inverse is given by* $\exp\left(z_{N+1}\sigma_c^{\mathbf{a}(N+1)}\right)$.

Based on this proposition, for a subset $I\subset\{1,\dots,N\}$, the general picture of analytic continuation is given by the following diagram:

(2.13)

$$\begin{array}{ccc} \text{Solutions of } \mathrm{GG}(A_I) & \xrightarrow[\sim]{\text{ext}} & \text{Solutions of } \mathrm{GG}(A) \text{ holomorphic in } z_{\bar{I}} \\ {\scriptstyle \text{analytic continuation}}\downarrow & & \downarrow{\scriptstyle \text{analytic continuation}} \\ \text{Solutions of } \mathrm{GG}(A_I) & \xrightarrow[\sim]{\text{ext}} & \text{Solutions of } \mathrm{GG}(A) \text{ holomorphic in } z_{\bar{I}}, \end{array}$$

where ext is the extension operator defined by $\mathrm{ext} = \exp\left(\sum_{j\in\bar{I}} z_j \sigma_c^{\mathbf{a}(j)}\right)$.

In order to choose a path of analytic continuation, we need to take into account the combinatorial structure of the regular triangulations.

3. Analytic continuation associated to a perestroika

Let us recall some basic notions. We first recall the notion of perestroika. For each regular triangulation T, there is a unique vertex v_T of $\mathrm{Sec}(A)$ such that the normal cone $N_{\mathrm{Sec}(A)}(v_T)$ of $\mathrm{Sec}(A)$ at v_T is equal to the cone $C_T \in \mathbb{R}^{1\times N}$ associated to the regular triangulation. For any two regular triangulation T and T', we say T is adjacent to T' if the corresponding vertices v_T and $v_{T'}$ are connected by an edge of $\mathrm{Sec}(A)$. The adjacency can be interpreted in a combinatorial way. We say $Z \subset \{1, \dots, N\}$ is a circuit if $\{\mathbf{a}(i)\}_{i\in Z}$ is a minimal linearly dependent subset of $\{\mathbf{a}(j)\}_{j=1}^N$. A subconfiguration $I \subset \{1, \dots, N\}$ is called a corank 1 configuration if the rank of $\mathrm{Ker}(A_I \times : \mathbb{Z}^{I\times 1} \to \mathbb{Z}^{n\times 1})$ is 1. If I is a corank 1 configuration, the corresponding subconfiguration $\{\mathbf{a}(i)\}_{i\in I}$ has only two regular triangulations. They are denoted by T_+ and T_-. This choice is not canonical but depends on the choice of the generator u of $L_{A_I} = \mathrm{Ker}(A_I \times : \mathbb{Z}^{I\times 1} \to \mathbb{Z}^{n\times 1})$. Namely, if we fix a generator u of L_{A_I}, we put $I_+ = \{i \mid u_i > 0\}$ and $I_- = \{i \mid u_i < 0\}$. Then T_+ (resp. T_-) is defined by $\{I \setminus \{i\}\}_{i\in I_+}$ (resp. $\{I \setminus \{i\}\}_{i\in I_-}$). We say that a polyhedral subdivision Q of A is an almost triangulation if any refinement of Q is a triangulation. The following proposition can be found in [DLRS10, Lemma 2.4.5] and [GKZ94, Chapter7, Theorem 2.10].

PROPOSITION 3.1. *A polyhedral subdivision Q of A is an almost triangulation if and only if each cell of Q has at most corank* 1 *and there is a unique circuit Z such that any corank* 1 *cell contains Z.*

PROPOSITION 3.2. *Let T and T' be two regular triangulations such that T is adjacent to T'. We denote by e the edge of* $\mathrm{Sec}(A)$ *connecting v_T and $v_{T'}$.*

Then, any weight vector $\omega \in$ *rel. int.* $N_{\mathrm{Sec}(A)}(e)$ *defines the same regular polyhedral subdivision* S. *Moreover,* S *is an almost triangulation whose refinements are given by* T *and* T'.

We have a precise description of the change of adjacent regular triangulations as follows. Suppose that a regular triangulation T is adjacent to another regular triangulation T'. Let Q be the intermediate regular polyhedral subdivision explained in the proposition above. We decompose Q as $Q = T_{irr} \cup \{I_s\}_s$ where T_{irr} consists of simplices and I_s are all corank 1 configuations. Since T (or T') is a refinement of Q and T is a triangulation, we see that each I_s has maximal space dimension n, i.e., convex hull $\{0, \{\mathbf{a}(i)\}_{i\in I_s}\}$ has a non-zero Euclidian volume. This implies that $\mathrm{Card}\, I_s = n+1$. Thus, if we denote by Z the common circuit, and by $T_+(I_s)$ (resp. $T_-(I_s)$) the two regular triangulations of I_s, we have $T = T_{irr} \cup \{T_+(I_s)\}_s$ and $T' = T_{irr} \cup \{T_-(I_s)\}_s$. This is also denoted by $T = T_{irr} \cup T_+(Z)$ and $T' = T_{irr} \cup T_-(Z)$. Note that T_{irr} can be empty. We call this process "perestroika".

As in Proposition 2.3, each regular triangulation T corresponds to a basis of solutions, say Ψ_T. In the situation above, we can naturally decompose Ψ_T (resp. $\Psi_{T'}$) as $\Psi_T = \Psi_{T_{irr}} \cup \Psi_{T_+(Z)}$ (resp. $\Psi_{T'} = \Psi_{T_{irr}} \cup \Psi_{T_-(Z)}$). According to the general strategy in [ST94, §1], we want to construct a path of anayltic continuation γ along which $\Psi_{T_{irr}}$ is invariant and $\Psi_{T_+(Z)}$ is transformed to $\Psi_{T_-(Z)}$.

The key to construct a path is the convergence of Mellin-Barnes integral. We fix a corank 1 configuration I. By the definition of corank 1 configuration, there is an element $u \in L_{A_I}$ such that $L_{A_I} = \mathbb{Z}u$. We put $I_{\geq 0} = \{j \in I \mid u_j \geq 0\}$, $I_+ = \{j \in I \mid u_j > 0\}$, and $I_- = \{j \in I \mid u_j < 0\}$. We fix a $j_0 \in I_+$ and put $\sigma = I \setminus \{j_0\}$. Consider an integral

$$I_\sigma(z_I; c) = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{\Gamma(-s) \prod_{i\in I_-} \Gamma\left({}^t\mathbf{e}_i A_\sigma^{-1}(c + \mathbf{a}(j_0)s)\right)}{\prod_{i\in\sigma\cap I_{\geq 0}} \Gamma\left(1 - {}^t\mathbf{e}_i A_\sigma^{-1}(c+\mathbf{a}(j_0)s)\right)} \times$$

$$(e^{\pi\sqrt{-1}\mathbf{1}_{I_-}} z_\sigma)^{-A_\sigma^{-1}(c+\mathbf{a}(j_0)s)} (e^{\pi\sqrt{-1}} z_{j_0})^s ds, \tag{3.1}$$

where C is a vertical contour from $-\sqrt{-1}\infty$ to $+\sqrt{-1}\infty$ separating two spirals of poles of Gamma functions in the integrand. Note that $(e^{\pi\sqrt{-1}\mathbf{1}_{I_-}} z_\sigma)^{A_\sigma^{-1}c} I_\sigma(z_I; c)$ depends only on circuit variables (z_{I_-}, z_{I_+}). Applying the difference operator of infinite order $\exp\left(\sum_{j\in\bar{I}} z_j \sigma_c^{\mathbf{a}(j)}\right)$ to $I_\sigma(e^{2\pi\sqrt{-1}\tilde{\mathbf{k}}_\sigma} z_\sigma, z_{j_0}; c)$, we obtain

$\psi^{I_-}_{I_{\geq 0}\setminus\{j_0\},\tilde{\mathbf{k}}_\sigma}(z;c)$.

Thus, we can apply the standard method of analytic continuation of Mellin-Barnes integral to the functions $I_\sigma(e^{2\pi\sqrt{-1}\tilde{\mathbf{k}}_\sigma}z_\sigma, z_{j_0}; c)$ (see [Sla66, Chapter3]) to obtain analytic continuations of $\psi^{I_-}_{I_{\geq 0}\setminus\{j_0\},\tilde{\mathbf{k}}_\sigma}(z;c)$. However, we should carefully choose the path γ so that the functions $\exp\left(\sum_{j\in\bar{I}} z_j\sigma_c^{\mathbf{a}(j)}\right) I_\sigma(e^{2\pi\sqrt{-1}\tilde{\mathbf{k}}_\sigma}z_\sigma, z_{j_0}; c)$ have a common domain of convergence for any $\sigma = I\setminus\{j_0\}$ and for a suitable choice of representatives $\tilde{\mathbf{k}}_\sigma$. Therefore, what remains to be checked is a Gevrey estimate of functions $I_\sigma(e^{2\pi\sqrt{-1}\tilde{\mathbf{k}}_\sigma}z_\sigma, z_{j_0}; c + A_{\bar{I}}\mathbf{m})$ along the path γ for any $\mathbf{m}\in\mathbb{Z}^{\bar{I}\times 1}$. For this purpose, we need to employ the so-called Erdélyi-Kober operator ([AI99]). In this paper, we only mention that it gives an integral representation of a difference operator from which we obtain the desired estimate.

We sketch the construction of the path of analytic continuation. We first take a point $-\log|z_{start}| \in (\omega_T + C_T)$. Then, we choose a suitable positive real number l so that $-\log|z_{end}| \overset{def}{=} -\log|z_{start}| + l(\varphi_T - \varphi_{T'}) \in (\omega_{T'} + C_{T'})$. Here φ_T and $\varphi_{T'}$ are GKZ vectors associated to regular triangulations T and T' ([GKZ94, §7.1.D]). Note that the secondary polytope is given by the convex hull of the set $\{-\varphi_T\}_{T:\text{regular triangulation}}$. We set $\gamma_1(t) = -\log|z_{start}| + lt(\varphi_T - \varphi_{T'})$ $(0 \leq t \leq 1)$. Then, we choose $\arg z$ along this path γ_1 so that the functions $\bigcup_{\sigma\in T_+}\left\{I_\sigma(e^{2\pi\sqrt{-1}\tilde{\mathbf{k}}_\sigma}z_\sigma, z_{j_0}; c)\right\}_{[\tilde{\mathbf{k}}_\sigma]\in\mathbb{Z}^{\sigma\times 1}/\mathbb{Z}^t A_\sigma}$ have a common domain of convergence. The existence of such a choice of an argument can be verified by a direct computation. When z runs over this path γ_1, we can show that there exists a vector $\omega_Q \in rel.int.C_Q$ so that if $-\log|z| \in (\omega_Q + \gamma_1)$, for any corank 1 configuration I in Q, and for any simplex $\sigma = I\setminus\{j_0\}$ with $j_0 \in I_+$, the function

$$\exp\left(\sum_{j\in\bar{I}} z_j\sigma_c^{\mathbf{a}(j)}\right) I_\sigma(e^{2\pi\sqrt{-1}\tilde{\mathbf{k}}_\sigma}z_\sigma, z_{j_0}; c) \tag{3.2}$$

are all convergent. We set $\gamma = \omega + \gamma_1$.

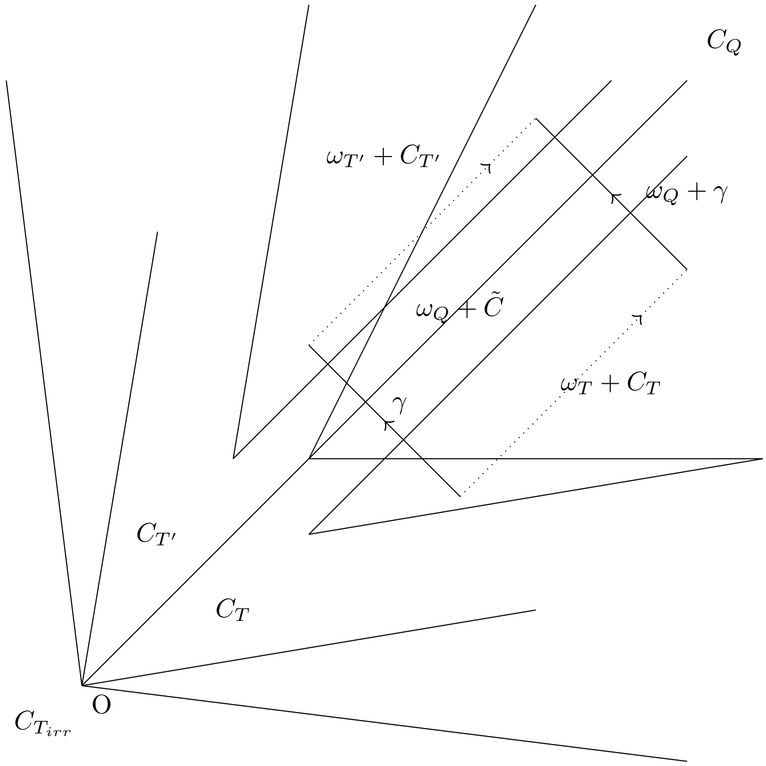

Figure 1. cones

The connection formula takes the following form.

Theorem 3.3. *In the setting above, take a corank* 1 *configuration* I *of* Q *and* $j_0 \in I_+$. *Put* $\sigma = I \setminus \{j_0\}$. *Suppose that the parameter* c *is generic so that* $\psi^{I_-}_{I_{\geq 0}\setminus\{j_0\},\tilde{\mathbf{k}}_\sigma}(z;c)$ *are independent for a choice of representative* $\{[\tilde{\mathbf{k}}_\sigma]\}$. *Then, along* γ, *one has a connection formula*

$$\psi^{I_-}_{I_{\geq 0}\setminus\{j_0\},\tilde{\mathbf{k}}_\sigma}(z;c) = \sum_{i_0\in I_-} \frac{1}{{}^t\mathbf{e}_{i_0}A_\sigma^{-1}\mathbf{a}(j_0)} \psi^{(I_-\setminus\{i_0\})\cup\{j_0\}}_{I_{\geq 0}\setminus\{j_0\},(\tilde{\mathbf{k}}_{\sigma\setminus\{i_0\}},\overset{j_0}{\check{0}})}(z;c). \tag{3.3}$$

Moreover, Γ*-series corresponding to* T_{irr} *are invariant after analytic continuation.*

Note that γ does not depend on the choice of corank 1 configuration I. Therefore, this gives rise to a connection formula between bases of solutions of $M_A(c)$.

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