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Spherical inversion for a small K-type on the split real Lie group of type G_2

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Dedicated to Professor Toshio Oshima on the occasion of his 70-th birthday

Abstract. We give an explicit formula for the Harish-Chandra c-function for a small K-type on a split real Lie group of type G_2. As an application we give an explicit formula for spherical inversion for this small K-type.

1. Introduction

Harmonic analysis on a Riemannian symmetric space G/K of the noncompact type is by now well developed (cf. [17]). A natural extension is to study harmonic analysis on homogeneous vector bundles over G/K. One of fundamental problems in harmonic analysis is to establish the Plancherel theorem. Harish-Chandra establishes a general theory of the Eisenstein integrals and the Plancherel theorem for noncompact real semisimple Lie groups (cf. [15, 18, 33, 34]). The Plancherel theorem on a homogeneous vector bundle over G/K associated with an irreducible representation π of K follows from Harish-Chandra's result by restricting the Plancherel measure to K-finite functions of type π. But it is a highly nontrivial and important problem to determine the Plancherel measure on the associated vector bundle as explicitly as in the case of the trivial K-type. There are several studies in this direction (cf. [8, 9, 11, 12, 16, 21, 30, 31]).

In our previous paper [21], we study elementary spherical functions on G with a small K-type π (in the sense of Wallach [33, §11.3]). Namely, we identify elementary spherical functions with the Heckman-Opdam hypergeometric function (cf. [16, 23]) and apply the inversion formula and the Plancherel formula for the hypergeometric Fourier transform ([22]) to obtain the inversion formula and the Plancherel formula for the π-spherical transform. But there is an exception in [21]. Namely, for a certain small K-type of a noncompact Lie group of type G_2,

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elementary spherical functions can not be expressed by the Heckman-Opdam hypergeometric function.

In this paper we give a complete treatment of harmonic analysis of π-spherical transform for each small K-type π of G_2. Namely, we give an explicit formula for the Harish-Chandra c-function $c^\pi(\lambda)$ and determine the Plancherel measure explicitly. The most continuous part of the Plancherel measure is $|c^\pi(\lambda)|^{-2}d\lambda$ on $\sqrt{-1}\mathfrak{a}^*$ and the other spectra with supports of lower dimensions are given explicitly by using residue calculus. As indicated by Oshima [24] and as was done for one-dimensional K-types by the second author [30], we could prove the inversion formula for the π-spherical transform in the case of G_2 by extending Rosenberg's method of a proof of the inversion formula in the case of the trivial K-type ([25]). Instead of doing this, we utilize general results on the Plancherel theorem and residue calculus on G due to Harish-Chandra and Arthur (cf. [15, 1, 18, 33]) and devote ourselves to the determination of the Plancherel measure.

This paper is organized as follows. In Section 2 we give general results for elementary π-spherical functions, the Harish-Chandra c-function, the inversion formula for π-spherical transform with respect to a small K-type π on a noncompact real semisimple Lie group of finite center.

In Section 3 we study the case of G_2. We give an explicit formula of the c-function (Theorem 3.3), the inversion formula, and the Plancherel formula (Theorem 3.5, Corollary 3.6) for each small K-type. In particular, they cover the small K-type that is not treated in [21].

2. Elementary spherical functions for small K-types

2.1. Notation

Let $\mathbb{N}$ denote the set of the nonnegative integers. Let G be a non-compact connected real semisimple Lie group of finite center and K a maximal compact subgroup of G. Let e denote the identity element of G. Lie algebras of Lie groups G, K, etc. are denoted by the corresponding German letter $\mathfrak{g}$, $\mathfrak{k}$, etc. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition and $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$. Let Σ denote the root system for $(\mathfrak{g}, \mathfrak{a})$. For $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha$ denote the corresponding root space and $\boldsymbol{m}_\alpha = \dim \mathfrak{g}_\alpha$. Fix a positive system $\Sigma^+ \subset \Sigma$ and let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ denote the set of simple roots in Σ^+. Define $\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ and $N = \exp \mathfrak{n}$. Then we have the Iwasawa decomposition $G = K \exp \mathfrak{a} N$. Define $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} \boldsymbol{m}_\alpha \alpha$.

Let W denote the Weyl group of Σ and s_i the reflection across $\alpha_i^\perp$ $(1 \le i \le r)$. We have $W \simeq M'/M$, where M' (resp. M) is the normalizer (resp. centralizer) of $\mathfrak{a}$ in K.

Define

$$\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \alpha(H) > 0 \text{ for all } \alpha \in \Sigma^+\},$$
$$\mathfrak{a}_{\mathrm{reg}} = \{H \in \mathfrak{a} \mid \alpha(H) \neq 0 \text{ for all } \alpha \in \Sigma^+\}.$$

We have the Cartan decomposition $G = K \exp \overline{\mathfrak{a}_+}\, K$.

Let $\langle\ ,\ \rangle$ denote the inner product on $\mathfrak{a}^*$ induced by the Killing form on $\mathfrak{g}$ and $||\ ||$ the corresponding norm. Define

$$\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^* \mid \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Sigma^+\}.$$

2.2. Elementary π-spherical function

In this subsection, we review elementary π-spherical functions for small K-types according to [21].

Let (π, V) be a small K-type, that is, $\pi|_M$ is irreducible. We call an $\mathrm{End}_{\mathbb{C}}V$-valued function f on G satisfying

$$f(k_1 g k_2) = \pi(k_2^{-1}) f(g) \pi(k_1^{-1}) \quad (k_1,\, k_2 \in K,\, g \in G)$$

a π-spherical function.

Let $\boldsymbol{D}^\pi$ denote the algebra of the invariant differential operators on the homogeneous vector bundle over G/K associated with π. Let $U(\mathfrak{g}_{\mathbb{C}})$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $U(\mathfrak{g}_{\mathbb{C}})^K$ the set of the $\mathrm{Ad}(K)$-invariant elements in $U(\mathfrak{g}_{\mathbb{C}})$. Let $J_{\pi^*} = \ker \pi^*$ in $U(\mathfrak{k}_{\mathbb{C}})$. We have

$$\boldsymbol{D}^\pi \simeq U(\mathfrak{g}_{\mathbb{C}})^K / U(\mathfrak{g}_{\mathbb{C}})^K \cap U(\mathfrak{g}_{\mathbb{C}}) J_{\pi^*}.$$

Let $S(\mathfrak{a}_{\mathbb{C}})$ denote the symmetric algebra of $\mathfrak{a}_{\mathbb{C}} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ and $S(\mathfrak{a}_{\mathbb{C}})^W$ the set of the W-invariant elements in $S(\mathfrak{a}_{\mathbb{C}})$. There exists an algebra homomorphism

$$\gamma^\pi : U(\mathfrak{g}_{\mathbb{C}})^K \to S(\mathfrak{a}_{\mathbb{C}})^W$$

with the kernel $U(\mathfrak{g}_{\mathbb{C}})^K \cap U(\mathfrak{g}_{\mathbb{C}}) J_{\pi^*}$ (cf. [33, Lemma 11.3.2, Lemma 11.3.3]). Notice that the homomorphism γ^π is independent of the choice of Σ^+. Thus we have the generalized Harish-Chandra isomorphism $\gamma^\pi : \boldsymbol{D}^\pi \xrightarrow{\sim} S(\mathfrak{a}_{\mathbb{C}})^W$. Therefore, any algebra homomorphism from $\boldsymbol{D}^\pi$ to $\mathbb{C}$ is of the form $D \mapsto \gamma^\pi(D)(\lambda)$ $(D \in \boldsymbol{D}^\pi)$ for some $\lambda \in {}_W\backslash \mathfrak{a}^*_{\mathbb{C}}$.

For $\lambda \in {}_W\backslash \mathfrak{a}^*_{\mathbb{C}}$ there exists a unique smooth π-spherical function $f = \phi^\pi_\lambda$ satisfying $f(e) = \mathrm{id}_V$ and $Df = \gamma^\pi(D)(\lambda) f$ $(D \in \boldsymbol{D}^\pi)$ (cf. [21, Theorem 1.4]). We call ϕ^π_λ the elementary π-spherical function. Since $\boldsymbol{D}^\pi$ contains an elliptic operator, ϕ^π_λ

is real analytic. Moreover, it has an integral representation

$$\phi^\pi_\lambda(g) = \int_K e^{(\lambda-\rho)(H(gk))}\pi(k\kappa(gk)^{-1})dk. \tag{1}$$

Here given $x \in G$, define $\kappa(x) \in K$ and $H(x) \in \mathfrak{a}$ by $x \in \kappa(x)e^{H(x)}N$. Notice that ϕ^π_λ is independent of the choice of Σ^+, though the right hand side of (1) depends on Σ^+ at first glance. Moreover, ϕ^π_λ depends holomorphically on $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

Formula (1) is a special case of the integral representations of elementary spherical functions (or more generally the Eisenstein integrals) given by Harish-Chandra (cf. [34, §6.2.2, §9.1.5], [8, (42)], [18, (14.20)]).

2.3. Harish-Chandra series

In this subsection, we review the Harish-Chandra expansion of the elementary spherical function according to [34, § 9.1]. We assume (π, V) is a small K-type.

Let $C^\infty(G,\pi,\pi)$ denote the space of the smooth π-spherical functions. If $f \in C^\infty(G,\pi,\pi)$ then $f|_A$ takes values in $\mathrm{End}_M V \simeq \mathbb{C}$. Hence we regard $\Upsilon^\pi(f) := f|_A \circ \exp$ as a scalar valued function on $\mathfrak{a}$. Let $C^\infty(\mathfrak{a})^W$ denote the space of the W-invariant smooth functions on $\mathfrak{a}$. The restriction map Υ^π gives an isomorphism $C^\infty(G,\pi,\pi) \xrightarrow{\sim} C^\infty(\mathfrak{a})^W$ ([21, Theorem 1.5]).

Let $\mathscr{R}$ be the unital algebra of functions on $\mathfrak{a}_{\mathrm{reg}}$ generated by $(1 \pm e^\alpha)^{-1}$ ($\alpha \in \Sigma^+$). For any $D \in U(\mathfrak{g}_{\mathbb{C}})^K$ there exists a unique W-invariant differential operator $\Delta^\pi(D) \in \mathscr{R} \otimes S(\mathfrak{a}_{\mathbb{C}})$ such that for any $f \in C^\infty(G,\pi,\pi)$

$$\Upsilon^\pi(Df) = \Delta^\pi(D)\Upsilon^\pi(f)$$

on $\mathfrak{a}_{\mathrm{reg}}$ ([21, Proposition 3.10]). We call $\Delta^\pi(D)$ the π-radial part of D. The function $\Phi = \Upsilon^\pi(\phi^\pi_\lambda)$ satisfies differential equations

$$\Delta^\pi(D)\Phi = \gamma^\pi(D)(\lambda)\Phi \quad (D \in U(\mathfrak{g}_{\mathbb{C}})^K). \tag{2}$$

Let $\mathbb{N}\Sigma^+$ denote the set of $\mu \in \mathfrak{a}^*$ of the form $\mu = n_1\alpha_1 + \cdots + n_r\alpha_r$ ($n_i \in \mathbb{N}$). For $\mu \in \mathbb{N}\Sigma^+ \setminus \{0\}$, let σ_μ denote the hyperplane

$$\sigma_\mu = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \langle 2\lambda - \mu, \mu\rangle = 0\}.$$

If $\lambda \notin \sigma_\mu$ for any $\mu \in \mathbb{N}\Sigma^+ \setminus \{0\}$, then there exists a unique convergent series solution

$$\Phi_\lambda(H) = e^{(\lambda-\rho)(H)} \sum_{\mu \in \mathbb{N}\Sigma^+} \Gamma_\mu(\lambda)e^{-\mu(H)} \quad (H \in \mathfrak{a}_+) \tag{3}$$

of (2) with $\Gamma_\mu(\lambda) \in \mathbb{C}$ and $\Gamma_0(\lambda) = 1$. This is a special case of [34, Theorem 9.1.4.1].

By using differential equations (2), apparent singularities of $\Phi_\lambda(H)$ as a function of λ is removable unless $\mu = n\alpha$ for some $n \in \mathbb{Z}_{>0}$ and $\alpha \in \Sigma^+$ (cf. [1, Corollary 6.3], see also [23, Lemma 6.5] and [16, Proposition 7.5]). For $\mu = n\alpha$, $\lambda \notin \sigma_\mu$ if and only if $\langle \lambda, \alpha^\vee \rangle \neq n$. Here $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$. Thus Φ_λ is defined if $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Sigma^+$.

If $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \Sigma^+$, then $\{\Phi_{w\lambda} \mid w \in W\}$ forms a basis of the solution space of (2) on $\mathfrak{a}_+$. Thus $\Upsilon^\pi(\phi^\pi_\lambda)$ is a linear combination of $\Phi_{w\lambda}$ $(w \in W)$. Since $\phi^\pi_{w\lambda} = \phi^\pi_\lambda$, there exists a constant $c^\pi(\lambda)$ such that

$$\Upsilon^\pi(\phi^\pi_\lambda)(H) = \sum_{w \in W} c^\pi(w\lambda)\Phi_{w\lambda}(H) \quad (H \in \mathfrak{a}_+). \tag{4}$$

2.4. Harish-Chandra c-function

In this subsection, we review the Harish-Chandra c-function. We refer to [26], [34, §9.1.6], [32, Chapter 8], and [28, §5] for details.

Let $H \in \mathfrak{a}_+$ and $\lambda \in \mathfrak{a}^*_\mathbb{C}$ satisfying $\operatorname{Re}\lambda \in \mathfrak{a}^*_+$. The leading coefficient $c^\pi(\lambda)$ of $\Upsilon^\pi(\phi^\pi_\lambda)$ at infinity in $A_+ = \exp \mathfrak{a}_+$ is given by the Harish-Chandra c-function (cf. [34, Theorem 9.1.6.1], [18, Theorem 14.7, (14.29)]):

$$\lim_{t\to\infty} e^{t(-\lambda+\rho)(H)}\Upsilon^\pi(\phi^\pi_\lambda)(e^{tH}) = c^\pi(\lambda), \tag{5}$$

$$c^\pi(\lambda) = \int_{\bar N} e^{-(\lambda+\rho)(H(\bar n))}\pi(\kappa(\bar n))d\bar n, \tag{6}$$

where the Haar measure on $\bar N$ is normalized so that

$$\int_{\bar N} e^{-2\rho(H(\bar n))}d\bar n = 1.$$

The integral (6) converges absolutely for $\operatorname{Re}\lambda \in \mathfrak{a}^*_+$ and extends to a meromorphic function on $\mathfrak{a}^*_\mathbb{C}$. Notice that $c^\pi(\lambda) \in \operatorname{End}_M V \simeq \mathbb{C}$.

Define

$$\Sigma_0 = \{\alpha \in \Sigma \mid \tfrac{1}{2}\alpha \notin \Sigma\}$$

and $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$. For $\alpha \in \Sigma_0^+$ let $\mathfrak{g}_{(\alpha)}$ denote the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. Put $\mathfrak{k}_\alpha = \mathfrak{k} \cap \mathfrak{g}_{(\alpha)}$, $\mathfrak{p}_\alpha = \mathfrak{p} \cap \mathfrak{g}_{(\alpha)}$, $\mathfrak{a}_\alpha = \mathfrak{a} \cap \mathfrak{g}_{(\alpha)}$, $\mathfrak{n}_\alpha = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$, and $\bar{\mathfrak{n}}_\alpha = \theta\mathfrak{n}_\alpha$. For $\alpha \in \Sigma_0^+$, let G_α, K_α, A_α, N_α, and $\bar N_\alpha$ denote the analytic subgroups of G corresponding to $\mathfrak{g}_{(\alpha)}$, $\mathfrak{k}_\alpha$, $\mathfrak{a}_\alpha$, $\mathfrak{n}_\alpha$, and $\bar{\mathfrak{n}}_\alpha$, respectively. We have the Iwasawa decomposition $G_\alpha = K_\alpha A_\alpha N_\alpha$. Put $\rho_\alpha = \frac{1}{2}(\boldsymbol{m}_\alpha + 2\boldsymbol{m}_{2\alpha})\alpha$. Let $d\bar n_\alpha$

denote the Haar measure on $\bar{N}_\alpha$ normalized so that

$$\int_{\bar{N}_\alpha} e^{-2\rho_\alpha(H(\bar{n}_\alpha))} d\bar{n}_\alpha = 1.$$

Let $w^* \in W$ be the element such that $w^*(\Sigma^+) = -\Sigma^+$. Let $w^* = s_{i_m} \cdots s_{i_2} s_{i_1}$ be a reduced expression, where m denotes the length of w^* and $1 \leq i_k \leq r$ $(1 \leq k \leq m)$. Put $\beta_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$. Then we have $\Sigma_0^+ = \{\beta_1, \beta_2, \ldots, \beta_m\}$ (cf. [17, Ch. IV Corollary 6.11]). We have the decomposition $\bar{N} = \bar{N}_{\beta_1} \cdots \bar{N}_{\beta_m}$, the product map being a diffeomorphism. Moreover, there exists a positive constant c_0 such that

$$d\bar{n} = c_0 \, d\bar{n}_{\beta_1} \cdots d\bar{n}_{\beta_m}. \tag{7}$$

For $\alpha \in \Sigma_0^+$ define

$$c_\alpha^\pi(\lambda) = \int_{\bar{N}_\alpha} e^{-(\lambda+\rho_\alpha)(H(\bar{n}_\alpha))} \pi(\kappa(\bar{n}_\alpha)) d\bar{n}_\alpha.$$

We have the following product formula ([26, Theorem 1.2], [32, §8.11.6], [34, §9.1.6], [28, Theorem 5.1]).

Theorem 2.1. $c^\pi(\lambda) = c_0 \, c_{\beta_1}^\pi(\lambda) \cdots c_{\beta_m}^\pi(\lambda)$.

For the case of the trivial K-type $\pi = \mathrm{triv}$, $c_{\beta_i}^{\mathrm{triv}}$ can be written explicitly by the classical Gamma function and we have the Gindikin and Karpelevič product formula for $c^{\mathrm{triv}}(\lambda)$ (cf. [14], see also [17, Ch IV, §6] and [34, §9.1.7]). Note that the constant c_0 in (7) is determined explicitly by the Gindikin and Karpelevič formula for $c^{\mathrm{triv}}(\lambda)$.

The c-function for a one-dimensional K-type π of a group G of Hermitian type is also given explicitly by the Gamma function ([27], [29]).

In [21] we give an explicit formula of $c^\pi(\lambda)$ for each simple G and each small K-type π, with one exception for G of type G_2 and a certain small K-type π. The method we use is to relate the π-elementary spherical function ϕ_λ^π with the Heckman-Opdam hypergeometric function, instead of computing the integral (6) by using Theorem 2.1. Heckman [16, Chapter 5] gives in this way an explicit formula of $c^\pi(\lambda)$ for a one-dimensional K-type π when the group G is of Hermitian type.

In § 3.2 we give an explicit formula of $c^\pi(\lambda)$ for G of type G_2 and each small K-type π by using Theorem 2.1.

2.5. π-spherical transform

Let dH denote the Euclidean measure on $\mathfrak{a}$ with respect to the Killing form. Define $\delta_{G/K} = \prod_{\alpha\in\Sigma^+} |2\sinh\alpha|^{\boldsymbol{m}_\alpha}$. We normalize the Haar measure dg on G so that

$$\int_G \psi(g)dg = \frac{1}{\# W}\int_{\mathfrak{a}} \psi(e^H)\,\delta_{G/K}(H)dH$$

for any compactly supported continuous K-bi-invariant function ψ on G (cf. [17, Ch. I, Theorem 5.8]).

Let $C_c^\infty(G,\pi,\pi)$ be the subspace of $C^\infty(G,\pi,\pi)$ consisting of the compactly supported smooth π-spherical functions. The π-spherical transform of $f \in C_c^\infty(G,\pi,\pi)$ is the $\mathrm{End}_M V \simeq \mathbb{C}$-valued function $f^\wedge$ on $\mathfrak{a}_{\mathbb{C}}^*$ defined by

$$f^\wedge(\lambda) = \int_G \phi_\lambda^\pi(g^{-1})f(g)dg. \tag{8}$$

The π-spherical transform $f \mapsto f^\wedge(\lambda)$ is a homomorphism from the commutative convolution algebra $C_c^\infty(G,\pi,\pi)$ to $\mathbb{C}$ (cf. [8]). It is a special case of the Fourier transform given by Arthur [1] (see also [6, §3]). By the identification $C_c^\infty(G,\pi,\pi) \simeq C_c^\infty(\mathfrak{a})^W$, the π-spherical transform $f^\wedge$ of $f \in C_c^\infty(\mathfrak{a})^W$ is given by

$$f^\wedge(\lambda) = \frac{1}{\# W}\int_{\mathfrak{a}} f(H)\Upsilon^\pi(\phi^\pi_{-\lambda})(H)\delta_{G/K}(H)\,dH. \tag{9}$$

We normalize the Haar measure $d\lambda$ on $\sqrt{-1}\mathfrak{a}^*$ so that the Euclidean Fourier transform and its inversion are given by

$$\tilde{f}(\lambda) = \int_{\mathfrak{a}} f(H)e^{-\lambda(H)}dH, \qquad f(H) = \int_{\sqrt{-1}\mathfrak{a}^*} \tilde{f}(\lambda)e^{\lambda(H)}d\lambda.$$

Let η_1 be a point in $-\overline{\mathfrak{a}_+^*}$ such that $c^\pi(-\lambda)^{-1}$ is a regular function of λ for $\mathrm{Re}\,\lambda \in \eta_1 - \overline{\mathfrak{a}_+^*}$. The existence of such η_1 follows from an explicit formula of the c-function for each small K-type, which is determined by [21] and §3 for G_2. It also follows from a general result on the Harish-Chandra c-function due to Cohn [10].

Let $F = f^\wedge$. Following [1, Chapter II, §1], define the function $F^\vee(H)$ on $\mathfrak{a}_+$ by

$$F^\vee(H) = \int_{\eta_1+\sqrt{-1}\mathfrak{a}^*} F(\lambda)\Phi_\lambda(H)c^\pi(-\lambda)^{-1}d\lambda. \tag{10}$$

The integral (10) converges and is independent of η_1 (cf. [1, Chapter II, §1]). $F^\vee$

defined above coincides with that given by Arthur, because ϕ^π_λ is W-invariant in λ and the Harish-Chandra μ-function associate with a minimal parabolic subgroup in our case is given by $c^\pi(\lambda)^{-1}c^\pi(-\lambda)^{-1}$ (cf. [33, § 10.5]). The following theorem is a special case of [1, Chapter III, Theorem 3.2]. It is also a special case of [4, Theorem 1.1], since G is a semisimple symmetric space for $G \times G$ (cf. [6]).

Theorem 2.2. *For $f \in C_c^\infty(\mathfrak{a})^W$ we have*

$$f(H) = (f^\wedge)^\vee(H) \quad (H \in \mathfrak{a}_+).$$

If $c^\pi(-\lambda)^{-1}$ is a regular function of λ for $\operatorname{Re}\lambda \in -\overline{\mathfrak{a}^*_+}$, then we can take $\eta_1 = 0$ and by (4) and W-invariance of $c^\pi(\lambda)^{-1}c^\pi(-\lambda)^{-1}$ in $\lambda \in \sqrt{-1}\mathfrak{a}^*$, we have

$$f(H) = \frac{1}{\#W}\int_{\sqrt{-1}\mathfrak{a}^*} f^\wedge(\lambda)\varUpsilon^\pi(\phi^\pi_\lambda)(H)|c^\pi(\lambda)|^{-2}d\lambda \quad (H \in \mathfrak{a}^*). \tag{11}$$

In [21, Corollary 7.6] we prove the formula (11) by using the inversion formula of the hypergeometric Fourier transform due to Opdam [22], under the assumption that π is a small K-type of a real simple G which is not in the following list:

(1) $\mathfrak{g} = \mathfrak{sp}(p,1)$, $\pi = \pi_n \circ \mathrm{pr}_2$ (π_n : n-dimensional irred. rep. of $\mathrm{Sp}(1)$),
(2) $\mathfrak{g} = \mathfrak{so}(2r,1)$,
$\pi = \pi^\pm_s$: irred. rep. of $\mathrm{Spin}(2r)$ with h.w. $(s/2, \cdots s/2, \pm s/2)$ ($s \in \mathbb{N}$),
(3) $\mathfrak{g} = \mathfrak{so}(p,q)$ $(p > q \geq 3,\ p$: even, q : odd$)$,
$\pi = \sigma \circ \mathrm{pr}_1$ (σ : one of half spin representation of $\mathrm{Spin}(p)$),
(4) $\mathfrak{g}$: Hermitian type, π : one dimensional K-type,
(5) $\mathfrak{g} = \mathfrak{g}_2$, $\pi = \pi_2$ (see §3 for the definition).

Though the case (3) is not covered by [21, Corollary 7.6], the formula (11) holds in this case, since $c^\pi(-\lambda)^{-1}$ is a regular function of λ for $\operatorname{Re}\lambda \in -\overline{\mathfrak{a}^*_+}$ as we mention in the final part of [21].

If the parameter of the small K-type is "large enough" in the cases (1), (2), and (4), then $c^\pi(-\lambda)^{-1}$ has singularities in $\operatorname{Re}\lambda \in -\overline{\mathfrak{a}^*_+}$ and we must take account of residues during the contour shift $\eta_1 + \sqrt{-1}\mathfrak{a}^* \to \sqrt{-1}\mathfrak{a}^*$. The most continuous part of the spectrum is given by the right hand side of (11). In addition, there are spectra with low dimensional supports. The residue calculus in the case (4) is done by [30]. For the cases (1) and (2), $\dim\mathfrak{a} = 1$ and the residue calculus is easy to proceed. Also these cases are covered by the inversion formula of the Jacobi transform (cf. [12, Appendix 1], [19]). See also [11] and [31] for the case (1).

We will discuss the case (5) in the next section.

3. The case of G_2

3.1. Notation and preliminary results

Let $\mathfrak{g}$ be the simple split real Lie algebra of type G_2 and G the connected simply connected Lie group with the Lie algebra $\mathfrak{g}$. G is the double cover of the adjoint group of $\mathfrak{g}$. Let K be a maximal compact subgroup of G and $\mathfrak{k}$ the Lie algebra of K. Then $K \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathfrak{k} \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Δ and Δ_K denote the root system for $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ and $(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, respectively. Then Δ is a root system of type G_2. We choose a positive system $\Delta^+ \subset \Delta$ so that its base contains a short compact root β_1. The other simple root, say β_2, is a long noncompact root. If we put $\Delta_K^+ = \Delta_K \cap \Delta^+$ then

$$\begin{aligned}
\Delta^+ &= \{\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2, 3\beta_1 + \beta_2, 3\beta_1 + 2\beta_2\}, \\
\Delta_K^+ &= \{\beta_1, 3\beta_1 + 2\beta_2\}.
\end{aligned}$$

We fix an inner product on $\sqrt{-1}\mathfrak{t}^*$ such that $(\beta_1, \beta_1) = 2$. Then $(\beta_2, \beta_2) = 6$ and $(\beta_1, \beta_2) = -3$. We let $K = K_1 \times K_2$ with $K_i \simeq \mathrm{SU}(2)$ $(i = 1, 2)$ assuming that β_1 (resp. $3\beta_1 + 2\beta_2$) is a root for $((\mathfrak{k}_1)_\mathbb{C}, (\mathfrak{t} \cap \mathfrak{k}_1)_\mathbb{C})$ (resp. $((\mathfrak{k}_2)_\mathbb{C}, (\mathfrak{t} \cap \mathfrak{k}_2)_\mathbb{C})$). Let pr_i denote the projection of K to K_i $(i = 1, 2)$.

The classification of the small K-types for G is given as follows:

Theorem 3.1 ([20, Theorem 1]). *The non-trivial small K-types are $\pi_1 := \sigma \circ \mathrm{pr}_1$ and $\pi_2 := \sigma \circ \mathrm{pr}_2$. Here σ is the two-dimensional irreducible representation of* $\mathrm{SU}(2)$.

A discrete series representation of G is an irreducible representation of G realized as a closed G-invariant subspace of the left regular representation on $L^2(G)$.

Lemma 3.2. *Let G be as above. Then no small K-type appears in any discrete series representation of G.*

Proof. If π is the trivial K-type or $\pi = \pi_1$, then it follows from the Plancherel formula for the π-spherical functions (cf. [17], [21]) that there are no discrete series representations having the K-type π.

Next let us discuss the case of π_2. The positive open chamber $(\sqrt{-1}\mathfrak{t}^*)^+$ for Δ_K^+ contains the following three open chambers for Δ:

$$\begin{aligned}
(\sqrt{-1}\mathfrak{t}^*)_1^+ &:= \{\lambda \in (\sqrt{-1}\mathfrak{t}^*)^+ \mid (\lambda, \beta_2) > 0\}, \\
(\sqrt{-1}\mathfrak{t}^*)_2^+ &:= \{\lambda \in (\sqrt{-1}\mathfrak{t}^*)^+ \mid (\lambda, \beta_2) < 0 \text{ and } (\lambda, \beta_1 + \beta_2) > 0\},
\end{aligned}$$

$$(\sqrt{-1}\mathfrak{t}^*)_3^+ := \{\lambda \in (\sqrt{-1}\mathfrak{t}^*)^+ \mid (\lambda, \beta_1 + \beta_2) < 0\}.$$

Let Δ_i^+ be the corresponding positive systems ($i = 1, 2, 3$). Note that $\Delta_1^+ = \Delta^+$. If we put $\delta_i = \frac{1}{2}\sum_{\beta\in\Delta_i^+}\beta$ then

$$\delta_1 = 5\beta_1 + 3\beta_2, \quad \delta_2 = 5\beta_1 + 2\beta_2, \quad \delta_3 = 4\beta_1 + \beta_2.$$

On the other hand, $\delta_K := \frac{1}{2}\sum_{\beta\in\Delta_K^+}\beta = 2\beta_1 + \beta_2$. Now suppose π_2 appears in a discrete series representation with Harish-Chandra parameter $\lambda \in \sqrt{-1}\mathfrak{t}^*$. We may assume $\lambda \in (\sqrt{-1}\mathfrak{t}^*)_i^+$ for $i = 1, 2$, or 3. Since the highest weight of π_2 is $\frac{3}{2}\beta_1 + \beta_2$, it follows from [2, Theorem 8.5] that

$$\frac{3}{2}\beta_1 + \beta_2 = \lambda + \delta_i - 2\delta_K + \sum_{\beta\in\Delta_i^+} n_\beta \beta \quad \text{for some } n_\beta \in \mathbb{N}.$$

If $i = 1$ then this reduces to

$$\lambda = \left(\frac{1}{2} - c_1\right)\beta_1 - c_2\beta_2 \quad \text{for some } c_1,\, c_2 \in \mathbb{N}.$$

Since $(\lambda, \beta_j) > 0$ ($j = 1, 2$), we have $1 - 2c_1 + 3c_2 > 0$ and $-\frac{3}{2} + 3c_1 - 6c_2 > 0$, which are impossible. If $i = 2$ or 3 then we can also deduce a contradiction in a similar way. □

3.2. Harish-Chandra c-function for G_2

The restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ is a root system of type G_2. For all $\alpha \in \Sigma$ we have $\mathfrak{g}_{(\alpha)} \simeq \mathfrak{sl}(2, \mathbb{R})$, since $\boldsymbol{m}_\alpha = \dim \mathfrak{g}_\alpha = 1$ and $2\alpha \notin \Sigma$.

We recall the c-function for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. Put

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\{h, e, f\}$ is a basis of $\mathfrak{sl}(2, \mathbb{R})$ and also forms an $\mathfrak{sl}_2$-triple. We put $\mathfrak{a} = \mathbb{R}h$ and $\mu = \lambda(h)$ for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. By [27, Remark 7.3], the c-function for $\mathfrak{sl}(2, \mathbb{R})$ with a one-dimensional $\mathfrak{k}$-type π of the weight $\nu \in \mathbb{Q}$ for $\sqrt{-1}(e - f)$ is given by

$$c^\pi(\lambda) = \frac{2^{1-\mu}\Gamma(\mu)}{\Gamma(\frac{1}{2}(\mu + 1 + \nu))\Gamma(\frac{1}{2}(\mu + 1 - \nu))}. \tag{12}$$

Now we come back to the case of G_2. For $\alpha \in \Sigma^+$ choose $e_\alpha \in \mathfrak{g}_\alpha$ so that $\{\alpha^\vee,\, e_\alpha,\, -\theta e_\alpha\}$ is an $\mathfrak{sl}_2$-triple. Put $t_\alpha := e_\alpha + \theta e_\alpha \in \sqrt{-1}\mathfrak{k}_\alpha$. If $\alpha \in \Sigma^+$ is a long root, then the possible weights of t_α for π_i ($i = 1, 2$) are $\pm\frac{1}{2}$ by [20, Lemma 4.2]. If $\alpha \in \Sigma^+$ is a short root, then the possible weights of t_α for π_1 (resp. π_2) are $\pm\frac{1}{2}$ (resp. $\pm\frac{3}{2}$) by [20, Lemma 4.3]. Since (12) remains unchanged if we replace ν by $-\nu$, $c^{\pi_i}_\alpha(\lambda)$ is a scalar operator for each $\alpha \in \Sigma$ and $i = 1, 2$.

Let Σ^+_{long} and $\Sigma^+_{\mathrm{short}}$ denote the sets of the long and short positive roots, respectively. Define $\lambda_\alpha = \langle \lambda, \alpha^\vee \rangle$ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $\alpha \in \Sigma^+$. It follows from Theorem 2.1, (12), and the proof of [17, Ch. IV, Theorem 6.13] that

$$
\begin{aligned}
c^{\mathrm{triv}}(\lambda) &= c_0 \prod_{\alpha\in\Sigma^+} \frac{2^{1-\lambda_\alpha}\Gamma(\lambda_\alpha)}{\Gamma(\frac{1}{2}\lambda_\alpha+\frac{1}{2})^2} = c_0 \prod_{\alpha\in\Sigma^+} \frac{\pi^{-\frac{1}{2}}\Gamma(\frac{1}{2}\lambda_\alpha)}{\Gamma(\frac{1}{2}\lambda_\alpha+\frac{1}{2})},\\
c^{\pi_1}(\lambda) &= c_0 \prod_{\alpha\in\Sigma^+} \frac{2^{1-\lambda_\alpha}\Gamma(\lambda_\alpha)}{\Gamma(\frac{1}{2}(\lambda_\alpha+\frac{3}{2}))\Gamma(\frac{1}{2}(\lambda_\alpha+\frac{1}{2}))} = c_0 \prod_{\alpha\in\Sigma^+} \frac{2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\Gamma(\lambda_\alpha)}{\Gamma(\lambda_\alpha+\frac{1}{2})},\\
c^{\pi_2}(\lambda) &= c_0 \prod_{\alpha\in\Sigma^+_{\mathrm{long}}} \frac{2^{1-\lambda_\alpha}\Gamma(\lambda_\alpha)}{\Gamma(\frac{1}{2}(\lambda_\alpha+\frac{3}{2}))\Gamma(\frac{1}{2}(\lambda_\alpha+\frac{1}{2}))} \prod_{\beta\in\Sigma^+_{\mathrm{short}}} \frac{2^{1-\lambda_\beta}\Gamma(\lambda_\beta)}{\Gamma(\frac{1}{2}(\lambda_\beta+\frac{5}{2}))\Gamma(\frac{1}{2}(\lambda_\beta-\frac{1}{2}))}\\
&= c_0 \prod_{\alpha\in\Sigma^+_{\mathrm{long}}} \frac{2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\Gamma(\lambda_\alpha)}{\Gamma(\lambda_\alpha+\frac{1}{2})} \prod_{\beta\in\Sigma^+_{\mathrm{short}}} \frac{2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\left(\lambda_\beta-\frac{1}{2}\right)\Gamma(\lambda_\beta)}{\Gamma(\lambda_\beta+\frac{3}{2})}.
\end{aligned}
$$

The value of the constant c_0 is determined by $c^{\mathrm{triv}}(\rho) = 1$. We have $c_0 = 2\pi^2$ by direct computation. Thus we have the following theorem.

THEOREM 3.3.

$$
\begin{aligned}
c^{\mathrm{triv}}(\lambda) &= \frac{2}{\pi} \prod_{\alpha\in\Sigma^+} \frac{\Gamma(\frac{1}{2}\lambda_\alpha)}{\Gamma(\frac{1}{2}\lambda_\alpha+\frac{1}{2})},\\
c^{\pi_1}(\lambda) &= \frac{16}{\pi} \prod_{\alpha\in\Sigma^+} \frac{\Gamma(\lambda_\alpha)}{\Gamma(\lambda_\alpha+\frac{1}{2})},\\
c^{\pi_2}(\lambda) &= \frac{16}{\pi} \prod_{\alpha\in\Sigma^+_{\mathrm{long}}} \frac{\Gamma(\lambda_\alpha)}{\Gamma(\lambda_\alpha+\frac{1}{2})} \prod_{\beta\in\Sigma^+_{\mathrm{short}}} \frac{\left(\lambda_\beta-\frac{1}{2}\right)\Gamma(\lambda_\beta)}{\Gamma(\lambda_\beta+\frac{3}{2})}.
\end{aligned}
$$

The formula for $c^{\mathrm{triv}}(\lambda)$ in Theorem 3.3 is a special case of the Gindikin-Karpelevič formula (cf. [14], [17, Ch. IV, Theorem 6.13]). The formula for $c^{\pi_1}(\lambda)$ is given in [21] by use of a different method. The formula for $c^{\pi_2}(\lambda)$ is new.

3.3. π-spherical transform

An inversion formula for the π-spherical transform is given by Theorem 2.2. We must shift the contour of integration from $\eta_1 + \sqrt{-1}\mathfrak{a}^*$ to $\sqrt{-1}\mathfrak{a}^*$ and get a globally defined function on $\mathfrak{a}$. We refer to [1, Ch. II, Ch. III] for the general residue scheme (see also [24, 3, 4, 5, 6, 7]).

For $\pi = \mathrm{triv}$ and π_1, $c^\pi(-\lambda)^{-1}$ is a regular function of λ for $\mathrm{Re}\,\lambda \in -\overline{\mathfrak{a}_+^*}$, hence the inversion formula is given by (11) for these small K-types (cf. [17, Ch IV, Theorem 7.5], [21, Corollary 7.6]).

For $\pi = \pi_2$, there appear singularities during the contour shift and we must take account of residues. The function $c^{\pi_2}(-\lambda)^{-1}$ for $\mathrm{Re}\,\lambda \in \mathfrak{a}_+^*$ has singularities along lines $\lambda_\beta = -\frac{1}{2}$ ($\beta \in \Sigma_{\mathrm{short}}^+$). Figure 1 illustrates singular lines $\lambda_{\alpha_1} = -\frac{1}{2}$, $\lambda_{\alpha_1+\alpha_2} = -\frac{1}{2}$, and $\lambda_{2\alpha_1+\alpha_2} = -\frac{1}{2}$ (dashed) for $c^{\pi_2}(-\lambda)^{-1}$ and $-\overline{\mathfrak{a}_+^*}$ (shaded region). Here α_1 and α_2 are the simple roots of Σ^+ ($||\alpha_1|| < ||\alpha_2||$). These singular lines divide $-\overline{\mathfrak{a}_+^*}$ into the following four regions (indicated in Figure 1):

$$
\begin{aligned}
&I\,:\,\lambda_{\alpha_1} < -\frac{1}{2}, \quad \lambda_{\alpha_2} \le 0,\\
&II\,:\,-\frac{1}{2} < \lambda_{\alpha_1} \le 0, \quad \lambda_{\alpha_1+\alpha_2} < -\frac{1}{2},\\
&III\,:\,-\frac{1}{2} < \lambda_{\alpha_1+\alpha_2}, \quad \lambda_{2\alpha_1+\alpha_2} < -\frac{1}{2}, \quad \lambda_{\alpha_2} \le 0,\\
&IV\,:\,\lambda_{2\alpha_1+\alpha_2} > -\frac{1}{2}, \quad \lambda_{\alpha_1} \le 0, \quad \lambda_{\alpha_2} \le 0.
\end{aligned}
$$

First $\eta_1 \in -\mathfrak{a}_+^*$ in (10) lies in the region *I*. We choose η_2, η_3, and η_4 in the regions *II, III*, and *IV*, respectively. We may take $\eta_4 = 0$. We shift the contour of integration from $\eta_1 + \sqrt{-1}\mathfrak{a}^*$ to $\eta_2 + \sqrt{-1}\mathfrak{a}^*$ and so on, and finally to $\eta_4 + \sqrt{-1}\mathfrak{a}^* = \sqrt{-1}\mathfrak{a}^*$, picking up residues. Define

$$F_i^\vee(H) = \int_{\eta_i+\sqrt{-1}\mathfrak{a}^*} f^\wedge(\lambda)\Phi_\lambda(H)c^{\pi_2}(-\lambda)^{-1}d\lambda \quad (1 \le i \le 4).$$

We regard $(\lambda_{\alpha_1}, \lambda_{3\alpha_1+2\alpha_2}) \in \mathbb{C}^2$ as a coordinate on $\mathfrak{a}_\mathbb{C}^*$. Define

$$c_1 = \frac{||\alpha_1||\,||3\alpha_1 + 2\alpha_2||}{4}. \tag{13}$$

For $\eta \in \mathfrak{a}^*$, the normalized measure $d\lambda$ on $\eta + \sqrt{-1}\mathfrak{a}^*$ is given by

$$d\lambda = \frac{c_1}{(2\pi\sqrt{-1})^2} d\lambda_{\alpha_1} d\lambda_{3\alpha_1+2\alpha_2}.$$

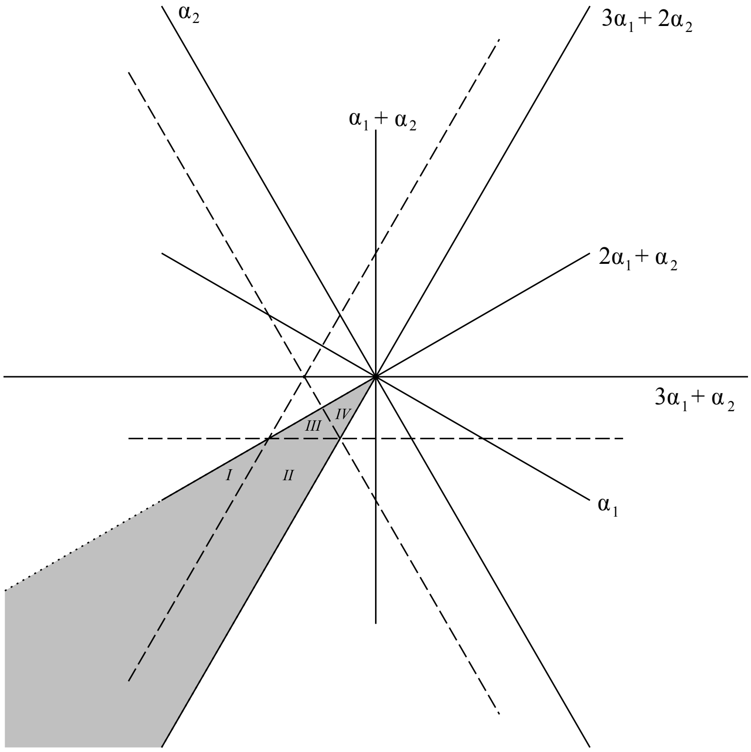

Figure 1. singular lines

First we change the contour of integration of $F^{\vee}(H) = F_1^{\vee}(H)$ from $\eta_1 + \sqrt{-1}\mathfrak{a}^*$ to $\eta_2 + \sqrt{-1}\mathfrak{a}^*$ with respect to the integration in the variable λ_{α_1}. By the explicit formula of $c^{\pi_2}(\lambda)$ in Theorem 3.3, $f^{\wedge}(\lambda)\Phi_\lambda(H)c^{\pi_2}(-\lambda)^{-1}$ has a possible simple pole during the change of the contour coming from the factor $(-\lambda_{\alpha_1} - \frac{1}{2})^{-1}$ of $c^{\pi_2}_{\alpha_1}(-\lambda)^{-1}$. Thus the difference $F_1^{\vee}(H) - F_2^{\vee}(H)$ is

$$-\frac{c_1}{2\pi\sqrt{-1}} \int_{\mu+\sqrt{-1}\mathbb{R}} \operatorname*{Res}_{\lambda_{\alpha_1}=-\frac{1}{2}} \bigl(f^{\wedge}(\lambda)\Phi_\lambda(H)c^{\pi_2}(-\lambda)^{-1}\bigr)d\lambda_{3\alpha_1+2\alpha_2}$$

for some μ with $\mu_{\alpha_1} = -\frac{1}{2}$, $\mu_{\alpha_2} < 0$. Next we move $\mu_{3\alpha_1+2\alpha_2}$ to 0 along the line $\lambda_{\alpha_1} = -\frac{1}{2}$. Singularities coming from $(-\lambda_\beta - \frac{1}{2})^{-1}$ ($\beta = \alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$) are on the walls and they are canceled by $\Gamma(-\lambda_\alpha)^{-1}$ ($\alpha = \alpha_2$, $\alpha_1 + \alpha_2$, respectively). Thus the integrand $\operatorname{Res}_{\lambda_{\alpha_1}=-\frac{1}{2}}\bigl(f^{\wedge}(\lambda)\Phi_\lambda(H)c^{\pi_2}(-\lambda)^{-1}\bigr)$ is regular for $\lambda_{3\alpha_1+2\alpha_2} \leq$

0. Hence we have

$$F_1^\vee(H) - F_2^\vee(H) = -\frac{c_1}{2\pi\sqrt{-1}} \int_{\sqrt{-1}\mathbb{R}} \operatorname*{Res}_{\lambda_{\alpha_1}=-\frac{1}{2}} \bigl(f^\wedge(\lambda)\Phi_\lambda(H)c^{\pi_2}(-\lambda)^{-1}\bigr) d\lambda_{3\alpha_1+2\alpha_2}.$$

Similarly, we have

$$F_2^\vee(H) - F_3^\vee(H) = -\frac{c_1}{2\pi\sqrt{-1}} \int_{\sqrt{-1}\mathbb{R}} \operatorname*{Res}_{\lambda_{\alpha_1+\alpha_2}=-\frac{1}{2}} \bigl(f^\wedge(\lambda)\Phi_\lambda(H)c^{\pi_2}(-\lambda)^{-1}\bigr) d\lambda_{3\alpha_1+\alpha_2},$$

$$F_3^\vee(H) - F_4^\vee(H) = -\frac{c_1}{2\pi\sqrt{-1}} \int_{\sqrt{-1}\mathbb{R}} \operatorname*{Res}_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}} \bigl(f^\wedge(\lambda)\Phi_\lambda(H)c^{\pi_2}(-\lambda)^{-1}\bigr) d\lambda_{\alpha_2}.$$

By summing up and changing variables, we have

$$\begin{aligned} &F_1^\vee(H) - F_4^\vee(H) \\ &\quad = -\frac{c_1}{2\pi\sqrt{-1}} \sum_{w\in\{e,s_1,s_2s_1\}} \int_{\sqrt{-1}\mathbb{R}} \operatorname*{Res}_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}} \bigl(f^\wedge(\lambda)\Phi_{w\lambda}(H)c^{\pi_2}(-w\lambda)^{-1}\bigr) d\lambda_{\alpha_2}. \end{aligned}$$

Let $W^{2\alpha_1+\alpha_2} = \{e, s_1, s_2, s_1s_2, s_2s_1, s_2s_1s_2\}$. By changing variables, we have

(14)

$$\begin{aligned} &F_1^\vee(H) - F_4^\vee(H) \\ &\quad = -\frac{c_1}{4\pi\sqrt{-1}} \sum_{w\in W^{2\alpha_1+\alpha_2}} \int_{\sqrt{-1}\mathbb{R}} \operatorname*{Res}_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}} \bigl(f^\wedge(\lambda)\Phi_{w\lambda}(H)c^{\pi_2}(-w\lambda)^{-1}\bigr) d\lambda_{\alpha_2}. \end{aligned}$$

Since $W^{2\alpha_1+\alpha_2} = \{w \in W \mid w(2\alpha_1+\alpha_2) \in \Sigma^+\}$, $c^{\pi_2}(w\lambda)|_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}} = 0$ for any $w \in W \setminus W^{2\alpha_1+\alpha_2}$ by Theorem 3.3. Notice that the Harish-Chandra expansion (4) is valid for $\lambda_{2\alpha_1+\alpha_2} = -\frac{1}{2}$, $\lambda_{\alpha_2} \in \sqrt{-1}\mathbb{R}\setminus\{0\}$. Hence

$$\Upsilon^{\pi_2}(\phi_\lambda^{\pi_2})|_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}} = \sum_{w\in W^{2\alpha_1+\alpha_2}} c^{\pi_2}(w\lambda)\Phi_{w\lambda}|_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}} \tag{15}$$

for $\lambda_{\alpha_2} \in \sqrt{-1}\mathbb{R}\setminus\{0\}$.

We write the c-function $c^{\pi_2}(\lambda)$ in Theorem 3.3 as

$$c^{\pi_2}(\lambda) = \frac{16}{\pi} c_l(\lambda)c_s(\lambda)$$

with

$$c_l(\lambda) = \prod_{\alpha \in \Sigma^+_{\text{long}}} \frac{\Gamma(\lambda_\alpha)}{\Gamma(\lambda_\alpha + \frac{1}{2})}, \qquad c_s(\lambda) = \prod_{\beta \in \Sigma^+_{\text{short}}} \frac{\left(\lambda_\beta - \frac{1}{2}\right)\Gamma(\lambda_\beta)}{\Gamma(\lambda_\beta + \frac{3}{2})}.$$

Notice that the functions $(c_l(\lambda)c_l(-\lambda))^{-1}$ and $(c_s(\lambda)c_s(-\lambda))^{-1}$ are W-invariant.

LEMMA 3.4. *We have*

$$(c_l(\lambda)c_l(-\lambda))^{-1}|_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}} = \frac{(4\lambda^3_{\alpha_2} - \lambda_{\alpha_2})\sin \pi\lambda_{\alpha_2}}{16\cos \pi\lambda_{\alpha_2}} \tag{16}$$

and

$$c_s(w\lambda)^{-1}|_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}} \operatorname*{Res}_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}} (c_s(-w\lambda)^{-1}) = \frac{36\lambda^2_{\alpha_2} - 1}{32\pi} \tag{17}$$

for any $w \in W^{2\alpha_1+\alpha_2}$.

PROOF. We show only (17) because (16) can be deduced in a similar way. Since the left hand side of (17) is the residue of the $(c_s(\lambda)c_s(-\lambda))^{-1}$ as a function of $\lambda_{2\alpha_1+\alpha_2}$ at $\lambda_{2\alpha_1+\alpha_2} = -\frac{1}{2}$, it suffices to show (17) for $w = 1$. By elementary calculation we have

$$\frac{\left(z - \frac{1}{2}\right)\Gamma(z)}{\Gamma(z + \frac{3}{2})} \cdot \frac{\left(-z - \frac{1}{2}\right)\Gamma(-z)}{\Gamma(-z + \frac{3}{2})} = -\frac{\cos \pi z}{z \sin \pi z} \quad (z \in \mathbb{C}).$$

Using $\lambda_{\alpha_1} = \frac{1}{2}\lambda_{2\alpha_1+\alpha_2} - \frac{3}{2}\lambda_{\alpha_2}$ and $\lambda_{\alpha_1+\alpha_2} = \frac{1}{2}\lambda_{2\alpha_1+\alpha_2} + \frac{3}{2}\lambda_{\alpha_2}$ we calculate

$$\begin{aligned}
&\operatorname*{Res}_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}} ((c_s(\lambda)c_s(-\lambda))^{-1}) \\
&\quad = \left(-\frac{z\sin \pi z}{\cos \pi z}\right)\Bigg|_{z=-\frac{1}{4}-\frac{3}{2}\lambda_{\alpha_2}} \left(-\frac{z\sin \pi z}{\cos \pi z}\right)\Bigg|_{z=-\frac{1}{4}+\frac{3}{2}\lambda_{\alpha_2}} \operatorname*{Res}_{z=-\frac{1}{2}} \left(-\frac{z\sin \pi z}{\cos \pi z}\right) \\
&\quad = \frac{1 - 36\lambda^2_{\alpha_2}}{16} \cdot \frac{\sin \pi\left(-\frac{1}{4} - \frac{3}{2}\lambda_{\alpha_2}\right)\sin \pi\left(-\frac{1}{4} + \frac{3}{2}\lambda_{\alpha_2}\right)}{\cos \pi\left(-\frac{1}{4} - \frac{3}{2}\lambda_{\alpha_2}\right)\cos \pi\left(-\frac{1}{4} + \frac{3}{2}\lambda_{\alpha_2}\right)} \cdot \left(-\frac{1}{2\pi}\right).
\end{aligned}$$

In the final expression the second factor reduces to 1. ☐

Thus we have the following inversion formula for π_2-spherical transform.

THEOREM 3.5. *For $f \in C_c^\infty(\mathfrak{a})^W$, we have*

$$f(H) = \frac{1}{12}\int_{\sqrt{-1}\mathfrak{a}^*} f^\wedge(\lambda)\Upsilon^{\pi_2}(\phi_\lambda^{\pi_2})(H)|c^{\pi_2}(\lambda)|^{-2}d\lambda$$
$$- \frac{c_1}{4\pi\sqrt{-1}}\int_{\sqrt{-1}\mathbb{R}} (f^\wedge(\lambda)\Upsilon^{\pi_2}(\phi_\lambda^{\pi_2}(H)))|_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}}\, p(\lambda_{\alpha_2})d\lambda_{\alpha_2}$$

for $H \in \mathfrak{a}$, where

$$p(\lambda_{\alpha_2}) = \operatorname*{Res}_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}}(c^{\pi_2}(\lambda)^{-1}c^{\pi_2}(-\lambda)^{-1}) = \frac{\pi(36\lambda_{\alpha_2}^2-1)(4\lambda_{\alpha_2}^3-\lambda_{\alpha_2})\sin\pi\lambda_{\alpha_2}}{2^{17}\cos\pi\lambda_{\alpha_2}}$$
$$= -2^{-17}\pi\left(36\left(\tfrac{\lambda_{\alpha_2}}{\sqrt{-1}}\right)^2+1\right)\left(4\left(\tfrac{\lambda_{\alpha_2}}{\sqrt{-1}}\right)^2+1\right)\left(\tfrac{\lambda_{\alpha_2}}{\sqrt{-1}}\right)\tanh\pi\left(\tfrac{\lambda_{\alpha_2}}{\sqrt{-1}}\right).$$

The Plancherel formula follows from Theorem 3.5 by a standard argument as in the case of π = triv (cf. the proof of [13, Theorem 6.4.2] and [17, Ch IV Theorem 7.5]).

COROLLARY 3.6. *For $f \in C_c^\infty(\mathfrak{a})^W$, we have*

$$\frac{1}{12}\int_{\mathfrak{a}}|f(H)|^2\delta_{G/K}(H)dH = \frac{1}{12}\int_{\sqrt{-1}\mathfrak{a}^*}|f^\wedge(\lambda)|^2|c^{\pi_2}(\lambda)|^{-2}d\lambda$$
$$-\frac{c_1}{4\pi\sqrt{-1}}\int_{\sqrt{-1}\mathbb{R}}|f^\wedge(\lambda)|_{\lambda_{2\alpha_1+\alpha_2}=-\frac{1}{2}}|^2\, p(\lambda_{\alpha_2})d\lambda_{\alpha_2}$$

As we see in § 3.1, no discrete spectrum appears in the inversion formula and the Plancherel formula. In addition to the most continuous spectrum, there is a contribution of a principal series representation associated with a maximal parabolic subgroup whose Levi part corresponds to a short restricted root.

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