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Confluence and versal unfolding of Pfaffian systems

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Abstract. A versal unfolding of a Pfaffian system with irregular singularities on the Riemann sphere is studied through its middle convolution. If the system is rigid, it is realized as a confluent limit of a rigid Fuchsian system. We show that the versal unfolding of a rigid Pfaffian system is extended to a versal KZ equation regarding singular points as variables. Appell's hypergeometric equations and their confluences are the simple examples.

1. Introduction

In [16] we introduce a versal unfolding of a single linear ordinary differential equation with unramified irregular singularities in $\mathbb{P}^1$, which is an embedding of the equation into a Fuchsian equation containing singular points as holomorphic parameters. We require that the *index of rigidity* is stable under the unfolding. For example, a versal unfolding of Hermite equation is an embedding into an equation with singular points $\frac{1}{t_1}$ and $\frac{1}{t_2}$ isomorphic to Gauss hypergeometric equation :

$$(1 - t_1x)(1 - t_2x)\tilde{u}'' + (\bar{\lambda}_1 + \bar{\lambda}_2x)\tilde{u}' + \bar{\mu}\bigl(\bar{\lambda}_2 - t_1t_2(\bar{\mu} + 1)\bigr)\tilde{u} = 0. \tag{1.1}$$

We note that the Riemann scheme of the unfolding is defined from that of the original equation and hence we know the procedure to get a Hermite equation from Gauss hypergeometric equation as a confluence limit. This enables us to study the original equation through the Fuchsian equation which we expect easier to be analyzed.

The existence of the unfolding of any irreducible equation with unramified irregular singularities is conjectured in [16]. We study there the middle convolution and versal addition acting on the versal unfolding and prove that the conjecture is stable under these operations of the original equation, which implies that the conjecture is true, for example, when the equation is a rigid single equation.

In this note we study versal unfolding of Pfaffian systems. A Pfaffian system

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on the Riemann sphere $\mathbb{P}^1$ is a differential equation

$$\frac{du}{dx} = \Bigl(\sum_{j=1}^{p}\sum_{i=1}^{r_j}\frac{A_{j,i}}{(x-a_j)^i} - \sum_{i=2}^{r_0}A_{0,i}x^{i-2}\Bigr)u. \tag{1.2}$$

Here $A_{j,i}$ are constant square matrices of size N and u is a vector of N unknown functions. The integer N is called the rank of the Pfaffian system. The points $a_1,\ldots,a_p$ and ∞ are singular points of the equation. If $r_0 = r_1 = \cdots = r_p = 1$, the differential equation (1.2) is called a *Fuchsian* system. Let $M_N(\mathbb{C})$ denote the space of square matrices of size N with entries in $\mathbb{C}$.

The versal unfolding of the equation (1.2) is

$$\begin{aligned}\frac{d\tilde u}{dx} = \Bigl(&\sum_{j=1}^{p}\sum_{i=1}^{r_j}\frac{\tilde A_{j,i}}{(x-a_{j,1})(x-a_{j,2})\cdots(x-a_{j,i})}\\ &-\sum_{i=2}^{r_0}\frac{\tilde A_{0,i}x^{i-2}}{(1-a_{0,2}x)(1-a_{0,2}x)\cdots(1-a_{0,i}x)}\Bigr)\tilde u.\end{aligned} \tag{1.3}$$

Here $\tilde A_{j,i} \in M_N(\mathbb{C})$ holomorphically depend on parameters $\mathbf{a} = (a_{0,2},\ldots,a_{0,r_0}, a_{1,1},\ldots,a_{1,r_1},\ldots,a_{p,1},\ldots,a_{p,r_p})$ which coincides with $A_{j,i}$ when $\mathbf{a} = (0,\ldots,0,a_1,\ldots,a_1,\ldots,a_p,\ldots,a_p)$. Moreover the index of rigidity (cf. §2) does not depend on the parameters.

If the points $a_{j,i}$ ($i = 1,\ldots,r_j$, $j = 1,\ldots,p$) and $\infty = a_{0,1}^{-1}$ and $a_{0,i}^{-1}$ ($i = 2,\ldots,r_0$) in $\mathbb{P}^1$ are mutually different, the equation (1.3) is a Fuchsian system

$$\frac{d\tilde u}{dx} = \Bigl(\sum_{j=1}^{p}\sum_{i=1}^{r_j}\frac{C_{j,i}}{x-a_{j,i}} + \sum_{i=2}^{r_0}\frac{C_{0,i}}{x-a_{0,i}^{-1}}\Bigr)\tilde u \tag{1.4}$$

with suitable matrices $C_{j,i} \in M_N(\mathbb{C})$.

As is given in §5 the versal Gauss Pfaffian system is

$$\frac{d\tilde u}{dx} = -\left(\frac{\begin{pmatrix}\lambda_1+t_1\mu & \lambda_2\\ \mu & t_2\mu\end{pmatrix}}{(1-t_1x)} + \frac{\begin{pmatrix}0 & 0\\ \lambda_1+t_2\mu & \lambda_2+t_2(t_2-t_1)\mu\end{pmatrix}x}{(1-t_1x)(1-t_2x)}\right)\tilde u. \tag{1.5}$$

Katz [8] introduces the operation called *middle convolution* for the study of rigid local systems and the middle convolution of the Fuchsian system

$$\frac{du}{dx} = \sum_{j=1}^{p}\frac{A_j}{x-a_j}u \tag{1.6}$$

is interpreted by [2] as an operation on matrices A_j. In §4–5 we define a middle convolution of the versal unfolding (1.3) of the equation (1.2) through the analytic continuation of that of the Fuchsian system (1.4). This assures the existence of versal unfolding in many cases. In particular the conjecture is true for a rigid Pfaffian system. Note that the confluent limit of the middle convolution defines that of the Pfaffian system (1.2) which has been introduced by several papers (cf. [9, 19, 21]).

Suppose the Fuchsian equation (1.6) is rigid. Then we regard the singular points a_j as variables and it is shown by [3] that the equation is extended to KZ (Knizhnik-Zamolodchikov) equation

$$\frac{\partial u}{\partial x_i} = \sum_{\substack{\nu \neq i \\ 0 \le \nu \le p}} \frac{A_{i,\nu}}{x_i - x_\nu} u \qquad (i = 0, \dots, p) \tag{1.7}$$

with $A_{0,j} = A_j$, $x_0 = x$ and $x_j = a_j$ $(j = 1, \dots, p)$. Here

$$A_{i,j} = A_{j,i},\ [A_{i,j}, A_{k,\ell}] = [A_{i,j}, A_{i,k} + A_{j,k}] = 0 \tag{1.8}$$

for mutually different indices i, j, k and ℓ. The spectral type of $A_{0,j}$ is given by [2] and that of $A_{i,j}$ with $i > 0$ is clarified by [15], where the relation (1.8) is essential. If the equation (1.2) is rigid, the versal unfolding (1.3) is also rigid and therefore (1.4) has an extension to a KZ equation. Then by an analytic continuation we have in §7 a versal confluence of the extended KZ equation where the extension of the original equation (1.2) to the equation with several variables is realized by a specialization of parameters. Hence the rigid equation (1.2) is extended to a Pfaffian system with the $p+1$ variables $(x_0, x_1, \dots, x_p) = (x, a_1, \dots, a_p)$.

2. Notation and preliminary results

For each singular point of the Pfaffian system (1.2) there exist N independent formal solutions. By a linear fractional transformation, we may assume the origin is a singular point. If the origin is a ramified irregular singular point, it becomes unramified under a new coordinate $y = x^{\frac{1}{p}}$ with a suitable positive integer p. We assume that the origin is a regular singular point or an unramified irregular singular point. Then the studies by Hukuhara, Turrittin and Levelt assure that the equation has formal solutions

$$u_\nu(x) = \Bigl(C_\nu x^{\lambda_{\nu,0}} \log^{k_\nu} x + \sum_{i=1}^{\infty} \sum_{j=0}^{N-1} C_{\nu,i,j} x^{\lambda_{\nu,0}+i} \log^j x\Bigr) \exp\Bigl(-\sum_{i=1}^{r-1} \frac{\lambda_{\nu,i}}{i! x^i}\Bigr) \tag{2.1}$$

so that the main terms

$$(C_\nu x^{\lambda_{\nu,0}} \log^{k_\nu} x) \exp\Bigl(-\sum_{i=1}^{r-1} \frac{\lambda_{\nu,i}}{i!x^i}\Bigr) \qquad (\nu = 1,\dots,N) \tag{2.2}$$

of the solutions are linearly independent (cf. [20]). Here $u_\nu(x)$, C_ν and $C_{\nu,i,j}$ are column vectors of size N. Then $\bigl\{\sum_{i=0}^{r-1} \lambda_{\nu,i} x^i \mid \nu = 1,\dots,N\bigr\}$ is called the set of characteristic exponents of the equation (1.2) (cf. [16]). For simplicity we assume $k_\nu = 0$ for $\nu = 1,\dots,N$. Suppose the multiplicity of a characteristic exponent $\lambda_\nu(x)$ equals m_ν. The the set

$$\bigl\{[\lambda_\nu(x)]_{m_\nu} \mid \nu = 1,\dots,n\bigr\} \ \text{ with } \ [\lambda_\nu(x)]_{m_\nu} = \begin{pmatrix} \lambda_\nu(x) \\ \vdots \\ \lambda_\nu(x) \end{pmatrix} \in \mathbb{C}[x]^{m_\nu} \tag{2.3}$$

is called the set of generalized characteristic exponents at the singular point if $k_\nu = 0$ in the above and $N = m_1 + \cdots + m_n$. We define maps

$$\pi_k : \{1,\dots,N\} \to \{1,\dots,N\} \tag{2.4}$$

for $k = 1, 2, \dots, r$ such that

$$\begin{aligned} &\pi_1(\nu) = \nu \quad (\nu = 1,\dots,N), \\ &\pi_k(\nu) = \pi_k(\nu') \Leftrightarrow \deg(\lambda_\nu(x) - \lambda_{\nu'}(x)) < k - 1 \quad (k = 2,\dots,r), \\ &\pi_k(\{1,\dots,N\}) = \{1,\dots,\pi_k(N)\}. \end{aligned}$$

Put

$$n_k := \pi_k(n), \quad m_i^{(k)} := \sum_{\pi_k(\nu)=i} m_\nu \quad (i = 1,\dots,n_k). \tag{2.5}$$

Then

$$\mathbf{m}^{(k)} : N = m_1^{(k)} + \cdots + m_{n_k}^{(k)} \quad (k = 1,\dots,r) \tag{2.6}$$

is a partition of N. Note that $n_1 = n$ and $m_i^{(1)} = m_i$. We may assume

$$1 \le \nu \le \nu' \le N \ \Rightarrow \ \pi_k(\nu) \le \pi_k(\nu') \tag{2.7}$$

by a permutation of the indices ν of $\lambda_\nu(x)$. Then we note that for any $k > 1$ and $\nu = 1, \dots, r_k$, there exists an integer ν' such that

$$m_1^{(k)} + \cdots + m_\nu^{(k)} = m_1^{(k-1)} + \cdots + m_{\nu'}^{(k-1)}$$

and $\mathbf{m}^{(k-1)}$ is a refinement of $\mathbf{m}^{(k)}$. The tuple of partitions $\{\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(r)}\}$ is called a spectral type of the equation at the singular point and represented by the series of figures separated by $|$ as $m_1^{(1)} \cdots m_{n_1}^{(1)} | m_1^{(2)} \cdots m_{n_2}^{(2)} | \cdots | m_1^{(r)} \cdots m_{n_r}^{(r)}$.

In this paper we assume the equation (1.2) has no unramified irregular singular points. Let $\{[\lambda_{j,\nu}]_{m_{j,\nu}} \mid \nu = 1, \dots, n_j\}$ be the set of generalized characteristic exponents at the singular points a_j for $j = 1, \dots, p$ and ∞ for $j = 0$ and the corresponding spectral types

$$\mathbf{m}_j = \{\mathbf{m}_j^{(1)}, \dots, \mathbf{m}_j^{(r_j)}\} \text{ with } \mathbf{m}_j^{(k)} : N = m_{j,1}^{(k)} + \cdots + m_{j,n_{j,k}}^{(k)} \quad (k = 1, \dots, r_j)$$

are the sets of partitions of N. Here we define the characteristic exponents at a singular point by applying a suitable coordinate transformation $x \mapsto x - a$ or $x \mapsto \frac{1}{x}$ so that the singular point corresponds to the origin of the new coordinate. Then the generalized Riemann scheme (GRS) is the table

$$\left\{\begin{matrix} x = \infty & x = a_1 & \cdots & x = a_p \\ [\lambda_{0,1}(x)]_{m_{0,1}} & [\lambda_{1,1}(x)]_{m_{1,1}} & \cdots & [\lambda_{p,1}(x)]_{m_{p,1}} \\ \vdots & \vdots & \cdots & \vdots \\ [\lambda_{0,n_0}(x)]_{m_{0,n_0}} & [\lambda_{1,n_1}(x)]_{m_{1,n_1}} & \cdots & [\lambda_{p,n_p}(x)]_{m_{p,n_p}} \end{matrix}\right\}. \tag{2.8}$$

In this paper we always assume the Fuchs relation

$$\sum_{j=0}^{p} \sum_{\nu=1}^{n_j} m_{j,\nu} \lambda_{j,\nu}(0) = 0, \tag{2.9}$$

which is satisfied if the system is Fuchsian or it is transformed into Hukuhara normal form at each singular point by the Gauge transformation with $GL(N, \mathbb{C}[\![x]\!])$. Here $\mathbb{C}[\![x]\!]$ is the ring of formal power series at the singular point. (Cf. [16] for single differential equations).

The set of spectral type of the equation is the set $\{\mathbf{m}_0, \dots, \mathbf{m}_p\}$ and is expressed by arranging the expression of $\mathbf{m}_j$ separated by ",". This notation is frequently used in [13] for a Fuchsian equation and given by [16] in general (cf. [16] for examples). The *unfolded Fuchsian spectral type* of $\mathbf{m}$ is the spectral type $\{\mathbf{m}_j^{(k)}\}_{1 \le k \le r_j,\ j = 0, \dots, p}$ of a Fuchsian differential equation with $(r_0 + \cdots + r_p)$ regular singular points, whose expression is obtained by replacing "$|$" by "," in the expression of $\mathbf{m}$. The spectral types of Gauss hypergeometric equation and its confluence called Kummer's equation are expressed by

$$11,11,11 \quad \text{and} \quad 11|11,11.$$

A given spectral type $\mathbf{m}$ is called *irreducibly realizable* if there exists an irreducible Pfaffian system (1.2) which has the GRS (2.8) for generic $\lambda_{j,\nu}$ satisfying (2.9). The condition so that $\mathbf{m}$ is irreducibly realizable is solved by [1] when (1.2) is Fuchsian, namely, $\lambda_{j,\nu}$ are constant, and by [4] in general allowing unramified irregular singularities. Then the dimension of parameters with a fixed GRS, which is called accessory parameters, equals $2 - \operatorname{idx}\mathbf{m}$. Here $\operatorname{idx}\mathbf{m}$, the index of rigidity, defined by [8] is

$$\begin{aligned}\operatorname{idx}\mathbf{m} &= 2N^2 - \sum_{j=0}^{p}\sum_{k=1}^{r_j}\Big(N^2 - \sum_{\nu=1}^{n_{j,k}}\big(m_{j,\nu}^{(k)}\big)^2\Big)\\ &= 2N^2 - \sum_{j=0}^{p}\Big(N^2 - \sum_{\nu=1}^{n_j} m_{j,\nu}^2\Big) - \sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\sum_{\nu'=1}^{n_j}\deg\big(\lambda_{j,\nu}-\lambda_{j,\nu'}\big),\end{aligned} \tag{2.10}$$

where we put $\deg 0 = 0$ (cf. [16, Lemma 3.5 i)]).

Since we examine matrices and functions with meromorphic parameters, we recall a certain result given by [12].

Let M be a C^∞-manifold and let U be a connected convex open subset of $\mathbb{C}^\ell$. We denote by $\mathcal{D}'(M)$ the space of Schwartz distributions on M. Here M may be a finite set. Suppose that meromorphic functions $f_1(\lambda),\dots,f_m(\lambda)$ of U valued in $\mathcal{D}'(M)$ are given. Moreover suppose there exists a non-zero holomorphic function $\phi(\lambda)$ on U such that $f_1,\dots,f_m$ are holomorphic on $U_\phi = \{\lambda\in U;\ \phi(\lambda)\neq 0\}$ and $\dim V_\lambda = m$ for any $\lambda\in U_\phi$ by putting $V_\lambda = \sum_{j=1}^m \mathbb{C}f_j(\lambda)$. For $\mu\in U$ we put

$$\begin{aligned}\bar V_\mu = \{f(0);\ &f \text{ is a holomorphic function on } \{t\in\mathbb{C};\ |t|<1\} \text{ valued in } \mathcal{D}'(M)\\ &\text{and there exists a holomorphic curve } c:\{t\in\mathbb{C};\ |t|<1\}\to U \text{ such that}\\ &c(t)\in U_\phi \text{ and } f(t)\in V_{c(t)} \text{ for } 0<|t|\ll 1 \text{ and } c(0)=\mu\}.\end{aligned}$$

DEFINITION 2.1. We call $\bar V_\mu$ the *closure* of the holomorphic family of the spaces V_λ $(\lambda\in U_\phi)$ at μ and $\bar V_\mu$ $(\mu\in U)$ the *completion* of the holomorphic family. It follows from [18, Proposition 2.21] that $\dim \bar V_\mu \geq m$. We define that a point $\mu\in U\setminus U_\phi$ is a *removable* (resp. *unremovable*) *singular point* if $\dim V_\mu = m$ (resp. $\dim V_\mu > m$).

THEOREM 2.2 ([12, Lemma 6.3]). *Retain the notation above. If μ is a removable singular point of the spaces V_λ, then there exist a neighborhood U_μ of μ and holomorphic functions $h_1(\lambda), \dots, h_m(\lambda)$ on U_μ valued in $\mathcal{D}'(M)$ such that they are linearly independent for any $\lambda \in U_\mu$. They span $\bar{V}_\lambda$ for any $\lambda \in U_\mu$.*

Conversely the existence of $\{h_1(\lambda), \dots, h_m(\lambda)\}$ with this property implies that μ is a removable singular point.

If there is no unremovable singular point in U, we may choose $U_\mu = U$ and we call the set of functions $\{h_1(\lambda), \dots, h_m(\lambda)\}$ a complete base of the holomorphic family. *If there is another complete base $\{\tilde{h}_1(\lambda), \dots, \tilde{h}_m(\lambda)\}$, then there exist holomorphic functions $c_{i,j}(\lambda)$ on U such that $\tilde{h}_i(\lambda) = \sum_{j=1}^m c_{i,j}(\lambda) h_j(\lambda)$ for $i = 1, \dots, m$. We note that there is no unremovable singular point when* $\dim U = 1$.

EXAMPLE 2.3. The set $\left\{f_i(x, \mathbf{a}) = \frac{1}{x - a_i} \mid i = 1, \dots, m\right\}$ is a holomorphic family of the functions $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{C}^m$ valued in rational functions of x, which has no unremovable singular point. The set $\Big\{h_i(x, \mathbf{a}) = \frac{1}{(x-a_1)(x-a_2)\cdots(x-a_i)} \mid i = 1, \dots, m\Big\}$ is a complete base of this family, which is essential in this note. Note that $\left\{\frac{x^i}{(x-a_1)\cdots(x-a_m)} \mid i = 0, \dots, m-1\right\}$ is also a complete base of this family.

Another important example without an unremovable singular point is $\left\{e^{\lambda_{\sigma(1)}x_1 + \cdots + \lambda_{\sigma(m)}x_m} \mid \sigma \in \mathfrak{S}_m\right\}$, which is examined in [11, §1]. Here $\mathfrak{S}_m$ is the permutation group of the set of indices $\{1, \dots, m\}$ and $(\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$.

3. An example of middle convolution

Consider a Pfaffian system

$$\frac{du}{dx} = \frac{A_1}{x}u + \frac{A_2}{x(x-c)}u \tag{3.1}$$

with A_1, $A_2 \in M_N(\mathbb{C})$. We will explain the middle convolution of this system which we expect to depend holomorphically on the parameter c in a neighborhood of 0. Since $\frac{1}{x(x-c)} = \frac{1}{c}\left(\frac{1}{x-c} - \frac{1}{x}\right)$, we have

$$\frac{du}{dx} = \frac{C_1}{x}u + \frac{C_2}{x-c}u \quad \text{with} \quad C_1 = A_1 - \tfrac{1}{c}A_2 \quad \text{and} \quad C_2 = \tfrac{1}{c}A_2, \tag{3.2}$$

namely

$$(3.3)\quad \begin{aligned} &\begin{pmatrix} \frac{1}{x} \\ \frac{1}{x(x-c)} \end{pmatrix} = T \begin{pmatrix} \frac{1}{x} \\ \frac{1}{x-c} \end{pmatrix} \quad \text{with} \quad T = \begin{pmatrix} 1 & 0 \\ -\frac{1}{c} & \frac{1}{c} \end{pmatrix} \quad \text{and} \quad S = T^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & c \end{pmatrix}, \\ &u' = (A_1, A_2) \begin{pmatrix} \frac{1}{x} \\ \frac{1}{x(x-c)} \end{pmatrix} u = (A_1, A_2) T \begin{pmatrix} \frac{1}{x} \\ \frac{1}{x-c} \end{pmatrix} u. \end{aligned}$$

Here and hereafter matrices are block matrices with elements in $M_N(\mathbb{C})$ if otherwise stated.

Apply the Riemann-Liouville integral to the solution $u(x)$ of (3.1):

$$\tilde{u}(x) := \begin{pmatrix} I_{\mu+1} \frac{u}{x} \\ I_{\mu+1} \frac{u}{x-c} \end{pmatrix} \quad \text{with} \quad (I_\mu v)(x) := \frac{1}{\Gamma(\mu)} \int_a^x v(t)(x-t)^{\mu-1} dx.$$

Here $\mu \in \mathbb{C}$ and a is a singular point of the system, namely, $a = 0$, c or ∞. Then $(I_{\mu+1} v)' = I_{\mu+1}(v')$ and $I_{\mu+1}(xv') = xI_{\mu+1}(v') - (\mu+1) I_{\mu+1}(v)$. Since

$$\begin{pmatrix} x(\frac{u}{x})' \\ (x-c)(\frac{u}{x-c})' \end{pmatrix} = \begin{pmatrix} (\frac{C_1 - 1}{x} + \frac{C_2}{x-c}) u \\ (\frac{C_1}{x} + \frac{C_2 - 1}{x-c}) u \end{pmatrix} = \begin{pmatrix} C_1 - 1 & C_2 \\ C_1 & C_2 - 1 \end{pmatrix} \begin{pmatrix} \frac{u}{x} \\ \frac{u}{x-c} \end{pmatrix},$$

$\tilde{u}(x)$ is a solution of the system

$$\frac{d\tilde{u}}{dx} = \frac{\begin{pmatrix} C_1 + \mu & C_2 \\ 0 & 0 \end{pmatrix}}{x} \tilde{u} + \frac{\begin{pmatrix} 0 & 0 \\ C_1 & C_2 + \mu \end{pmatrix}}{x-c} \tilde{u} = \left(\begin{pmatrix} \frac{1}{x} \\ \frac{1}{x-c} \end{pmatrix} (C_1, C_2) + \begin{pmatrix} \frac{\mu}{x} & \\ & \frac{\mu}{x-c} \end{pmatrix} \right) \tilde{u}.$$

This system is the *convolution* of (3.2) defined by [2].

Putting

$$(3.4)\quad \hat{u}(x) := \begin{pmatrix} I_{\mu+1} \frac{u}{x} \\ I_{\mu+1} \frac{u}{x(x-c)} \end{pmatrix},$$

we will examine the equation satisfied by $\hat{u}(x)$. Since $\hat{u}(x) = T\tilde{u}(x)$, we have

$$\begin{aligned} \hat{u}' &= T \left(\begin{pmatrix} \frac{1}{x} \\ \frac{1}{x-c} \end{pmatrix} (C_1, C_2) + \mu \begin{pmatrix} \frac{1}{x} & \\ & \frac{1}{x-c} \end{pmatrix} \right) S \hat{u} \\ &= \left(\begin{pmatrix} \frac{1}{x} \\ \frac{1}{x(x-c)} \end{pmatrix} (A_1, A_2) + \mu T \begin{pmatrix} \frac{1}{x} & \\ & \frac{1}{x} + \frac{c}{x(x-c)} \end{pmatrix} S \right) \hat{u} \end{aligned}$$

$$= \left(\begin{pmatrix} \frac{1}{x} \\ \frac{1}{x(x-c)} \end{pmatrix} (A_1, A_2) + \frac{\mu}{x} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \frac{c\mu}{x(x-c)} \begin{pmatrix} 1 & 0 \\ -\frac{1}{c} & \frac{1}{c} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & c \end{pmatrix} \right) \hat{u}$$

$$= \frac{\begin{pmatrix} A_1 + \mu & A_2 \\ 0 & \mu \end{pmatrix}}{x} \hat{u} + \frac{\begin{pmatrix} 0 & 0 \\ A_1 + \mu & A_2 + c\mu \end{pmatrix}}{x(x-c)} \hat{u}.$$

Hence we have the convolution

$$\frac{d\hat{u}}{dx} = \Bigl(\frac{\hat{A}_1}{x} + \frac{\hat{A}_2}{x(x-c)}\Bigr)\hat{u}, \quad \hat{A}_1 = \begin{pmatrix} A_1 + \mu & A_2 \\ 0 & \mu \end{pmatrix}, \ \hat{A}_2 = \begin{pmatrix} 0 & 0 \\ A_1 + \mu & A_2 + c\mu \end{pmatrix} \tag{3.5}$$

of the system (3.1) satisfied by (3.4).

Put $\mathcal{K} := \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \;\middle|\; \begin{pmatrix} A_1 & A_2 \\ A_2 & cA_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \right\}$. Then for $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{K}$, we have

$$A_1 v_1 + A_2 v_2 = A_2(v_1 + cv_2) = 0,$$

$$(\hat{A}_1 - \mu) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0,$$

$$\hat{A}_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A_1 + \mu & A_2 + c\mu \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mu \begin{pmatrix} 0 \\ v_1 + cv_2 \end{pmatrix} \in \mathcal{K}.$$

Moreover $\mathcal{L} := \left\{ \begin{pmatrix} v_0 \\ 0 \end{pmatrix} \;\middle|\; (A_1 + \mu)v_0 = 0 \right\}$ satisfies $\hat{A}_1 \mathcal{L} = \hat{A}_2 \mathcal{L} = \{0\}$. Then $\hat{A}_1$ and $\hat{A}_2$ induce linear transformations on the quotient space $\bar{V} = \mathbb{C}^{2N}/(\mathcal{K}+\mathcal{L})$. Let $\bar{A}_1$ and $\bar{A}_2$ be matrices corresponding to the induced linear transformations on $\bar{V}$ with respect a suitable base of $\bar{V}$. We define that the system

$$\bar{u}' = \Bigl(\frac{\bar{A}_1}{x} + \frac{\bar{A}_2}{x(x-c)}\Bigr)\bar{u} \tag{3.6}$$

is the *middle convolution* of (3.1). We are interested in the case when A_1 and A_2 and $\mathcal{K} + \mathcal{L}$ holomorphically depend on c and hence so are $\bar{A}_1$ and $\bar{A}_2$.

Remark 3.1. Suppose A_1 and A_2 holomorphically depend on c. Then there exist a non-negative integer k and a base $\{w_1, \ldots, w_k\}$ of $\mathcal{K} + \mathcal{L}$ for a generic c. We may assume w_j holomorphically depend on c and then Theorem 2.2 assures the existence of a base $\{\tilde{w}_1, \ldots, \tilde{w}_k\}$ of $\mathcal{K} + \mathcal{L}$ for a generic c such that $\tilde{w}_1, \ldots, \tilde{w}_k$ holomorphically depend on c and moreover they are linearly independent for any c. Then $A_\nu \tilde{w}_j = \sum_{i=1}^k a_{\nu,i,j} \tilde{w}_i$ with holomorphic functions $a_{\nu,i,j}$ of c for $\nu = 1, 2$

and therefore the space $\sum_{j=1}^{k}\mathbb{C}\tilde{w}_j$ is invariant by A_1 and A_2 for any c. Then we define $\hat{V}=\mathbb{C}^{2N}/\sum_{j=1}^{k}\mathbb{C}\tilde{w}_j$ and get the middle convolution with $(\bar{A}_1,\bar{A}_2)$ which holomorphically depends on c.

4. Middle convolution – a special case

First we examine the convolution of the system

$$\frac{du}{dx}=\sum_{i=1}^{r}\frac{A_i}{(x-a_1)\cdots(x-a_i)}u \tag{4.1}$$

of rank N. It is essentially same as in the preceding section but the calculation is a little more complicated. Put

$$p_i(x)=\frac{1}{x-a_i},\quad q_i(x)=p_1(x)\cdots p_i(x). \tag{4.2}$$

Note that $\{q_1(x),\dots,q_r(x)\}$ is a complete base of the holomorphic family $\{p_1(x),\dots,p_r(x)\}$ with respect to $(a_1,\dots,a_r)\in\mathbb{C}^r$ (cf. Example 2.3).

For a solution $u(x)$ of the system, the convolution of this system is the equation satisfied by the function

$$\hat{u}(x)=\begin{pmatrix}I_{\mu+1}(q_1u)\\ \vdots\\ I_{\mu+1}(q_ru)\end{pmatrix}\quad\text{with}\quad I_\mu(v)(x)=\frac{1}{\Gamma(\mu)}\int_c^x v(t)(x-t)^{\mu-1}dt.$$

Here $c=a_j$ or ∞ and μ is a complex parameter. Note that

$$\begin{aligned}\frac{1}{x-a_i}&=\sum_{j=1}^{i}\frac{\prod_{\nu=1}^{j-1}(a_i-a_\nu)}{(x-a_1)\cdots(x-a_j)},\\ \frac{1}{(x-a_1)\cdots(x-a_i)}&=\sum_{j=1}^{i}\frac{\prod_{\substack{1\le\nu\le i\\ \nu\ne j}}(a_j-a_\nu)^{-1}}{x-a_j}.\end{aligned} \tag{4.3}$$

Hence we have

(4.4)

$$\mathbf{p} = S_r(\mathbf{a})\mathbf{q},\quad S_r(\mathbf{a}) := (S_{i,j}(\mathbf{a}))_{\substack{1\le i\le r,\\ 1\le j\le r}},\quad S_{i,j}(\mathbf{a}) := \begin{cases} 0 & (i<j),\\ s_{i,\{1,\ldots,j-1\}}(\mathbf{a}) & (i\ge j),\end{cases}$$

$$\mathbf{q} = T_r(\mathbf{a})\mathbf{p},\quad T_r(\mathbf{a}) := (T_{i,j}(\mathbf{a}))_{\substack{1\le i\le r,\\ 1\le j\le r}},\quad T_{i,j}(\mathbf{a}) := \begin{cases} 0 & (i<j),\\ t_{j,\{1,\ldots,i\}\setminus\{j\}}(\mathbf{a}) & (i\ge j)\end{cases}$$

by denoting

$$\mathbf{p} = \begin{pmatrix} p_1\\ \vdots\\ p_r\end{pmatrix},\ \mathbf{q} = \begin{pmatrix} q_1\\ \vdots\\ q_r\end{pmatrix}, \tag{4.5}$$

$$s_{i,\{j_1,,\ldots,j_k\}}(\mathbf{a}) = \prod_{\nu=1}^{k}(a_i - a_{j_\nu})\ \text{ and }\ t_{i,\{j_1,\ldots,j_k\}}(\mathbf{a}) = s_{i,\{j_1,\ldots,j_k\}}(\mathbf{a})^{-1}. \tag{4.6}$$

Putting $\tilde{u}(x) = \begin{pmatrix} I_{\mu+1}(p_1u)\\ \vdots\\ I_{\mu+1}(p_ru)\end{pmatrix}$ and $(C_1,\ldots,C_r) = (A_1,\ldots,A_r)T_r(\mathbf{a})$, $u(x)$ and $\tilde{u}(x)$ are solutions of

$$\frac{du}{dx} = \sum_{i=1}^{r}\frac{C_i}{x-a_i}u$$

and

$$\frac{d\tilde{u}}{dx} = \bigl(\mathbf{p}(C_1,\ldots,C_r) + \mu\operatorname{Diag}(p_1,\ldots,p_r)\bigr)\tilde{u},$$

respectively. Here $\mathbf{p}$ is a column vector of elements in $M_N(\mathbb{C})$ and $\operatorname{Diag}(c_1,\ldots,c_r)$ denote the block diagonal matrix with the i-th diagonal element c_i. If c_i are scalars, they are also identified with scalar matrices in $M_N(\mathbb{C})$. Hence $\hat{u}(x) = T_r(\mathbf{a})\tilde{u}(x)$ is a solution of the system

$$\begin{aligned}\frac{d\hat{u}}{dx} &= T_r(\mathbf{a})\bigl(\mathbf{p}(C_1,\ldots,C_r) + \mu\operatorname{Diag}(p_1,\ldots,p_r)\bigr)S_r(\mathbf{a})\bigr)\hat{u}\\ &= \bigl(\mathbf{q}(A_1,\ldots,A_r) + \mu T_r(\mathbf{a})\operatorname{Diag}(p_1,\ldots,p_r)S_r(\mathbf{a})\bigr)\hat{u},\end{aligned}$$

which is the convolution of the system (4.1). The system equals

$$\frac{d\hat{u}}{dx} = \sum_{j=1}^{r} \frac{\hat{A}_j}{(x-a_1)\cdots(x-a_j)}\hat{u}, \tag{4.7}$$

where we define

$$D_r(\mathbf{a}) := T_r(\mathbf{a})\operatorname{Diag}(p_1,\ldots,p_r)S_r(\mathbf{a}) = \sum_{j=1}^{r} D_{r,j}(\mathbf{a})q_j, \tag{4.8}$$

$$\hat{A}(x) := \mathbf{q}(A_1,\ldots,A_r) + \mu D_r(\mathbf{a}) = \sum_{j=1}^{r} \hat{A}_j q_j. \tag{4.9}$$

Here $D_{r,j}(\mathbf{a})$ and $\hat{A}_j$ are matrices of size rN which do not depend on x and

$$D_r(\mathbf{a}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ t_{1,2} & t_{2,1} & 0 & 0 & \cdots \\ t_{1,23} & t_{2,13} & t_{3,12} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & p_3 & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & s_{2,1} & 0 & 0 & \cdots \\ 1 & s_{3,1} & s_{3,12} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \tag{4.10}$$

$$p_1 = q_1,\ p_2 = q_1 + s_{2,1}q_2,\ p_3 = q_1 + s_{3,1}q_2 + s_{3,12}q_3, \ldots \tag{4.11}$$

and for simplicity we denote $s_{3,12} = s_{3,\{1,2\}}(\mathbf{a})$ etc. Then, for example, we have

$$D_{r,1} = 1_{rN},\quad D_{r,2} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & s_{2,1} & 0 & \cdots \\ 0 & 1 & s_{3,1} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},\quad D_{r,3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & s_{3,1} & s_{3,12} & 0 & 0 & \cdots \\ 0 & 1 & s_{4,1}+s_{4,2} & s_{4,12} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{4.12}$$

by (4.6), (4.8), (4.10) and (4.11).

EXAMPLE 4.1. When $r = 3$, the convolution of the system (4.1) equals

$$
\begin{aligned}
\frac{d\hat{u}}{dx} &= \frac{\begin{pmatrix} A_1+\mu & A_2 & A_3 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}}{x-a_1}\hat{u} + \frac{\begin{pmatrix} 0 & 0 & 0 \\ A_1+\mu & A_2+(a_2-a_1)\mu & A_3 \\ 0 & \mu & (a_3-a_1)\mu \end{pmatrix}}{(x-a_1)(x-a_2)}\hat{u} \\
&+ \frac{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_1+\mu & A_2+(a_3-a_1)\mu & A_3+(a_3-a_1)(a_3-a_2)\mu \end{pmatrix}}{(x-a_1)(x-a_2)(x-a_3)}\hat{u}.
\end{aligned} \tag{4.13}
$$

Moreover under the notation in Lemma 4.2 ii) we have

$$
K = \begin{pmatrix} A_1 & A_2 & A_3 \\ A_2 & A_3 + s_{2,1}A_2 & s_{3,1}A_3 \\ A_3 & s_{3,1}A_3 & s_{3,12}A_3 \end{pmatrix}.
$$

Let $D_{r,k,i,j}(\mathbf{a})$ be the (i,j) element of $D_{r,k}(\mathbf{a})$. Then

$$
\begin{aligned}
D_{r,k,i,j}(\mathbf{a}) &= \sum_{\nu=1}^{r} T_{i,\nu}(\mathbf{a})S_{\nu,k}(\mathbf{a})S_{\nu,j}(\mathbf{a}) = D_{r,j,i,k}(\mathbf{a}) \\
&= \sum_{\nu=\max\{k,j\}}^{i} t_{\nu,\{1,2,\dots,i\}\setminus\{\nu\}} s_{\nu,\{1,2,\dots,k-1\}} s_{\nu,\{1,\dots,j-1\}}, \\
&= \sum_{\nu=\max\{k,j\}}^{i} t_{\nu,\{k,k+1,\dots,i\}\setminus\{\nu\}} s_{\nu,\{1,\dots,j-1\}}
\end{aligned} \tag{4.14}
$$

$$
D_{r,k,i,j}(t\mathbf{a}) = t^{j+k-i-1} D_{r,k,i,j}(t\mathbf{a}) \quad (\forall t \in \mathbb{C} \setminus \{0\}). \tag{4.15}
$$

LEMMA 4.2. i) *$D_{r,k,i,j}(\mathbf{a})$ are homogeneous polynomials of $\mathbf{a}$ with degree $j+k-i-1$. In particular*

$$
D_{r,k,i,j}(\mathbf{a}) = \begin{cases} 0 & (i<k \quad or \quad i<j \quad or \quad i>j+k-1), \\ 1 & (i=j+k-1). \end{cases}
$$

ii) *Put* $K = \begin{pmatrix} (A_1,\dots,A_r)D_{r,1}(\mathbf{a}) \\ \vdots \\ (A_1,\dots,A_r)D_{r,r}(\mathbf{a}) \end{pmatrix}$ *and* $\mathcal{L} = \left\{ \begin{pmatrix} v_0 \\ 0 \\ \vdots \end{pmatrix} \in \mathbb{C}^{rN} \;\middle|\; (A_1+\mu)v_0 = 0 \right\}$.
Then $\mathcal{K} = \{\mathbf{v} \in \mathbb{C}^{rN} \mid K\mathbf{v} = 0\}$ *and* $\mathcal{L}$ *satisfies* $\hat{A}_j\mathcal{K} \subset \mathcal{K}$ *and* $\hat{A}_j\mathcal{L} = \{0\}$ *for*

$j = 1, \dots, r$.

Proof. i) Put $\pi(\mathbf{a}) = \prod_{1\le i<j\le r}(a_i - a_j)$. Then $\pi(\mathbf{a})^L D_r(\mathbf{a})$ is a matrix of polynomials of $\mathbf{a}$ for a suitable integer L. We may put $L = 0$ if we show that $D_r(\mathbf{a})$ is holomorphic with respect to $\mathbf{a}$ in a small neighborhood U_{ij} of a generic point of a hypersurface defined by $a_i = a_j$. Put $\tilde{q}_j = p_i p_j$ and $\tilde{q}_\nu = p_\nu$ for $\nu \ne j$. Since the transformation of $(q_1, \dots, q_r)$ to $(\tilde{q}_1, \dots, \tilde{q}_r)$ and that of $(\tilde{q}_1, \dots, \tilde{q}_r)$ to $(q_1, \dots, q_r)$ holomorphically depend on $\mathbf{a} \in U_{ij}$, it is sufficient to show that the same claim is valid for the system $u' = \sum_{j=1}^r \tilde{q}_j B_j u$. It follows from Example 2.3 that we may assume $(i, j) = (1, 2)$ and in this case the corresponding convolution $(\tilde{B}_1, \dots, \tilde{B}_r)$ is given by

$$\sum_{i=1}^r \tilde{q}_j \tilde{B}_j = \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \\ \vdots \\ \tilde{q}_r \end{pmatrix} (B_1,\, B_2,\, B_3, \dots,\, B_r) + \mu \begin{pmatrix} \tilde{q}_1 & 0 & & & \\ \tilde{q}_2 & \tilde{q}_1 + (a_2 - a_1)\tilde{q}_2 & & & \\ & & \tilde{q}_3 & & \\ & & & \ddots & \\ & & & & \tilde{q}_r \end{pmatrix}.$$

Hence $\tilde{B}_j$ are polynomial functions of $\mathbf{a}$ and we can conclude that $D_{r,k}(\mathbf{a})$ are also polynomials of $\mathbf{a}$.

If $i > j + k - 1$, the identity (4.15) implies $D_{r,k,i,j}(\mathbf{a}) = 0$.

Lastly we determine the constant $D_{r,k,j+k-1,j}(\mathbf{a}) = D_{r,j,j+k-1,k}(\mathbf{a})$. Then we may assume $j \le k$ and

$$D_{r,k,j+k-1,j}(\mathbf{a}) = \sum_{\nu=k}^{j+k-1} t_{\nu,\{k,k+1,\dots,j+k-1\}\setminus\{\nu\}} s_{\nu,\{1,\dots,j-1\}},$$

which equals 1 when $j = 1$. If $j > 1$, we can restrict this function on $a_{j+k-1} = a_{j-1}$ and we have

$$D_{r,k,j+k-1,j}(\mathbf{a}) = \sum_{\nu=k}^{j+k-2} t_{\nu,\{k,k+1,\dots,j+k-2\}\setminus\{\nu\}} s_{\nu,\{1,\dots,j-2\}},$$

which equals 1 by the induction on j. Thus we have completed the proof.

ii) $\mathcal{K}_0 = \{\tilde{v} \in \mathbb{C}^{rN} \mid \mathrm{Diag}(C_1, \dots, C_r)\tilde{v} = 0\}$ satisfies $\tilde{A}_j \mathcal{K}_0 \subset \mathcal{K}_0$ for $j = 1, \dots, r$. Hence $\mathcal{K}_1 := \{\hat{v} \in \mathbb{C}^{rN} \mid \mathrm{Diag}(C_1, \dots, C_r) S_r(\mathbf{a})\hat{v} = 0\}$ satisfies $\hat{A}_j \mathcal{K}_1 \subset \mathcal{K}_1$ for $j = 1, \dots, r$. Since $C_i = \sum_{\nu=1}^r A_\nu T_{\nu,i}$, the (i, j) block of $\mathrm{Diag}(C_1, \dots, C_r) S_r(\mathbf{a})$ equals $\sum_{\nu=1}^r A_\nu T_{\nu,i}(\mathbf{a}) S_{i,j}(\mathbf{a})$ and that of

${}^tS_r(\mathbf{a})\,\mathrm{Diag}(C_1,\dots,C_r)S_r(\mathbf{a})$ equals

$$\sum_{\nu=1}^{r}\sum_{k=1}^{r} S_{k,i}(\mathbf{a})A_\nu T_{\nu,k}(\mathbf{a})S_{k,j}(\mathbf{a}) = \sum_{\nu=1}^{r} A_\nu D_{r,i,\nu,j}(\mathbf{a})$$

and therefore $\mathcal{K} = \mathcal{K}_1$. It is clear that $\hat{A}_i\mathcal{L} = \{0\}$ for $i = 1,\dots,r$. □

Owing to Lemma 4.2 we have the following remark.

Remark 4.3. If we restrict to $a_1 = \cdots = a_r = a$, the convolution of the system

$$\frac{du}{dx} = \sum_{j=1}^{r} \frac{A_j}{(x-a)^j}u \tag{4.16}$$

is give by

$$\frac{d\hat{u}}{dx} = \sum_{j=1}^{r} \frac{\hat{A}_j}{(x-a)^j}\hat{u} \quad \text{with} \quad \hat{A}_j = \begin{array}{r} \\ \\ j) \\ \\ \\ \\ \end{array}\!\!\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A_1+\mu & A_2 & A_3 & \cdots & A_r \\ 0 & \mu & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix} \tag{4.17}$$

and

$$K = \begin{pmatrix} A_1 & A_2 & \cdots & A_{r-1} & A_r \\ A_2 & A_3 & \cdots & A_r & 0 \\ \vdots & & \iddots & & \vdots \\ A_r & 0 & \cdots & 0 & 0 \end{pmatrix} = \Big(A_{i+j-1}\Big)_{\substack{1\le i\le r\\ 1\le j\le r}} \quad \text{with} \quad A_\nu = 0 \text{ for } \nu > r. \tag{4.18}$$

5. Middle convolution

In this section we examine the middle convolution of the general system

$$\begin{aligned} \frac{du}{dx} = \Bigg(&\sum_{j=1}^{p}\sum_{i=1}^{r_j} \frac{A_{j,i}}{(x-a_{j,1})(x-a_{j,2})\cdots(x-a_{j,i})} \\ &- \sum_{i=2}^{r_0} \frac{A_{0,i}x^{i-2}}{(1-a_{0,2}x)(1-a_{0,3}x)\cdots(1-a_{0,i}x)}\Bigg)u. \end{aligned} \tag{5.1}$$

By a linear fractional transformation of the variable x the system (5.1) is transformed into a system (5.1) with $r_0 = 1$, namely, a system (5.1) without the term $\Sigma_{i=2}^{r_0}$ in the second line of (5.1). Then restricting the system (4.1) in a neighborhood of the point $a_1 = \cdots = a_{r_1} = a'_1$, $a_{r_1+1} = \cdots = a_{r_1+r_2-1} = a'_2, \ldots,$ $a_{r_1+\cdots+r'_p-1} = \cdots = a_p = a_{p'}$ with $a'_i \neq a'_j$ $(1 \le i < j \le p')$, we have the middle convolution of the general system (5.1) as a special case of middle convolutions studied in the previous section, which is sufficient to many purposes.

But we want to have an explicit form of the middle convolution of (5.1). Hence we will examine the convolution of the system

$$\frac{du}{dx} = -\sum_{i=2}^{r} \frac{A'_i x^{i-2}}{(1-a_2x)(1-a_3x)\cdots(1-a_ix)} u \tag{5.2}$$

as in the system (4.1). Put

$$\tilde{p}_i(x) = \frac{-1}{x - a_i^{-1}}, \quad \tilde{q}_i(x) = x^{i-1}\prod_{\nu=2}^{i}(1-a_\nu x)^{-1}. \tag{5.3}$$

Then we have

$$\tilde{p}_i(x) = y\frac{a_i}{y-a_i}, \quad \tilde{q}_i(x) = y\prod_{\nu=2}^{i}\frac{1}{y-a_\nu} \quad \text{with} \quad y = \frac{1}{x}. \tag{5.4}$$

Putting $\mathbf{a}' = (a_2, \ldots, a_r)$, we have

(5.5)

$$\mathbf{p}' = S'_r(\mathbf{a}')\mathbf{q}',\ S'_r(\mathbf{a}') := (S'_{i,j})_{\substack{2\le i\le r,\\ 2\le j\le r}} \quad S'_{i,j}(\mathbf{a}') := \begin{cases} 0 & (i<j),\\ s'_{i+1,\{2,\ldots,j\}}(\mathbf{a}') & (i \ge j),\end{cases}$$

$$\mathbf{q}' = T'_r(\mathbf{a}')\mathbf{p}',\ T'_r(\mathbf{a}') := (T'_{i,j})_{\substack{2\le i\le r,\\ 2\le j\le r}} \quad T'_{i,j}(\mathbf{a}') := \begin{cases} 0 & (i<j),\\ t'_{i+1,\{2,\ldots,i+1\}\setminus\{j+1\}}(\mathbf{a}') & (i \ge j)\end{cases}$$

by denoting

$$\mathbf{p}' = \begin{pmatrix} p'_2 \\ \vdots \\ p'_r \end{pmatrix}, \ \mathbf{q}' = \begin{pmatrix} q'_2 \\ \vdots \\ q'_r \end{pmatrix}, \tag{5.6}$$

$$s'_{i,\{j_1,\ldots,j_k\}}(\mathbf{a}') = a_i\prod_{\nu=1}^{k}(a_i - a_{j_\nu}) \ \text{ and } \ t'_{i,\{j_1,\ldots,j_k\}}(\mathbf{a}') = s'_{i,\{j_1,\ldots,j_k\}}(\mathbf{a}')^{-1}. \tag{5.7}$$

Then the convolution of the system (5.2) equals

$$\frac{d\hat{u}}{dx} = \big(\mathbf{q}'(A_2', \cdots, A_r') + D_r'(\mathbf{a}')\big)\hat{u},$$

where

$$D_r'(\mathbf{a}') := T_r'(\mathbf{a})\,\mathrm{Diag}(p_2', \dots, p_r')S_r'(\mathbf{a}') = \sum_{k=2}^{r} D_{r,k}'(\mathbf{a}')q_k', \tag{5.8}$$

namely,

$$D_r'(\mathbf{a}') = \begin{pmatrix} t_2' & 0 & 0 & 0 & \cdots \\ t_{2,3}' & t_{3,2}' & 0 & 0 & \cdots \\ t_{2,34}' & t_{3,243}' & t_{3,12}' & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \mathrm{Diag}(p_2', p_3', q_4', \cdots) \begin{pmatrix} s_2' & 0 & 0 & \cdots & \cdots \\ s_3' & s_{3,2}' & 0 & \cdots & \cdots \\ s_4' & s_{4,2}' & s_{4,23}' & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix},$$
$$p_2' = s_2'q_2, \quad p_3' = s_3'q_2' + s_{3,2}'q_3', \quad p_4' = s_4'q_2' + s_{4,2}'q_3' + s_{4,23}'q_4', \dots .$$

Let $D_{r,k,i,j}'$ be the (i,j) element of $D_{r,k}'$. Then we remark that

$$\begin{aligned} D_{r,k,i,j}'(\mathbf{a}') &= \sum_{\nu=1}^{r} T_{i,\nu}'(\mathbf{a}')S_{\nu,k}'(\mathbf{a}')S_{\nu,j}'(\mathbf{a}') = D_{r,j,i,k}'(\mathbf{a}') \\ &= \sum_{\nu=\max\{k,\,j\}}^{i} t_{\nu+1,\{k+1,k+2,\dots,i+1\}\setminus\{\nu+1\}} s_{\nu+1,\{2,\dots,j\}}', \end{aligned} \tag{5.9}$$

$$T_{i,\nu}'(\mathbf{a}')S_{\nu,j}'(\mathbf{a}') = T_{i-1,\nu-1}(\mathbf{a}')S_{\nu-1,j-1}(\mathbf{a}') \tag{5.10}$$

and we get the following lemma by the same way as in the proof of the previous lemma.

LEMMA 5.1. i) *$D_{r,k,i,j}'(\mathbf{a}')$ are homogeneous polynomials of $\mathbf{a}'$ with degree $j+k-i$. In particular*

$$D_{r,k,i,j}'(\mathbf{a}') = \begin{cases} 0 & (i<k \quad or \quad i<j \quad or \quad i>j+k), \\ 1 & (i=j+k). \end{cases}$$

ii) *Put*

$$K := \begin{pmatrix} (A_2',\dots,A_r')D_{r-1,1}(\mathbf{a}') \\ \vdots \\ (A_2',\dots,A_r')D_{r-1,r-1}(\mathbf{a}') \end{pmatrix}, \quad \mathcal{K} := \bigl\{\mathbf{v} \in \mathbb{C}^{(r-1)N} \bigm| K\mathbf{v} = 0\bigr\},$$

$$\mathcal{L} := \left\{ \begin{pmatrix} {\scriptstyle (-a_3)\cdots(-a_r)v} \\ {\scriptstyle (-a_4)\cdots(-a_r)v} \\ \vdots \\ {\scriptstyle v} \end{pmatrix} \in \mathbb{C}^{(r-1)N} \;\middle|\; \Bigl(\sum_{j=2}^{r}\bigl(\prod_{\nu=j+1}^{r} a_\nu\bigr)A_j + \bigl(\prod_{\nu=2}^{r} a_\nu\bigr)\mu\Bigr)v = 0 \right\}.$$

Then we have $\hat{A}_j\mathcal{K} \subset \mathcal{K}$ *and* $\hat{A}_j\mathcal{L} = \{0\}$ *for* $j = 1,\dots,r$.

EXAMPLE 5.2. When $r = 4$,

$$D_{4,2}'(\mathbf{a}') = \begin{pmatrix} a_2 & 0 & 0 \\ 1 & a_3 & 0 \\ 0 & 1 & a_4 \end{pmatrix}, \; D_{4,3}'(\mathbf{a}') = \begin{pmatrix} 0 & 0 & 0 \\ a_3 & a_3(a_3-a_2) & 0 \\ 1 & a_3+a_4-a_2 & a_4(a_4-a_2) \end{pmatrix},$$

$$D_{4,4}'(\mathbf{a}') = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_4 & a_4(a_4-a_2) & a_4(a_4-a_2)(a_4-a_3) \end{pmatrix}$$

and the convolution of the system (5.2) with a parameter μ equals

$$\begin{aligned}\frac{d\tilde{u}}{dx} = &-\frac{\begin{pmatrix} A_2'+a_2\mu & A_3' & A_4' \\ \mu & a_3\mu & 0 \\ 0 & \mu & a_4\mu \end{pmatrix}}{1-a_2x}\tilde{u} - \frac{\begin{pmatrix} 0 & 0 & 0 \\ A_2'+a_3\mu & A_3'+a_3(a_3-a_2)\mu & A_4' \\ \mu & (a_3+a_4-a_2)\mu & a_4(a_4-a_2)\mu \end{pmatrix}x}{(1-a_2x)(1-a_3x)}\tilde{u} \\ &-\frac{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_2'+a_4\mu & A_3'+a_4(a_4-a_2)\mu & A_4'+a_4(a_4-a_2)(a_4-a_3)\mu \end{pmatrix}x^2}{(1-a_2x)(1-a_3x)(1-a_4x)}\tilde{u}.\end{aligned}$$

Under the notation in Lemma 5.1, we have

$$K = \begin{pmatrix} A_2' & A_3' & A_4' \\ A_3' & (a_3-a_2)A_3'+A_4' & (a_4-a_2)A_4' \\ A_4' & (a_4-a_2)A_4' & (a_4-a_2)(a_4-a_3)A_4' \end{pmatrix},$$

$$\mathcal{L} = \left\{ \begin{pmatrix} a_3a_4v \\ -a_4v \\ v \end{pmatrix} \;\middle|\; (a_3a_4A_2' - a_4A_3' + A_4' + a_2a_3a_4\mu)v = 0 \right\}.$$

If we put $a_2 = \cdots = a_r = 0$, the convolution of the system

$$\frac{du}{dx} = -\sum_{j=2}^{r} A'_j x^{j-2} u \tag{5.11}$$

is given by

$$\frac{d\tilde{u}}{dx} = -\sum_{j=2}^{r} \tilde{A}'_j x^{j-2} u \quad \text{with} \quad \tilde{A}'_j = \begin{matrix} \\ \\ j-1) \\ \\ \\ \\ \end{matrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A'_2 & A'_3 & A'_4 & \cdots & A'_r \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & \mu & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix} \tag{5.12}$$

and $\mathcal{L} \subset \mathcal{K}$.

Combining the results we have studied, we have the convolution

$$\begin{aligned} \frac{d\hat{u}}{dx} = \Bigg(& \sum_{j=1}^{p} \sum_{i=1}^{r_j} \frac{\hat{A}_{j,i}}{(x-a_{j,1})(x-a_{j,2})\cdots(x-a_{j,i})} \\ & - \sum_{i=2}^{r_0} \frac{\hat{A}_{0,i} x^{i-2}}{(1-a_{0,2}x)(1-a_{0,3}x)\cdots(x-a_{0,i}x)} \Bigg) \hat{u}. \end{aligned} \tag{5.13}$$

of the general system (5.1) by the following theorem.

THEOREM 5.3 (CONVOLUTION). *The convolution* (5.13) *of the Pfaffian system* (5.1) *is given as follows. Put*

$$\begin{aligned} &\mathbf{a}_0 = (a_{0,2}, \ldots, a_{0,r_0}) \quad \textit{and} \quad \mathbf{a}_j = (a_{i,1}, \ldots, a_{i,r_j}) \quad \textit{for} \quad j = 1, \ldots, p, \\ &q_{0,i} = -\prod_{\nu=2}^{i} (1-a_{0,\nu}x)^{-1} \quad \textit{and} \quad q_{j,i} = \prod_{\nu=1}^{i} (x-a_{i,\nu})^{-1} \quad \textit{for} \quad j = 1, \ldots, p, \\ &\mathbf{q}_0 = \begin{pmatrix} q_{0,2} \\ \vdots \\ a_{0,r_0} \end{pmatrix}, \ \mathbf{q}_j = \begin{pmatrix} q_{j,1} \\ \vdots \\ a_{j,r_j} \end{pmatrix} \quad \textit{for} \quad j = 1, \ldots, p \quad \textit{and} \quad \mathbf{q} = \begin{pmatrix} \mathbf{q}_0 \\ \vdots \\ \mathbf{q}_p \end{pmatrix}, \\ &\mathbf{A} = (A_{0,2}, \ldots, A_{0,r_0}, A_{1,1}, \ldots, A_{1,r_1}, \ldots, A_{p,1}, \ldots, A_{p,r_p}), \\ &\mathbf{D}_0(\mathbf{a}_0) = -\sum_{i=2}^{r_0} D'_{r_0,i}(\mathbf{a}_0) q_{0,i} \quad \textit{and} \quad \mathbf{D}_j(\mathbf{a}_j) = \sum_{i=1}^{r_j} D_{r_j,i}(\mathbf{a}_j) q_{j,i} \quad (j = 1, \ldots, p), \\ &\mathbf{D}(\mathbf{a}) = \mathrm{Diag}\big(\mathbf{D}_0(\mathbf{a}_0), \ldots, \mathbf{D}_r(\mathbf{a}_r)\big). \end{aligned}$$

Here $D_{r_j,i}(\mathbf{a}_j)$ *(resp.* $D'_{r_0,i}(\mathbf{a}_0)$*) are derfined by* (4.8) *(resp.* (5.8)*) with* (4.4) *(resp.* (5.5)*). Then the convolution* (5.13) *is*

$$\frac{d\hat{u}}{dx} = \Big(\mathbf{q}\mathbf{A} + \mu\mathbf{D}(\mathbf{a})\Big)\hat{u}. \tag{5.14}$$

Put

$$K_0 = \begin{pmatrix} (A_{0,2},\dots,A_{0,r_0})D_{r_0-1,1}(\mathbf{a}_0) \\ \vdots \\ (A_{0,2},\dots,A_{0,r_0})D_{r_0-1,r_0-1}(\mathbf{a}_0) \end{pmatrix},\ K_j = \begin{pmatrix} (A_{j,1},\dots,A_{j,r_j})D_{r_j,1}(\mathbf{a}_j) \\ \vdots \\ (A_{j,1},\dots,A_{j,r_j})D_{r_j,r_j}(\mathbf{a}_j) \end{pmatrix},$$

$$K = \mathrm{Diag}(K_0, K_1, \dots, K_p),\ \mathcal{K} = \{\mathbf{v} \in \mathbb{C}^{(r_0+\cdots+r_p-1)N} | Kv = 0\},$$

$$A_\infty = \sum_{i=2}^{r_0}\Big(\prod_{\nu=i+1}^{r_0} a_{0,\nu}\Big)A_{0,i} + \Big(\prod_{\nu=2}^{r_0} a_{0,\nu}\Big)\Big(\mu + \sum_{j=1}^{p}\sum_{i=1}^{r_j} A_{j,i}\Big),$$

$$\mathcal{L}_0 = \{v \in \mathbb{C}^N \mid A_\infty v = 0\},\ \mathcal{L} = \left\{ \begin{array}{r} \\ \\ \\ \\ \\ r_0 \rightarrow \\ \\ \\ r_0+r_1 \rightarrow \\ \\ \\ \end{array} \begin{pmatrix} (-a_{0,3})\cdots(-a_{0,r_0})v \\ (-a_{0,4})\cdots(-a_{0,r_0})v \\ \vdots \\ (-a_{0,r_0})v \\ v \\ -(-a_{0,2})\cdots(-a_{0,r_0})v \\ 0 \\ \vdots \\ -(-a_{0,2})\cdots(-a_{0,r_0})v \\ 0 \\ \vdots \end{pmatrix} \ \middle|\ v \in \mathcal{L}_0 \right\}.$$

In the above definition of $\mathcal{L}$*, the* i*-th block equals* $(-a_{0,i+2})\cdots(-a_{0,r_0})v \in \mathbb{C}^N$ *for* $1 \le i < r_0$ *and the* $(r_0+\cdots+r_j)$*-th block equals* $-(-a_{0,2})\cdots(-a_{0,r_0})v$ *for* $j = 0,\dots,p-1$ *and the other* i*-th blocks with* $i > r_0$ *are* $0 \in \mathbb{C}^N$*. Then we have*

$$\hat{A}_{j,i}\mathcal{K} \subset \mathcal{K} \quad and \quad \hat{A}_{j,i}\mathcal{L} = \{0\} \quad for \quad i = 1+\delta_{j,0},\dots,r_j \quad and \quad j = 0,\dots,p. \tag{5.15}$$

Here $\delta_{j,k} = 1$ *if* $j = k$ *and* 0 *otherwise.*

DEFINITION 5.4 (MIDDLE CONVOLUTION). In the above theorem, $\hat{A}_{j,i}$ induce linear transformations on the quotient space $\bar{V} = \mathbb{C}^{(r_0+\cdots+r_p-1)N}/(\mathcal{K}+\mathcal{L})$. Let $\bar{A}_{j,i}$ be matrices corresponding to the induced linear transformations on $\bar{V}$ with respect a suitable base of $\bar V$. We define that the system

$$\begin{aligned}\frac{d\bar u}{dx} = \Bigl(&\sum_{j=1}^{p}\sum_{i=1}^{r_j}\frac{\bar A_{j,i}}{(x-a_{j,1})(x-a_{j,2})\cdots(x-a_{j,i})}\\ &-\sum_{i=2}^{r_0}\frac{\bar A_{0,i}x^{i-2}}{(1-a_{0,2}x)(1-a_{0,3}x)\cdots(x-a_{0,i}x)}\Bigr)\bar u\end{aligned} \tag{5.16}$$

is the *middle convolution* mc_μ of (3.1) with a parameter $\mu \in \mathbb{C}$.

Suppose $A_{j,i}$ in the system (5.1) holomorphically depend on $\mathbf{a}$. Since $\mathcal{K}$ and $\mathcal{L}$ in Theorem 5.3 depend on $\mathbf{a}$, we denote them by $\mathcal{K}(\mathbf{a})$ and $\mathcal{L}(\mathbf{a})$, respectively. Let k and ℓ be the dimensions of $\mathcal{K}(\mathbf{a})$ and $\mathcal{L}(\mathbf{a})$ for a generic $\mathbf{a}$ satisfying $a_{0,2}\cdots a_{0,r} \neq 0$. Note that $\mathcal{K}(\mathbf{a}) \cap \mathcal{L}(\mathbf{a}) = \{0\}$ in this case. Then Theorem 2.2 implies the relation $\dim(\mathcal{K}(0) + \mathcal{L}(0)) \geq k + \ell$ and the following proposition.

PROPOSITION 5.5. *Retain the notation above.*

i) *Fix a generic* $\mathbf{a}$ *and replace* $a_{j,i}$ *by* $a_{j,1} + t(a_{j,i} - a_{j,1})$ *for* $j = 0, \dots, p$ *and* $i = 1, \dots, r_j$ *with* $a_{0,1} = 0$. *Then we may assume that* $\tilde{\mathcal{K}} = \mathcal{K} + \mathcal{L}$ *holomorphically depends on* $t \in U_\epsilon \setminus \{0\}$ *with* $U_\epsilon = \{t \in \mathbb{C} \mid |t| < \epsilon\}$ *and* $\epsilon > 0$, *which we denote by* $\tilde{\mathcal{K}}(t)$. *Then Theorem 2.2 assures that we can uniquely define* $\tilde{\mathcal{K}}(0)$ *so that* $\tilde{\mathcal{K}}(t)$ *holomorphically depends on* $t \in U_\epsilon$. *Using this single parameter* t, *we can define a middle convolution which holomorphically depend on* $t \in U_\epsilon$ *(cf. Remark 3.1).*

ii) *Let* U *be an open subset of the space of parameter* $\mathbf{a}$ *such that* $A_{j,i}$ *are holomorphic with respect to the parameter* $\mathbf{a} \in U$. *If the dimension of* $\mathcal{L}(\mathbf{a}) + \mathcal{K}(\mathbf{a})$ *is constant for* $\mathbf{a} \in U$, *then* $\bar A_{j,i}$ *are holomorphic with respect to* $\mathbf{a} \in U$. *This is assured if the middle convolution* (5.16) *defined in i) is irreducible.*

REMARK 5.6. i) By the restriction $a_{0,2} = \cdots = a_{0,r_0} = 0$ and $a_{j,1} = \cdots = a_{j,r_j} = a_j$ we get the middle convolution of the system (1.2) which coincides with the middle convolution defined by [19].

ii) Proposition 5.5 i) may relate to Remark 6.2 ii).

iii) Calculation of $D(\mathbf{a})$ in (5.14) is implemented in a computer program in [17].

6. Versal unfolding and confluence

Note that the identity (4.3) implies that the Riemann scheme of the equation

$$\frac{du}{dx} = \frac{\lambda_k}{(x-a_1)(x-a_2)\cdots(x-a_k)}u$$

equals

$$\left\{\begin{matrix} x = a_i & (i = 1, \dots, k) \\ \frac{\lambda_k}{s_{i,\{1,\dots,k\}}(a_1,\dots,a_k)} & \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} x = a \\ \lambda_k x^{k-1} \end{matrix}\right\} \quad (a_1 = \cdots = a_k = a),$$

which is an example of versal unfolding of the equation $u' = \frac{\lambda}{(x-a)^k} u$.

Let (2.8) be GRS of the system (1.2). Put

$$\lambda_{j,\nu}(x) = \lambda_{j,\nu,0} + \lambda_{j,\nu,1}x + \cdots + \lambda_{j,\nu,r_j-1}x^{r_j-1} \in \mathbb{C}[x] \quad (\nu = 1, \dots, n_j,\ j = 0, \dots, p),$$

$$a_{0,1} = 0, \quad a'_{j,i} = \begin{cases} a_{j,i}^{-1} & (j = 0,\ i = 1, \dots, r_0), \\ a_{j,i} & (1 \le j \le p,\ i = 1, \dots, r_j). \end{cases}$$

Then the versal unfolding of GRS (2.8) defined by [16] equals

$$\left\{\begin{matrix} x = a'_{j,i} & (i = 1, \dots, r_j,\ j = 0, \dots, p) \\ [\lambda_{j,1}^{(i)}]_{m_{j,1}^{(i)}} & \\ \vdots & \\ [\lambda_{j,n_{j,i}}^{(i)}]_{m_{j,n_{j,r}}^{(i)}} & \end{matrix}\right\}$$

and we require that the versal unfolding (1.3) of the system (1.2) has this GRS. Here

$$\lambda_{j,\nu}^{(r)}(\mathbf{a}) := \sum_{k=1}^{r_j} \frac{\tilde\lambda_{j,\ell_{j,r,\nu},k}(\mathbf{a})}{s_{r,\{1,\dots,k\}}} \quad (0 \le j \le p,\ 1 \le r \le r_j,\ 1 \le \nu \le n_{j,i}), \tag{6.1}$$

$$\tilde\lambda_{j,\nu,k}(\mathbf{a}_0) = \lambda_{j,\nu,k} \quad \text{and} \quad \sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}\tilde\lambda_{j,\nu,0}(\mathbf{a}) = 0 \tag{6.2}$$

with $a_{0,1} = 0$ and $\mathbf{a}_0$ is the point defined by $a_{0,i} = 0$ $(i = 1, \dots, r_0)$ and $a_{j,i} = a_j$ $(i = 1, \dots, r_j$ and $j = 1, \dots, p)$. See [16, Example 4.4] for an example of the unfolding for a single differential equation. Replacing $[\lambda]_{(m)}$ in the example by $[\lambda]_m$, we have GRS of a Pfaffian system from that of a single differential equation.

For a function $\phi(x)$ of x, the addition $\mathrm{Ad}(\phi)$ is an automorphism of the ring of linear differential operators induced from the map $u(x) \mapsto \phi(x)u(x)$. Then

$\mathrm{Ad}(\phi)\left(\frac{d}{dx}\right) = \frac{d}{dx} - \frac{\phi'}{\phi}$ and by the versal additions

$$
(6.3)\quad
\begin{aligned}
&\mathrm{AdV}^0_{(c_0,\dots,c_m)}(\lambda_0,\dots,\lambda_m) := \mathrm{Ad}\Bigl(\exp\Bigl(\int_c^x \sum_{k=0}^m \frac{\lambda_k ds}{\prod_{\nu=0}^k (s-c_\nu)}\Bigr)\Bigr) \quad (c\in\mathbb{C}),\\
&\mathrm{AdV}^0_{(c_0,\dots,c_m)}(\lambda_0,\dots,\lambda_m)\Bigl(\frac{d}{dx}\Bigr) = \frac{d}{dx} - \sum_{k=0}^m \frac{\lambda_k}{\prod_{\nu=0}^k (x-c_\nu)},
\end{aligned}
$$

$$
(6.4)\quad
\begin{aligned}
&\mathrm{AdV}_{(\frac{1}{c_1},\dots,\frac{1}{c_m})}(\lambda_1,\dots,\lambda_m) := \mathrm{Ad}\Bigl(\exp\Bigl(-\sum_{k=1}^m \int_0^x \frac{\lambda_k s^{k-1} ds}{\prod_{i=1}^k (1-c_i s)}\Bigr)\Bigr),\\
&\mathrm{AdV}_{(\frac{1}{c_1},\dots,\frac{1}{c_m})}(\lambda_1,\dots,\lambda_m)\Bigl(\frac{d}{dx}\Bigr) = \frac{d}{dx} + \sum_{k=1}^m \frac{\lambda_k x^{k-1}}{\prod_{i=1}^k (1-c_i x)}
\end{aligned}
$$

introduced by [13, §2.3], we can shift $\lambda_{j,\nu}$ by any polynomial $f_j(x)$ with $\deg f_j(x) < r_j$ except for the constant terms of $\lambda_{0,\nu}$ (cf. [16]). Then applying suitable versal additions, GRS of the resulting universal unfolding satisfies

$$
(6.5)\qquad \deg \lambda_{0,1}^{(1)} < 1 \ \text{ and } \ \lambda_{j,1}^{(i)} = 0 \ \ (i = \delta_{0,j}+1,\dots,r_j,\ j=0,\dots,p).
$$

Let $\mathbf{m} = \{\mathbf{m}_j^{(i)}\}$ be a spectral type of a differential equation whose irregular singularities are unramified. We define three condition for $\mathbf{m}$.

1. $\mathbf{m}$ is *irreducibly realizable* : There exists an irreducible differential equation with a generic Riemann scheme corresponding to the spectral type $\mathbf{m}$.

2. $\mathbf{m}$ is *versally realizable* : $\mathbf{m}$ is irreducibly realizable and the corresponding equation has a versal unfolding.

3. $\mathbf{m}$ has a *Fuchsian base* : There exists an irreducible Fuchsian differential equation with a generic Riemann scheme corresponding to the unfolded Fuchsian spectral type $\{\mathbf{m}_j^{(i)}\}$.

In this paper, we say that the system (1.2) is irreducible if and only if the condition $A_{j,i}V \subset V$ $(i=1,\dots,r_j,\ j=1,\dots,p)$ for a subspace V of $\mathbb{C}^N$ implies $V=\{0\}$ or $V=\mathbb{C}^N$. This irreducibility is weaker than the irreducibility as a $\mathcal{D}$-module which is examined in [13, 14].

Conjecture 6.1 ([16]). *The above three conditions are equivalent.*

Remark 6.2. i) In [16] we mainly examine when the equations are single equations with arbitrary orders. In this note we examine when the equations are Pfaffian systems. The necessary and sufficient condition for **1** is given by [4] and that for **3** is given by [1]. In fact, these conditions are interpreted to the problem characterizing the roots of Kac-Moody root systems by suitable correspondences between spectral types $\mathbf{m}$ and elements of the root lattices. Then the equivalence of the conditions **1** and **3** is reduced to a problem of roots in two Kac-Moody root systems. Note that our middle convolutions and versal additions with holomorphic parameters $\mathbf{a}$ are compatible with those for equations with irregular singularities when $\mathbf{a} = \mathbf{a}_0$ and also with those for Fuchsian equations when $\mathbf{a}$ and GRS of the equations are generic. Therefore these operations keep the above three conditions for the spectral type $\mathbf{m}$.

We note that these operations induce the action of the Weyl groups of certain Kac-Moody root systems. Hence each condition is invariant in the orbit of the group action of a Weyl group. We remark that [7] shows that there are only finite orbits with the same index of rigidity and there is a unique spectral type with the minimal rank in each orbit, which is called a *basic spectral type* (cf. [16, §5]).

Comparing the correspondence of the roots (cf. [4, 7]), the answer of the Deligne-Simpson problem given by [4] assures that the condition **3** implies the condition **1**, which is explained in [5]. The author originally wanted to solve the Deligne-Simpson problem for single equations with irregular singularities by showing the equivalence of the conditions **1** and **3**. But this implication and the Deligne-Simpson problem are still unsolved for single differential equations. Here the Deligne-Simpson problem is to characterize GRS with an irreducible differential equation.

ii) Kawakami [10] shows that the system (1.2) satisfying $r_j \le 2$ for $j = 0, \dots, p$ has a versal unfolding. Hence we can conclude that the conjecture is true for the Pfaffian systems satisfying that the Poincaré rank is not larger than 1 at every singular points. Moreover the versal unfolding of some basic spectral types $\mathbf{m}$ with $\operatorname{idx} \mathbf{m} \ge -2$ are also given in [6].

iv) If $\mathbf{m}$ is rigid, then $\mathbf{m}$ is an orbit of the Weyl group containing the trivial equation $u' = 0$ of rank 1. Hence the conjecture is true and we have the following theorem (cf. [16, §5]).

Theorem 6.3. *Any irreducible rigid Pfaffian system* (1.2) *is realized as a specialization of its versal unfolding* (1.3) *which is constructed by successive applications of versal additions and middle convolutions to the trivial equation $v' = 0$ whose rank equals 1.*

7. Several variables

Consider the differential equation

$$\tag{7.1} \begin{aligned} \frac{\partial u}{\partial x_0} = \Bigl(&\sum_{j=1}^{p}\sum_{i=1}^{r_j} \frac{A_{j,i}}{(x_0 - x_j - e_{j,1})(x_0 - x_j - e_{j,2})\cdots(x_0 - x_j - e_{j,i})} \\ &- \sum_{i=2}^{r_0} \frac{A_{0,i}x_0^{i-2}}{(1 - a_{0,2}x_0)(1 - a_{0,3}x_0)\cdots(1 - a_{0,i}x_0)}\Bigr)u. \end{aligned}$$

In this section we assume that $A_{i,j}$ do not depend on $(x_0, x_1, \ldots, x_p)$ but holomorphically depend on $\mathbf{a}$ and $\mathbf{e}$. Here $\mathbf{a} = (a_{0,2}, \ldots, a_{0,r_0})$ and $\mathbf{e} = (e_{1,1}, \ldots e_{1,r_1}, \ldots, e_{p,r_p})$ with $(\mathbf{a}, \mathbf{e}) \in U \subset \mathbb{C}^{r_0+r_1+\cdots+r_p-1}$ and U is a neighborhood of the origin. Put

$$\tag{7.2} \pi(\mathbf{a}, \mathbf{e}) = \Bigl(\prod_{j=1}^{p} \prod_{1\le i<k\le r_j} (e_{j,k} - e_{j,i})\Bigr)\Bigl(\prod_{i=2}^{r_0} a_{0,i}\Bigr)\Bigl(\prod_{1\le i<k\le r_0} (a_{0,k} - a_{0,i})\Bigr),$$

$$\tag{7.3} U' = \{(\mathbf{a}, \mathbf{e}) \in U \mid \pi(\mathbf{a}, \mathbf{e}) \neq 0\}.$$

Note that the equation (7.1) is a Fuchsian system of ordinary differential equation with the variable x_0 if $(\mathbf{a}, \mathbf{e}) \in U'$.

We examine its extension to a Pfaffian system

$$\tag{7.4} \frac{\partial u}{\partial x_j} = A_j(x)u \quad (j = 0, \ldots, p)$$

with the integrability condition

$$\tag{7.5} \frac{\partial A_j(x)}{\partial x_i} + A_j(x)A_i(x) = \frac{\partial A_i(x)}{\partial x_j} + A_i(x)A_j(x) \qquad (0 \le i < j \le p)$$

which is equivalent to (1.8) for the system (1.7). Here $A_j(x)$ are suitable square matrices of size N whose entries are rational functions of x and $A_0(x)$ is the coefficient of u in the right hand side of (7.1).

In particular we will show the existence and construction of the extension when the system (7.1) is rigid. According to the usual expression of Pfaffian forms, the system (7.4) equals

$$\tag{7.6} du = \sum_{i=0}^{p} A_i(x)u\,dx_i.$$

Theorem 7.1. i) *Put* $x_{0,i} = a_{0,j}^{-1}$ *for* $i = 2, \ldots, r_0$ *and* $x_{j,i} = x_j + e_{j,i}$ *for* $i = 1, \ldots, r_j$ *and* $j = 1, \ldots, p$. *Suppose that when* $(\mathbf{a}, \mathbf{e}) \in U'$, *the Fuchsian system* (7.1) *with singularities at* $x_{j,i}$ *and* ∞ *is extended to a KZ system with variables* $(x_0, x_1, \ldots, x_p)$ *and that the KZ system is holomorphically extended for* $(\mathbf{a}, \mathbf{e}) \in U$. *Then the middle convolution of* (7.1) *defined in* §5 *has the same property.*

ii) *If the equation* (5.1) *is a versal unfolding of the irreducible rigid equation* (1.2), *then the equation* (5.1) *is extended to a Pfaffian system with variables* $(x, x_1, \ldots, x_p)$ *and a holomorphic parameter* $(\mathbf{a}, \mathbf{e}) \in U$.

Proof. i) The transformation of the equation (7.1) to a Fuchsian system (1.6) with p replaced by $R = r_0 + \cdots + r_p - 1$ is holomorphic with respect to $(\mathbf{a}, \mathbf{e}) \in U'$ and the transformation between $\tilde{u}$ and $\hat{u}$ is also holomorphic with respect to $(\mathbf{a}, \mathbf{e}) \in U'$. Let $\hat{u}$ be the generator of the convolution of the equation (7.1) given in the previous section. Then applying Definition 7.3 to the Fuchsian system, we have a Pfaffian system $d\hat{u} = \sum_{j=0}^{p} \hat{A}_j(x)\hat{u}\, dx_j$ for $(\mathbf{a}, \mathbf{e}) \in U'$. Note that $\hat{A}_j(x)$ are holomorphic for $(\mathbf{a}, \mathbf{e}) \in U'$ and meromorphic with respect to $(\mathbf{a}, \mathbf{e}) \in U$.

Then we show that the parameter $(\mathbf{a}, \mathbf{e})$ of $\hat{A}_j(x)$ is holomorphically extended to U. It is sufficient to show that $\hat{A}_j(x)$ are holomorphic with respect to the parameter $(\mathbf{a}, \mathbf{e})$ in a neighborhood of the generic point of $U \setminus U'$. There are the following three types of the generic point of $U \setminus U'$.

Case 1. There exist j_o, i_o and i'_o such that $e_{j_o,i_o} = e_{j_o,i'_o}$ and $1 \le j_o \le p$ and $1 \le i_o < i'_o \le r_{j_o}$.

Case 2. There exist i_o and i'_o such that $a_{0,i_o} = a_{0,i'_o}$ and $2 \le i_o < i'_o \le r_o$.

Case 3. There exists i_o such that $a_{0,i_o} = 0$ and $2 \le i_o \le r_o$.

When the parameter is in a neighborhood of a generic point classified in Case 1 or Case 2, then as in the proof of Lemma 4.2 i), we can reduce the regularity for the equation (7.1) with $r_1 = 2$ and $r_j = 1 \quad (j = 0, 2, \ldots, p)$. We examine this case in Example 7.5 when $p = 2$. The result for $p = 1$ or $p > 2$ is essentially same as this case.

When the parameter is in a neighborhood of a generic point classified in Case 3, we can reduce the regularity for the equation (7.1) with $r_0 = 2$ and $r_j = 1$ $(j = 1, \ldots, p)$. We examine this case in Example 7.6 when $p = 2$. The result for $p = 0,\ 1$ or $p > 2$ is essentially same as this case.

Thus we see that $\hat{A}_j(x)$ are holomorphic with respect to the parameter $(\mathbf{a}, \mathbf{e}) \in U$. Let $\mathcal{K}$ and $\mathcal{L}$ be the subspaces of $\mathbb{C}^{RN}$ to define the middle convolution of (7.1) which are given in Theorem 5.3. Since $\hat{A}_j(x)\big(\mathcal{K} + \mathcal{L}\big) \subset \mathcal{K} + \mathcal{L}$ for $j = 1, \ldots, p$ and $(\mathbf{a}, \mathbf{e}) \in U'$ and moreover $\mathcal{K} + \mathcal{L}$ holomorphically depends on $(\mathbf{a}, \mathbf{e}) \in U$, we have the middle convolution of (7.4) with the holomorphic parameter $(\mathbf{a}, \mathbf{e}) \in U$.

ii) As is stated in Theorem 6.3, the versal unfolding of an irreducible rigid differential equation (7.1) is obtained by successive applications of versal additions and middle convolutions to the trivial equation. Hence the claim ii) follows from the claim i). □

Remark 7.2. i) Using a complete base of

$$\Big\{\frac{1}{x_j+e_{j,i}-x_0},\ \frac{1}{x_j-x_k+e_{j,i}-e_{k,\nu}},\ \frac{1}{1-a_{0,\nu'}(x_j+e_{j,i})}$$
$$\mid i=1,\dots,r_j,\ k\in\{1,\dots,p\}\setminus\{j\},\ \nu=1,\dots,r_k,\ \nu'=2,\dots,r_0\Big\},$$

the holomorphic dependence of $A_j(x)$ with respect to $(\mathbf{a},\mathbf{e})$ means that of the coefficients of linear combination of this base in the expression of $A_j(x)$.

ii) By the restriction $e_{j,i}=0$ for $i=1,\dots,r_j$ and $j=1,\dots,p$ and $a_{0,i}=0$ for $i=2,\dots,r_0$, we have the extension of the rigid system (1.2) to a Pfaffian system with $p+1$ variables $(x_0,x_1,\dots,x_p)=(x,a_1,\dots,a_p)$.

Definition 7.3 ([2, 3]). The convolution

$$\frac{\partial\tilde{u}}{\partial x_i}=\sum_{\substack{\nu\neq i\\ 0\le\nu\le p}}\frac{\tilde{A}_{i,\nu}}{x_i-x_\nu}u\qquad(i=0,\dots,p)\tag{7.7}$$

of the KZ equation (1.7) is defined by

$$\begin{array}{cl} & \begin{array}{ccccc} \phantom{A_{0,1}} & & j & & \phantom{A_{0,p}} \end{array}\\ \tilde{A}_{0,j}=\ j & \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0\\ \vdots & \cdots & \vdots & \cdots & \vdots\\ A_{0,1} & \cdots & A_{0,j}+\mu & \cdots & A_{0,p}\\ \vdots & \cdots & \vdots & \cdots & \vdots\\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}\in M(pN,\mathbb{C})\qquad(1\le j\le p) \end{array}$$
$$=\Big((A_{0,t}+\mu\delta_{s,t})\delta_{s,j}\Big)_{\substack{1\le s\le p\\ 1\le t\le p}},$$

$$\tilde{A}_{i,j} = \begin{array}{cc} & \begin{array}{ccccccc} & & i & & j & & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} A_{i,j} & & & & & & \\ & \ddots & & & & & \\ & & A_{i,j}+A_{0,j} & & -A_{0,j} & & \\ & & & \ddots & & & \\ & & -A_{0,i} & & A_{i,j}+A_{0,i} & & \\ & & & & & \ddots & \\ & & & & & & A_{i,j} \end{pmatrix} \end{array} \in M(pN,\mathbb{C}) \quad (1 \le i < j \le p)$$
$$= \Big(A_{i,j}\delta_{s,t} + A_{0,j}\delta_{s,i}(\delta_{t,i} - \delta_{t.j}) + A_{0,i}\delta_{s,j}(\delta_{t,j} - \delta_{t,i})\Big)_{\substack{1\le s\le p \\ 1\le t\le p}}.$$

Putting $\mathcal{K} = \big\{u \in \mathbb{C}^{pN} \mid \mathrm{Diag}(A_{0,1},\dots,A_{0,p})u = 0\big\}$ and $\mathcal{L} = \big\{\begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} \in \mathbb{C}^{pN} \mid (A_{0,1} + \cdots + A_{0,p} + \mu)v = 0\big\}$, we have $\tilde{A}_{i,j}(\mathcal{K}+\mathcal{L}) \subset (\mathcal{K}+\mathcal{L})$ for $0 \le i < j \le p$.

The above $\tilde{A}_{i,j}$ are obtained as follows (cf. [3]). Putting $u_j = \frac{u}{x_0 - x_j}$ and $\tilde{u}_j = I_{\mu+1}(u_j)$, we have

$$\begin{aligned}\frac{\partial u_j}{\partial x_i} &= \frac{1}{x_0 - x_j}\frac{\partial u_j}{\partial x_i} + \delta_{i,j}\Big(\frac{\partial}{\partial x_i}\frac{1}{x_0 - x_i}\Big)u \\ &= \frac{A_{i,0}u}{(x_0 - x_j)(x_i - x_0)} + \sum_{1\le\nu\le p,\,\nu\ne i}\frac{A_{i,\nu}u_j}{x_i - x_\nu} - \delta_{i,j}\Big(\frac{\partial}{\partial x_0}\frac{1}{x_0 - x_i}\Big)u.\end{aligned}$$

If $i \ne j$, then

$$\frac{u}{(x_0 - x_i)(x_0 - x_j)} = \frac{1}{x_i - x_j}\Big(\frac{u}{x_0 - x_i} - \frac{u}{x_0 - x_j}\Big) = \frac{u_i - u_j}{x_i - x_j}$$

and

$$\frac{\partial \tilde{u}_j}{\partial x_i} = \frac{A_{i,0}\tilde{u}_j}{x_i - x_j} - \frac{A_{i,0}\tilde{u}_i}{x_i - x_j} + \sum_{0\le\nu\le p,\,\nu\ne i}\frac{A_{i,\nu}\tilde{u}_j}{x_i - x_\nu} \quad (i \ne j). \tag{7.8}$$

If $i = j$, we have

$$I_{\mu+1}\Big(\big(\frac{\partial}{\partial x_0}\frac{1}{x_0 - x_i}\big)u\Big) = \frac{\partial}{\partial x_0}I_{\mu+1}(u_i) - I_{\mu+1}\Big(\frac{1}{x_0 - x_i}\frac{\partial u}{\partial x_0}\Big)$$

$$= \frac{\partial \tilde{u}_i}{\partial x_0} - I_{\mu+1}\Big(\sum_{\nu=1}^{p} \frac{A_{0,\nu} u}{(x_0 - x_i)(x_0 - x_\nu)}\Big)$$
$$= \frac{\partial \tilde{u}_i}{\partial x_0} - I_{\mu+1} \frac{A_{0,i} u}{(x_0 - x_i)^2} - \sum_{1\le\nu\le p,\, \nu\ne i} I_{\mu+1}\Big(\frac{A_{0,\nu}(u_i - u_\nu)}{x_i - x_\nu}\Big)$$

and

$$\frac{\partial \tilde{u}_i}{\partial x_i} = \sum_{\nu=1}^{p} \frac{A_{i,0} + \mu\delta_{i,\nu}}{x_i - x_0}\tilde{u}_\nu + \sum_{1\le\nu\le p,\, \nu\ne i} \frac{A_{0,\nu}(\tilde{u}_i - \tilde{u}_\nu)}{x_i - x_\nu} + \sum_{1\le\nu\le p,\, \nu\ne i} \frac{A_{i,\nu}\tilde{u}_i}{x_i - x_\nu}. \tag{7.9}$$

Example 7.4. Note that the convolution of the KZ equation

$$du = \sum_{i=0}^{3}\Big(\sum_{\nu\in\{0,1,2,3\}\setminus\{i\}} \frac{C_{i,\nu}}{x_i - x_\nu}\Big) u dx_i \tag{7.10}$$

is given by Definition 7.3, which has the generator $\tilde{u}$.

For $i = 0$ or $i = 3$ we have

$$\begin{aligned}\frac{\partial u}{\partial x_i} &= \Big(\frac{A_{i,1}}{x_i - x_1} + \frac{A_{i,2}}{(x_i - x_1)(x_i - x_2)} + \frac{A_{i,3-i}}{x_i - x_{3-i}}\Big)u \\ &= \Big(\frac{C_{i,1}}{x_i - x_1} + \frac{C_{i,2}}{x_i - x_2} + \frac{C_{i,3-i}}{x_i - x_{3-i}}\Big)u.\end{aligned} \tag{7.11}$$

Here

$$C_{i,1} = A_{i,1} + \frac{A_{i,2}}{x_1 - x_2},\ C_{i,2} = \frac{A_{i,2}}{x_2 - x_1},\ C_{i,3-i} = A_{i,3-i},\ \hat{u} = T\hat{u},$$
$$S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & x_2 - x_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ T := S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{x_1 - x_2} & \frac{1}{x_2 - x_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We rewrite the convolution according to the generator $\tilde{u}$ defined in §5 for the ordinary differential equation (7.11) with $i = 0$. Hence

$$\frac{\partial \hat{u}}{\partial x_2} = \hat{A}_3(x)\hat{u} \ \text{ with } \ \hat{A}_3(x) = T\Big(\frac{\tilde{C}_{3,0}}{x_3 - x_0} + \frac{\tilde{C}_{3,1}}{x_3 - x_1} + \frac{\tilde{C}_{3,2}}{x_3 - x_2}\Big)S.$$

Putting

$$\hat{A}_3(x) = \frac{\hat{A}_{3,0}}{x_3 - x_0} + \frac{\hat{A}_{3,1}}{x_3 - x_1} + \frac{\hat{A}_{3,2}}{(x_3 - x_1)(x_3 - x_2)},$$

we have

$$
\begin{aligned}
\hat{A}_{3,0} &= T\tilde{C}_{0,3}S = \hat{A}_{0,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{0,1} & A_{0,2} & A_{0,3}+\mu \end{pmatrix}, \\
\hat{A}_{3,1} &= T\bigl(\tilde{C}_{3,1}+\tilde{C}_{3,2}\bigr)S \\
&= T\begin{pmatrix} C_{3,1}+C_{3,2}+C_{3,0} & 0 & -C_{3,0} \\ 0 & C_{3,1}+C_{3,2}+C_{3,0} & -C_{3,0} \\ -C_{1,0} & -C_{2,0} & C_{3,1}+C_{3,2}+C_{1,0}+C_{2,0} \end{pmatrix} S \\
&= \begin{pmatrix} A_{3,1}+A_{3,0} & 0 & -A_{3,0} \\ 0 & A_{3,1}+A_{3,0} & 0 \\ -A_{1,0} & -A_{2,0} & A_{3,1}+A_{1,0} \end{pmatrix}, \\
\hat{A}_{3,2} &= T\bigl((x_2-x_1)\tilde{C}_{3,2}\bigr)S = (x_2-x_1)T\begin{pmatrix} C_{3,2} & 0 & 0 \\ 0 & C_{3,2}+C_{3,0} & -C_{3,0} \\ 0 & -C_{2,0} & C_{3,2}+C_{2,0} \end{pmatrix} S \\
&= \begin{pmatrix} A_{3,2} & 0 & 0 \\ A_{3,0} & A_{3,2}+(x_2-x_1)A_{0,3} & -A_{0,3} \\ -A_{2,0} & (x_1-x_2)A_{0,2} & A_{3,2}+A_{0,2} \end{pmatrix}.
\end{aligned}
$$

Hence $\hat{A}_{3,j}(x)$ holomorphically depend on the difference of (x_1, x_2) if so are $A_{3,j}(x)$.

Next we examine an example which regards an irregular singular point as a variable. Put $(x, x_1, x_2, x_3) = (x, y+e_1, y+e_2, x_3)$ for the equation (7.10) and fix x_0, x_3, e_1 and e_2 as constants. Then the Pfaffian system (7.10) equals

$$
\begin{aligned}
du &= \frac{C_{01}u}{x_1-x_0}dx_1 + \frac{C_{12}u}{x_1-x_2}dx_1 + \frac{C_{13}u}{x_1-x_3}dx_1 + \frac{C_{02}u}{x_2-x_0}dx_2 + \frac{C_{12}u}{x_2-x_1}dx_2 + \frac{C_{23}u}{x_2-x_3}dx_2 \\
&= \frac{C_{01}u}{y+e_1-x_0}dy + \frac{C_{13}u}{y+e_1-x_3}dy + \frac{C_{02}u}{y+e_2-x_0}dy + \frac{C_{23}u}{y+e_2-x_3}dy \\
&= \sum_{i\in\{0,3\}} \Bigl(\frac{C_{i1}+C_{i2}}{y+e_1-x_i} - \frac{s_{21}C_{i2}}{(y+e_1-x_i)(y+e_2-x_i)}\Bigr)udy \quad \text{with} \quad s_{21} = e_2 - e_1
\end{aligned}
$$

and Definition 7.3 implies

$$
\begin{aligned}
d\tilde{u} = \Biggl(&\frac{\left(\begin{smallmatrix} C_{01}+\mu & C_{02} & C_{03} \\ C_{01} & C_{02}+\mu & C_{03} \\ 0 & 0 & 0 \end{smallmatrix}\right)}{y+e_1-x_0} - \frac{s_{21}\left(\begin{smallmatrix} 0 & 0 & 0 \\ C_{01} & C_{02}+\mu & C_{03} \\ 0 & 0 & 0 \end{smallmatrix}\right)}{(y+e_1-x_0)(y+e_2-x_0)} \\
&+ \frac{\left(\begin{smallmatrix} C_{03}+C_{13}+C_{23} & 0 & -C_{03} \\ 0 & C_{03}+C_{13}+C_{23} & -C_{03} \\ -C_{01} & -C_{02} & C_{01}+C_{02}+C_{03} \end{smallmatrix}\right)}{y+e_1-x_3} - \frac{s_{21}\left(\begin{smallmatrix} C_{23} & 0 & 0 \\ 0 & C_{23}+C_{03} & -C_{03} \\ 0 & -C_{02} & C_{23}+C_{02} \end{smallmatrix}\right)}{(y+e_1-x_3)(y+e_2-x_3)}\Biggr)\tilde{u}dy.
\end{aligned}
$$

For simplicity we allow to denote C_{ij} and A_{ij} in place of $C_{i,j}$ and $A_{i,j}$, respectively, if we do not expect any confusion. Put

$$
A_{1i} = C_{i1} + C_{i2},\ A_{2,i} = -s_{21}C_{i2} \quad (i = 0,\,3) \quad \text{and} \quad S = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & s_{21} & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right).
$$

Using $\hat{u} = S^{-1}\tilde{u}$, we have

$$d\hat{u} = \sum_{i\in\{0,3\}} \Big(\frac{\hat{A}_{1,i}}{y+e_1-x_i} - \frac{\hat{A}_{2,i}}{(y+e_1-x_i)(y+e_2-x_i)}\Big)\hat{u}dy$$

with

$$\hat{A}_{10} = \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \hat{A}_{20} = \begin{pmatrix} 0 & 0 & 0 \\ A_{01}+\mu & A_{02}+(e_2-e_1)\mu & A_{03} \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.12}$$

$$\hat{A}_{13} = \begin{pmatrix} A_{13}+A_{03} & 0 & -A_{03} \\ 0 & A_{13}+A_{03} & 0 \\ -A_{01} & -A_{02} & A_{13}+A_{01} \end{pmatrix}, \tag{7.13}$$

$$\hat{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 \\ A_{03} & A_{23}+(e_2-e_1)A_{03} & -A_{03} \\ -A_{02} & -(e_2-e_1)A_{02} & A_{23}+A_{02} \end{pmatrix}. \tag{7.14}$$

Combining this result with Example 7.4 we have the following example.

Example 7.5. The convolution of the versal KZ equation

$$\begin{cases} \dfrac{\partial u}{\partial x_0} = \Big(\dfrac{A_{0,1}}{x_0-x_1} + \dfrac{A_{0,2}}{(x_0-x_1)(x_0-x_1-e_2)} + \dfrac{A_{0,3}}{x_0-x_3}\Big)u, \\ \dfrac{\partial u}{\partial x_1} = \Big(\dfrac{A_{1,0}}{x_1-x_0} - \dfrac{A_{2,0}}{(x_1-x_0)(x_1-x_0+e_2)} \\ \qquad\qquad + \dfrac{A_{1,3}}{x_1-x_3} - \dfrac{A_{2,3}}{(x_1-x_3)(x_1-x_3+e_2)}\Big)u, \\ \dfrac{\partial u}{\partial x_3} = \Big(\dfrac{A_{3,0}}{x_3-x_0} + \dfrac{A_{3,1}}{x_3-x_1} + \dfrac{A_{3,2}}{(x_3-x_1)(x_3-x_1-e_2)}\Big)u \end{cases} \tag{7.15}$$

with

$$A_{i,j} = A_{j,i} \quad (0 \le i < j \le 3) \tag{7.16}$$

is obtained by replacing $(u, A_{i,j})$ by $(\hat{u}, \hat{A}_{i,j})$ in (7.15), where $\hat{A}_{i,j}$ satisfy the same relation as (7.16) and are given by (7.12), (7.13), (7.14) with $e_1 = 0$ and

$$\hat{A}_{03} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{01} & A_{02} & A_{03}+\mu \end{pmatrix}. \tag{7.17}$$

If $A_{i,j}$ are complex numbers, the above convolution gives the equation of Appell's F_1 and its confluence.

Putting $x_3 = a^{-1}$, the generator u of the Pfaffian system (7.10) satisfies

$$\begin{aligned}
&\frac{\partial u}{\partial x_0} = \Big(\frac{C_{0,1}}{x_0 - x_1} + \frac{C_{0,2}}{x_0 - x_2} + \frac{C_{0,3}}{x_0 - a^{-1}}\Big)u = \Big(\frac{A_{0,1}}{x_0 - x_1} + \frac{A_{0,2}}{x_0 - x_2} - \frac{A_{0,3}}{1 - ax_0}\Big)u,\\
&\frac{\partial u}{\partial x_1} = \Big(\frac{C_{1,0}}{x_1 - x_0} + \frac{C_{1,2}}{x_1 - x_2} + \frac{C_{1,3}}{x_1 - a^{-1}}\Big)u = \Big(\frac{A_{1,0}}{x_1 - x_0} + \frac{A_{1,2}}{x_1 - x_2} - \frac{A_{1,3}}{1 - ax_1}\Big)u,\\
&A_{0,2} = C_{0,2},\ A_{i,1-i} = C_{i,1-i},\ A_{03} = aC_{0,3} \quad (i = 0,\, 1).
\end{aligned}$$

Put $S = \mathrm{Diag}(1, 1, a)$ and $\hat{u} = S\tilde{u}$ with the generator $\tilde{u}$ of the convolution of the system (7.10) and

$$\frac{\partial \hat{u}}{\partial x_1} = \Big(\frac{\hat{A}_{1,0}}{x_1 - x_0} + \frac{\hat{A}_{1,2}}{x_1 - x_2} - \frac{\hat{A}_{1,3}}{1 - ax_1}\Big)\hat{u}.$$

Then we have

$$\hat{A}_{10} = S^{-1}\begin{pmatrix} C_{01} + \mu & C_{02} & C_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} S = \begin{pmatrix} A_{01} + \mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.18}$$

$$\hat{A}_{12} = S^{-1}\begin{pmatrix} C_{12} + C_{02} & -C_{02} & 0 \\ -C_{01} & C_{12} + C_{01} & 0 \\ 0 & 0 & C_{12} \end{pmatrix} S = \begin{pmatrix} A_{12} + A_{02} & -A_{02} & 0 \\ -A_{01} & A_{12} + A_{01} & 0 \\ 0 & 0 & A_{12} \end{pmatrix}, \tag{7.19}$$

$$\hat{A}_{13} = aS^{-1}\begin{pmatrix} C_{13} + C_{03} & 0 & -C_{03} \\ 0 & C_{13} & 0 \\ -C_{01} & 0 & C_{13} + C_{01} \end{pmatrix} S = \begin{pmatrix} A_{13} + A_{03} & 0 & -aA_{03} \\ 0 & A_{13} & 0 \\ -A_{01} & 0 & A_{13} + aA_{01} \end{pmatrix}. \tag{7.20}$$

Thus we have the following example.

Example 7.6. The convolution of the KZ equation

$$\begin{cases}
\dfrac{\partial u}{\partial x_0} = \Big(\dfrac{A_{0,1}}{x_0 - x_1} + \dfrac{A_{0,2}}{x_0 - x_2} - \dfrac{A_{0,3}}{1 - ax_0}\Big)u,\\
\dfrac{\partial u}{\partial x_1} = \Big(\dfrac{A_{1,0}}{x_1 - x_0} + \dfrac{A_{1,2}}{x_1 - x_2} - \dfrac{A_{1,3}}{1 - ax_1}\Big)u,\\
\dfrac{\partial u}{\partial x_2} = \Big(\dfrac{A_{2,0}}{x_2 - x_0} + \dfrac{A_{2,1}}{x_1 - x_2} - \dfrac{A_{2,3}}{1 - ax_2}\Big)u,
\end{cases} \tag{7.21}$$

$$A_{i,j} = A_{j,i} \quad (0 \le i < j \le 3)$$

is obtained by replacing $(u, A_{i,j})$ by $(\hat{u}, \hat{A}_{i,j})$. Here $\hat{A}_{i,j}$ are given by (7.18), (7.19), (7.20) and

$$\hat{A}_{02} = \begin{pmatrix} 0 & 0 & 0 \\ A_{01}+\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{A}_{03} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{01} & A_{02} & A_{03}+a\mu \end{pmatrix},$$

$$\hat{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{23}+A_{03} & -aA_{03} \\ 0 & -A_{02} & A_{23}+aA_{02} \end{pmatrix}.$$

Remark 7.7. If $A_{i,j}$ are complex numbers in (7.21), the above convolution gives the equation of Appell's F_1 and its confluence, which corresponds to the convolution in Example 7.5 by the correspondence $x \mapsto \frac{1}{x-x_1} + x_1$.

The following example have two irregular singular points.

Example 7.8. We examine the extension of the rigid equation (1.2) with $p = 2$, $r_0 = 1$, $r_1 = 2$ and $r_2 = 3$, namely,

$$\frac{du}{dx} = \Bigl(\sum_{\nu=1}^{2} \frac{A_{0,\nu}}{(x-x_1)^\nu} + \sum_{\nu=1}^{3} \frac{A_{0,\nu+2}}{(x-x_2)^\nu}\Bigr)u.$$

Then the extension is

$$(7.22) \qquad \begin{cases} \dfrac{\partial u}{\partial x_0} = \Bigl(\displaystyle\sum_{\nu=1}^{2} \frac{A_{0,\nu}}{(x_0-x_1)^\nu} + \sum_{\nu=1}^{3} \frac{A_{0,\nu+2}}{(x_0-x_2)^\nu}\Bigr)u, \\ \dfrac{\partial u}{\partial x_1} = \Bigl(\displaystyle\sum_{\nu=1}^{2} \frac{(-1)^{\nu-1}A_{\nu,0}}{(x_1-x_0)^\nu} + \sum_{\nu=1}^{4} \frac{A_{1,\nu+2}}{(x_1-x_2)^\nu}\Bigr)u, \\ \dfrac{\partial u}{\partial x_2} = \Bigl(\displaystyle\sum_{\nu=1}^{3} \frac{(-1)^{\nu-1}A_{\nu+2,0}}{(x_2-x_0)^\nu} + \sum_{\nu=1}^{4} \frac{(-1)^{\nu-1}A_{\nu+2,1}}{(x_2-x_1)^\nu}\Bigr)u, \end{cases}$$
$$A_{i,j} = A_{j,i}$$

with $x_0 = x$, which is obtained by successive applications of middle convolutions and additions to the trivial equation. It is calculated through the convolution of the above equation given by

$$\hat{A}_{01} = \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} & A_{04} & A_{05} \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{A}_{02} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ A_{01}+\mu & A_{02} & A_{03} & A_{04} & A_{05} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{A}_{03} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_{01} & A_{02} & A_{03}+\mu & A_{04} & A_{05} \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & \mu \end{pmatrix}, \quad \hat{A}_{04} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_{01} & A_{02} & A_{03}+\mu & A_{04} & A_{05} \\ 0 & 0 & 0 & \mu & 0 \end{pmatrix},$$

$$\hat{A}_{05} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_{01} & A_{02} & A_{03}+\mu & A_{04} & A_{05} \end{pmatrix},$$

$$\hat{A}_{13} = \begin{pmatrix} A_{13}+A_{03} & 0 & -A_{03} & -A_{0,4} & -A_{05} \\ 0 & A_{13}+A_{03} & 0 & 0 & 0 \\ -A_{01} & -A_{02} & A_{13}+A_{01} & 0 & 0 \\ 0 & 0 & 0 & A_{13}+A_{01} & A_{02} \\ 0 & 0 & 0 & 0 & A_{13}+A_{01} \end{pmatrix},$$

$$\hat{A}_{14} = \begin{pmatrix} A_{14}+A_{04} & 0 & -A_{04} & -A_{05} & 0 \\ -A_{03} & A_{14}+A_{04} & A_{03} & A_{04} & A_{05} \\ A_{02} & 0 & A_{14}-A_{02} & 0 & 0 \\ -A_{01} & -A_{02} & A_{14}+A_{01} & -A_{02} & 0 \\ 0 & 0 & 0 & A_{01} & A_{14}-A_{02} \end{pmatrix},$$

$$\hat{A}_{15} = \begin{pmatrix} A_{15}+A_{05} & 0 & -A_{05} & 0 & 0 \\ -2A_{04} & A_{15}+A_{05} & 2A_{04} & 2A_{05} & 0 \\ 0 & 0 & A_{15} & 0 & 0 \\ 2A_{02} & 0 & -2A_{02} & A_{15} & 0 \\ -A_{01} & -A_{02} & A_{01} & -2A_{02} & A_{15} \end{pmatrix}, \quad \hat{A}_{16} = \begin{pmatrix} A_{16} & 0 & 0 & 0 & 0 \\ -3A_{05} & A_{16} & 3A_{05} & 0 & 0 \\ 0 & 0 & A_{16} & 0 & 0 \\ 0 & 0 & 0 & A_{16} & 0 \\ 3A_{02} & 0 & -3A_{02} & 0 & A_{16} \end{pmatrix}.$$

This result is obtained by the versal unfolding and its extension to a KZ equation explained in this section. Here we only give the result restricting the parameters of the equation with irregular singularities. The explicit expression of the versal unfolding of the system such as (7.22) is a little complicated in general, which is calculated by the function `mcvm()` in [17].

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