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A characterization of the monodromy group of Gauss hypergeometric equation

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Abstract. We give a characterization of the monodromy group of the second order linear Fuchsian differential equation on the Riemann sphere which has three singular points.

1. Introduction

Hilbert's twenty-first problem asks the existence of a linear differential equation of Fuchsian class with given singular points and monodromy group. Since the existence is not always true when the equation is single, the original problem was formulated as the problem of the existence of the first order Fuchsian system of Schlesinger canonical form with given singular points and monodromy group, which is called Riemann-Hilbert problem. Plemelj (cf. [Pl]) and independently Birkhoff gave affirmative answers. But their arguments were not sufficient and in 1990 Bolibrukh [Bo] gave a counterexample. Then the problem was affirmatively proved by Bolibrukh [Bo2] and Kostov [Ko] if the the monodromy group is irreducible.

In this paper we consider single linear Fuchsian differential equations of higher order. When the spectral type of the monodromy group is not rigid, the problem is not true in general because the number of accessory parameters is not sufficient. But it is proved in [O1] that the problem is affirmative if the monodromy group is rigid and irreducible. Also it is shown in [O1, Example 2.2] that the problem is not affirmative if the monodromy group is 'rigid' but reducible.

In this paper we give a characterization of the monodromy group of a Fuchsian differential equation of the second order with three singular points. In this case the equation is essentially Gauss hypergeometric equation

$$x(1-x)u'' + \big(\gamma - (\alpha+\beta+1)x\big)u' - \alpha\beta u = 0 \tag{1}$$

and hence the result is classically known (cf. [IKSY, Chapter 2, Corollary 4.3.4], [KS]). Here we give it by a simple argument based on a result in [O2], which

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studies Gauss hypergeometric equation only by an elementary calculus without any integration.

2. A characterization

Let

$$u'' + a(x)u' + b(x)u = 0 \tag{2}$$

be a Fuchsian differential equation of the second order with three singular points c_0, c_1 and c_2 in the Riemann sphere. To study the monodromy group of this equation we may assume $(c_0, c_1, c_2) = (0, 1, \infty)$ by a linear fractional transformation. Then $x(1-x)a(x)$ and $x^2(1-x)^2b(x)$ are polynomials of degree at most 1 and 2, respectively. Hence the equation (2) has 5 parameters.

Let

$$\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} & ;\ x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} & \end{Bmatrix} \tag{3}$$

be the Riemann scheme of the equation (2). Then we have the Fuchs relation

$$\lambda_{0,1} + \lambda_{0,2} + \lambda_{1,1} + \lambda_{1,2} + \lambda_{\infty,1} + \lambda_{\infty,2} = 1. \tag{4}$$

Since Gauss hypergeometric equation is characterized by its Riemann scheme

$$\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & \alpha & ;\ x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix},$$

the equation (2) is obtained from (1) by the gauge transformation

$$u \mapsto x^{\lambda}(1-x)^{\mu}u$$

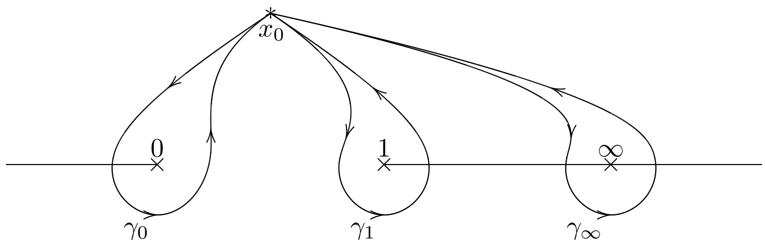
with

$$\begin{cases} \lambda_{0,1} = \lambda, & \lambda_{1,1} = \mu, & \lambda_{\infty,1} = \alpha - \lambda - \mu, \\ \lambda_{0,2} = 1-\gamma+\lambda, & \lambda_{1,2} = \gamma-\alpha-\beta+\mu, & \lambda_{\infty,2} = \beta - \lambda - \mu \end{cases} \tag{5}$$

and the explicit form of (2) is

$$
\begin{aligned}
(6)\qquad & u'' - \frac{\lambda_{0,1}+\lambda_{0,2}-1}{x}u' + \frac{\lambda_{1,1}+\lambda_{1,2}-1}{1-x}u' \\
& + \frac{\lambda_{0,1}\lambda_{0,2}}{x^2}u + \frac{\lambda_{1,1}\lambda_{1,2}}{(1-x)^2}u + \frac{\lambda_{0,1}\lambda_{0,2}+\lambda_{1,1}\lambda_{1,2}-\lambda_{\infty,1}\lambda_{\infty,2}}{x(1-x)}u = 0.
\end{aligned}
$$

Let (u_1, u_2) be a base of local solutions of the equation (6) at a generic point x_0 as in the following figure. Let γ_p be closed paths starting from x_0 and circling around the point $x = p$ once in a counterclockwise direction for $p = 0$, 1 and ∞, respectively, as follows.



Let $\gamma_p u_j$ be the local solutions in a neighborhood of x_0 obtained by the analytic continuation of u_j along γ_p, respectively. Then there exist $M_p \in GL(2,\mathbb{C})$ satisfying $(\gamma_p u_1, \gamma_p u_2) = (u_1, u_2)M_p$. Here $GL(2,\mathbb{C})$ is the group of invertible matrices of size 2 with entries in $\mathbb{C}$. The matrices M_p are called the *local generator matrices of monodromy* of the equation (6) and the subgroup of $GL(2,\mathbb{C})$ generated by M_0, M_1 and M_∞ is called the *monodromy group*. We note that the eigenvalues of M_p are $e^{2\pi\sqrt{-1}\lambda_{p,1}}$ and $e^{2\pi\sqrt{-1}\lambda_{p,2}}$ and moreover we have

$$
(7)\qquad M_\infty M_1 M_0 = I_2 \text{ (the identity matrix)}
$$

and if we differently choose x_0 and (u_1, u_2), the set of local generator matrices of monodromy (M_0, M_1, M_∞) changes into $(gM_0g^{-1}, gM_1g^{-1}, gM_\infty g^{-1})$ with a certain $g \in GL(2,\mathbb{C})$. If there exists a subspace V of $\mathbb{C}^2$ such that $\{0\} \subsetneqq V \subsetneqq \mathbb{C}^2$ and $M_pV \subset V$ for $p = 0$, 1, ∞, then we say that the monodromy of the equation (6) is *reducible*. If it is not reducible, it is called *irreducible*.

Definition 2.1. *For $(A_0, A_1, A_2) \in GL(2,\mathbb{C})^3$ we put*

$$
C_{A_0,A_1,A_2} := \{(gA_0g^{-1}, gA_1g^{-1}, gA_2g^{-1}) \mid g \in GL(2,\mathbb{C}))\}
$$

and

$$\widetilde{M} := \left\{ C_{M_0,M_1,M_\infty} \;\middle|\; \begin{array}{c} M_0,\ M_1 \text{ and } M_\infty \text{ are local generator matrices of} \\ \text{the equation } (6) \text{ with } \lambda_{i.j} \in \mathbb{C} \text{ satisfying } (4) \end{array} \right\}$$

under the above notation.

Then the following theorem is our characterization of $\widetilde{M}$.

THEOREM 2.2. *Under the above notation, $\widetilde{M}$ is a subset of*

$$\widetilde{X} := \{C_{A_0,A_1,A_2} \mid A_i \in GL(2,\mathbb{C}) \ \ (i = 0,1,2) \text{ and } A_2A_1A_0 = I_2\}$$

characterized by

$$\begin{aligned} \widetilde{X} \setminus \widetilde{M} = &\left\{ C_{A_0,A_1,A_2} \;\middle|\; A_i = \begin{pmatrix} a_i & 0 \\ 0 & a_i' \end{pmatrix},\ a_i \neq a_i' \ \ (i = 0,1,2) \text{ and } \ A_2A_1A_0 = I_2 \right\} \\ &\cup \left\{ C_{A_0,A_1,A_2} \;\middle|\; A_i = \begin{pmatrix} a_i & b_i \\ 0 & a_i \end{pmatrix},\ b_i \neq 0 \ \ (i = 0,1,2) \text{ and } \ A_2A_1A_0 = I_2 \right\}. \end{aligned}$$

Remark 2.3. i) *The above theorem is equivalent to* [IKSY, Chapter 2, Corollary 4.3.4].

ii) *The characterization of the monodromy group of the Gauss hypergeometric equation* (1) *is obtained by Theorem 2.2 imposing the condition*

$$\operatorname{rank}(A_i - 1) \leq 1 \text{ for } i = 0 \text{ and } 1. \tag{8}$$

3. Proof of the theorem

We will show Theorem 2.2 by the following result in [O2].

THEOREM 3.1 ([O2, Theorem 8]). *Retain the notation in the previous section. Let M_0, M_1 and M_∞ be local monodromy matrices of the equation* (6) *with* (4).

i) (M_0, M_1, M_∞) *is irreducible if and only if*

$$\lambda_{0,1} + \lambda_{1,\nu} + \lambda_{\infty,\nu'} \notin \mathbb{Z} \quad (\forall \nu,\ \nu' \in \{1,2\}). \tag{9}$$

ii) *Suppose*

$$\lambda_{0,2} + \lambda_{1,2} + \lambda_{\infty,\nu} \notin \{0,-1,-2,\ldots\} \quad (\nu = 1,\ 2). \tag{10}$$

We may assume

$$\lambda_{p,1} - \lambda_{p,2} \notin \{1, 2, 3, \ldots\} \quad (p = 0,\ 1) \tag{11}$$

by one or both of the permutations $\lambda_{0,1} \leftrightarrow \lambda_{0,2}$ *and* $\lambda_{1,1} \leftrightarrow \lambda_{1,2}$ *if necessary.*

When

$$\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,\nu} \notin \mathbb{Z} \quad (\nu = 1,\ 2), \tag{12}$$

there exists $g \in GL(2, \mathbb{C})$ *such that the monodromy matrices satisfy*

$$(gM_0g^{-1}, gM_1g^{-1}) = \left(\begin{pmatrix} e^{2\pi i\lambda_{0,2}} & b_0 \\ 0 & e^{2\pi i\lambda_{0,1}} \end{pmatrix}, \begin{pmatrix} e^{2\pi i\lambda_{1,1}} & 0 \\ b_1 & e^{2\pi i\lambda_{1,2}} \end{pmatrix}\right) \tag{13}$$

with

$$\begin{aligned} b_0 &= 2e^{-\pi i\lambda_{\infty,2}} \sin \pi(\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,2}), \\ b_1 &= 2e^{-\pi i\lambda_{\infty,1}} \sin \pi(\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,1}). \end{aligned} \tag{14}$$

When (12) *is not valid, we have* (13) *with a certain* $g \in GL(2, \mathbb{C})$ *and*

$$\begin{aligned} b_0 &= \begin{cases} 1 & \textit{if}\ \ \lambda_{0,1} + \lambda_{1,2} + \lambda_{\infty,\nu} \notin \{0, -1, -2, \ldots\} \quad (\nu = 1,\ 2), \\ 0 & \textit{otherwise}, \end{cases} \\ b_1 &= \begin{cases} 1 & \textit{if}\ \ \lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,\nu} \notin \{0, -1, -2, \ldots\} \quad (\nu = 1,\ 2), \\ 0 & \textit{otherwise}. \end{cases} \end{aligned} \tag{15}$$

Note that the Fuchs relation (4) *implies* $b_0b_1 = 0$ *in this case.*

iii) *Under a change of indices* $\lambda_{p,\nu} \mapsto \lambda_{\sigma(p),\sigma_p(\nu)}$ *with suitable permutations* $(\sigma, \sigma_0, \sigma_1, \sigma_\infty) \in \mathfrak{S}_3 \times \mathfrak{S}_2^3$ *we have* (10) *and* (11). *Here* $\mathfrak{S}_3$ *and* $\mathfrak{S}_2$ *are identified with the permutation groups of* $\{0, 1, \infty\}$ *and* $\{1, 2\}$*, respectively.*

Proof of Theorem 2.2. We say $(A_0, A_1, A_2) \in GL(2, \mathbb{C})^3$ is irreducible if and only if there exists no subspace V of $\mathbb{C}^2$ such that $\{0\} \subsetneqq V \subsetneqq \mathbb{C}^2$ and $A_pV \subset V$ for $p = 0,\ 1,\ 2$.

Let $(A_0, A_1, A_\infty) \in GL(2, \mathbb{C})^3$ with $A_\infty A_1 A_0 = I_2$. We can choose $\lambda_{i,j} \in \mathbb{C}$ such that they satisfy (4) and the set of eigenvalues of A_i are $\{e^{2\pi\sqrt{-1}\lambda_{i,1}}, e^{2\pi\sqrt{-1}\lambda_{i,2}}\}$ for $i = 0$, 1 and ∞, respectively.

First we suppose (A_0, A_1, A_∞) is irreducible. We will prove $C_{A_0,A_1,A_\infty} \in \widetilde{M}$. Since the eigenvector v_0 of A_0 with the eigenvalue $e^{2\pi\sqrt{-1}\lambda_{0,2}}$ and the eigenvector v_1 of A_1 with the eigenvalue $e^{2\pi\sqrt{-1}\lambda_{1,2}}$ are linearly independent, there exists $g \in$

$GL(2,\mathbb{C})$ satisfying

$$(16)\quad (gA_0g^{-1}, gA_1g^{-1}) = \left(\begin{pmatrix} e^{2\pi\sqrt{-1}\lambda_{0,2}} & b_0 \\ 0 & e^{2\pi\sqrt{-1}\lambda_{0,1}} \end{pmatrix}, \begin{pmatrix} e^{2\pi\sqrt{-1}\lambda_{1,1}} & 0 \\ b_1 & e^{2\pi\sqrt{-1}\lambda_{1,2}} \end{pmatrix}\right)$$

with suitable complex numbers b_0 and b_1. Here the irreducibility of (A_0, A_1, A_∞) implies $b_0b_1 \neq 0$. Since $\operatorname{trace} A_1A_0 = \operatorname{trace} A_\infty^{-1}$, we have

$$\begin{aligned}
\operatorname{trace} A_1A_0 &= e^{2\pi i(\lambda_{0,2}+\lambda_{1,1})} + b_0b_1 + e^{2\pi i(\lambda_{0,1}+\lambda_{1,2})} = e^{-2\pi i\lambda_{\infty,1}} + e^{-2\pi i\lambda_{\infty,2}}, \\
b_0b_1 &= e^{-2\pi i\lambda_{\infty,1}} + e^{-2\pi i\lambda_{\infty,2}} - e^{2\pi i(\lambda_{0,2}+\lambda_{1,1})} - e^{2\pi i(\lambda_{0,1}+\lambda_{1,2})} \\
&= e^{-2\pi i\lambda_{\infty,2}}(e^{2\pi i(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,2})} - 1)(e^{2\pi i(\lambda_{0,1}+\lambda_{1,2}+\lambda_{\infty,2})} - 1) \\
&= e^{\pi i(\lambda_{0,1}+\lambda_{0,2}+\lambda_{1,1}+\lambda_{1,2})}\bigl(2i\sin\pi(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,2})\bigr) \\
&\quad \cdot \bigl(2i\sin\pi(\lambda_{0,1}+\lambda_{1,2}+\lambda_{\infty,2})\bigr) \\
&= 4e^{-\pi i(\lambda_{\infty,1}+\lambda_{\infty,2})}\sin\pi(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,2})\sin\pi(\lambda_{0,2}+\lambda_{1,1}+\lambda_{\infty,1}).
\end{aligned}$$

The condition $b_0b_1 \neq 0$ implies that in (16) we may choose b_0 and b_1 by (14) for a suitable matrix $g \in GL(2,\mathbb{C})$. Hence Theorem 3.1 ii) assures $C_{A_0,A_1,A_\infty} \in \widetilde{M}$.

Now we suppose (A_0, A_1, A_∞) is reducible. Then there exists a simultaneous eigenvector v_0 of A_0 and A_1, therefore

$$\begin{aligned}
\widetilde{X}' &:= \{C_{A_0,A_1,A_2} \mid (A_0,A_1,A_2) \in GL(2,\mathbb{C})^3 \text{ is reducible and } A_2A_1A_0 = I_2\} \\
&= \left\{ C_{A_0,A_1,A_2} \,\middle|\, A_i = \begin{pmatrix} a_i & b_i \\ 0 & a_i' \end{pmatrix} \quad (i=0,1,2) \text{ and } A_2A_1A_0 = I_2 \right\}
\end{aligned}$$

and $\widetilde{X} \setminus \widetilde{M} \subset \widetilde{X}'$. Note that if we fix $i \in \{0,1,2\}$ and there exists an eigenvector v_1 of A_i with $v_1 \notin \mathbb{C}v_0$, we may assume that A_i is a diagonal matrix in the above.

According to the number $N = \#\bigl\{i \in \{0,1,2\} \mid a_i = a_i'\bigr\}$, we can divide $\widetilde{X}'$ into $\widetilde{X}_N$ with $N = 0$, 1 and 3:

$$\widetilde{X}' = \widetilde{X}_0 \sqcup \widetilde{X}_1 \sqcup \widetilde{X}_3.$$

Suppose $C_{A_0,A_1,A_2} \in \widetilde{X}_0$. Then A_1 is diagonalizable and according to the simultaneous diagonaizability of (A_0, A_1, A_2), we divide $\widetilde{X}_0$ as follows.

$$\widetilde{X}_0 := \widetilde{X}_{0,0} \sqcup \widetilde{X}_{0,1},$$

$$\widetilde{X}_{0,\nu} := \left\{ C_{A_0,A_1,A_2} \;\middle|\; \begin{array}{l} A_i = \begin{pmatrix} a_i & b_i \\ 0 & a_i' \end{pmatrix},\ a_i \neq a_i' \ \ (i=0,1,2) \\ A_2A_1A_0 = I_2,\ b_0 = \nu \text{ and } b_1 = 0 \end{array} \right\}.$$

According to the existence of the scalar matrix in $\{A_0, A_1, A_2\}$, we have

$$\widetilde{X}_1 = \widetilde{X}_{1,0} \sqcup \widetilde{X}_{1,1},$$

$$\widetilde{X}_{1,\nu} := \left\{ C_{A_0,A_1,A_2},\ C_{A_1,A_2,A_0},\ C_{A_2,A_0,A_1} \;\middle|\; \begin{array}{l} A_0 = \begin{pmatrix} a_0 & \nu \\ 0 & a_0 \end{pmatrix},\ A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_1' \end{pmatrix}, \\ A_2A_1A_0 = I_2 \text{ and } a_1 \neq a_1' \end{array} \right\}$$

Considering the number of scalar matrices in $\{A_0, A_1, A_2\}$, we have

$$\widetilde{X}_3 = \widetilde{X}_{3,3} \sqcup \widetilde{X}_{3,1} \sqcup \widetilde{X}_{3,0},$$

$$\widetilde{X}_{3,3} := \big\{ (a_0I_2, a_1I_2, a_2I_2) \mid a_0a_1a_2 = 1 \big\},$$

$$\widetilde{X}_{3,1} := \left\{ C_{A_0,A_1,A_2},\ C_{A_2,A_0,A_1},\ C_{A_1,A_2,A_0} \;\middle|\; \begin{array}{l} A_0 = \begin{pmatrix} a_0 & 1 \\ 0 & a_0 \end{pmatrix},\ A_1 = a_1I_2, \\ A_2A_1A_0 = I_2 \end{array} \right\},$$

$$\widetilde{X}_{3,0} := \left\{ C_{A_0,A_1,A_2} \;\middle|\; \begin{array}{l} A_i = \begin{pmatrix} a_i & b_i \\ 0 & a_i \end{pmatrix} \ \ (i=0,1,2) \\ A_2A_1A_0 = I_2 \text{ and } b_0b_1b_2 \neq 0 \end{array} \right\}.$$

We give examples of local monodromy matrices M_0 and M_1 of the equation (6) with the Riemann scheme (3). Here we assume $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ and denote $(A, B) \sim (A', B')$ if there exists $g \in GL(2, \mathbb{C})$ satisfying $A' = gAg^{-1}$ and $B' = gBg^{-1}$.

	$\begin{Bmatrix} \lambda_{0,1} & \lambda_{1,1} & \lambda_{\infty,1} \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{\infty,2} \end{Bmatrix}$	(M_0, M_1)
$X_{1,0}$	$\begin{Bmatrix} 0 & 0 & 0 \\ 1 & \lambda & -\lambda \end{Bmatrix}$	$\sim \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & e^{2\pi i\lambda} \end{pmatrix} \right) \sim \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\lambda} \end{pmatrix} \right)$
$X_{1,1}$	$\begin{Bmatrix} 0 & 0 & 0 \\ 0 & \lambda & 1-\lambda \end{Bmatrix}$	$\sim \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\lambda} \end{pmatrix} \right)$
$X_{3,3}$	$\begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \end{Bmatrix}$	$\sim \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$
$X_{3,1}$	$\begin{Bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{Bmatrix}$	$\sim \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$

These 4 examples satisfy (10) and (11) but do not satisfy (12). The last column in the above table follows from the last claim of Theorem 3.1 ii).

Now we note the following facts. If $C_{A,B,(BA)^{-1}} \in \widetilde{M}$, then $C_{B,A,(AB)^{-1}} \in \widetilde{M}$ and $C_{A,(AB)^{-1},B} \in \widetilde{M}$ because of the symmetry of the equation (6) determined by the Riemann scheme (3) and moreover $C_{aA,bB,(abBA)^{-1}} \in \widetilde{M}$ for $(a,b) \in (\mathbb{C} \setminus \{0\})^2$ because of a suitable gauge transformation $u \mapsto x^\lambda(1-x)^\mu u$.

It follows from these facts that the above $X_{i,j}$ shows $\widetilde{X}_{i,j} \subset \widetilde{M}$ for $(i,j) = (1,0)$, $(1,1)$, $(3,3)$ and $(3,1)$.

Suppose there exist complex numbers a_0 and a_1 satisfying $(M_0 - a_0)^2 = (M_1 - a_1)^2 = 0$ and moreover $C_{M_0,M_1,(M_1M_0)^{-1}} \in \widetilde{X}'$. Then (12) is not valid and Theorem 3.1 ii) and iii) imply that at least one of M_0, M_1 and M_1M_0 is a scalar matrix. Hence $\widetilde{X}_{3,0} \subset \widetilde{X} \setminus \widetilde{M}$.

Note that

$$\begin{aligned}&\{(a_0, a_0', a_1, a_1', a_2, a_2') \mid \{a_i, a_i'\} \text{ are the sets of eigenvalues of } A_i \text{ for } C_{A_0,A_1,A_2} \in \widetilde{M}\}\\ &\quad = \{(c_1, \ldots, c_6) \in \mathbb{C}^6 \mid c_1c_2c_3c_4c_5c_6 = 1\}.\end{aligned}$$

Suppose $C_{M_0,M_1,M_\infty} \in \widetilde{X}_0$. We may assume $\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,1} \in \mathbb{Z}$. Owing to the Fuchs relation (4), we may moreover assume

$$\lambda_{0,2} + \lambda_{1,1} + \lambda_{\infty,1} \in \{0, -1, -2, \ldots\}$$

and hence

$$\lambda_{0,1} + \lambda_{1,2} + \lambda_{\infty,2} \in \{1, 2, 3, \ldots\}.$$

Since any one of M_1 and M_0 is not a scalar matrix, we have

$$\lambda_{0,2} + \lambda_{1,2} + \lambda_{\infty,1} \notin \mathbb{Z} \quad \text{and} \quad \lambda_{0,2} + \lambda_{1,2} + \lambda_{\infty,2} \notin \mathbb{Z}.$$

In this case (10) and (11) are valid and (12) is not valid. Hence Theorem 3.1 ii) assures $b_0 = 1$ and $b_1 = 0$ in (13). Thus we have $\widetilde{X}_0 \cap \widetilde{M} = \widetilde{X}_{0,1}$.

Combining the facts we have proved, we have the theorem. □

Remark 3.2. *Using our classification of elements of $\widetilde{X}$, it is easy to show*

$$\begin{aligned}\widetilde{X} = \Big\{&C_{M_0,M_1,M_\infty} \mid M_0,\ M_1 \textit{ and } M_\infty \textit{ are local monodromy matrices of}\\ &\textit{the equation } \frac{du}{dx} = \left(\frac{A_0}{x} + \frac{A_1}{1-x}\right)u \ \textit{ with } A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix},\\ &(a_i, b_i, c_i, d_i) \in \mathbb{C}^4 \ \textit{ for } \ i = 0 \textit{ and } 1\Big\}.\end{aligned}$$

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