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Uniformizing model of type A_3 and almost Belyi functions

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Abstract. We first review the result by K. Saito on the uniformization of the discriminant loci for the case of the root system of type A_3. Then we formulate a generalization of his result and construct three examples which satisfy the uniformizing model after K. Saito.

1. Introduction

The discriminant $\Delta(A_3)$ of the polynomial $f(t) = t^4 + x_1t^2 + x_2t + x_3$ has many interesting properties. Among others we focus our attention on the relation

$$-\frac{1}{108}\Delta(A_3) = L_0(x)^3 + M_0(x)^2,$$

where L_0, M_0 are polynomials (cf. (2), (3) in the main text). K. Saito ([3]) constructed a uniformization of the complement of $\Delta(A_3) = 0$ of $\mathbf{C}^3$ with the help of this relation. Noting that $\Delta(A_3) = 0$ is also a free divisor, we are led to ask the question whether there is a polynomial of three variables which defines a free divisor and has a kind of the relation similar to (3). If there is a polynomial satisfying these two conditions, it is expected to develop an argument analogues to [3]. The author treated a generalization to the following case, namely, the case where there exists a triplet of functions $\{F, L, M\}$ of three variables satisfying

(A1) F, L, M are polynomials of (x_1, x_2, x_3).

(A2) $F = 0$ is a free divisor in the sense of [4].

(A3:p) $F = L^p + M^2$ for some positive integer p.

See [6], [7] for details. It is underlined here that if $p = 3$, it is hard to find the triplet $\{F, L, M\}$ satisfying (A1), (A2), (A3:3) except the case of the discriminant $\Delta(A_3)$ treated by K. Saito.

In this paper we discuss a generalization of the above Saito's result in another direction. Namely we discuss the possibility of the existence of a triplet $\{F, L, M\}$ of three variables when Condition (A1) is changed to (A1)' below while keeping Conditions (A2), (A3:p):

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(A1)' F is a polynomial but both L, M are algebraic functions of (x_1, x_2, x_3).

If there is a triplet $\{F, L, M\}$ with Conditions (A1)', (A2), (A3:p), it is possible to develop the argument in $\{F, L, M\}$ similar to that in the case of the discriminant of type A_3 (cf. [3]).

Among others, it is one of basic problems to construct such triplets $\{F, L, M\}$ with Conditions (A1)', (A2), (A3:p). The idea to perform this purpose employed in this paper is the use of a generalization of Belyi functions, that is, the so called "almost Belyi functions" introduced by A. V. Kitaev [2]. Then it is possible to construct examples of the triplets at least in the case $p = 3$. We explain the reason why the research on the almost Belyi function can be applied to the construction of the triplet $\{F, L, M\}$ with Conditions (A1)', (A2), (A3:3) briefly. Some of almost Belyi functions are realized in the following manner.

There are four weighted homogeneous polynomials h_1, h_2, h_3, h_4 of three variables x_1, x_2, x_3 with Conditions (E1), (E2):

(E1) $h_2 = 0$ defines a free divisor in $\mathbf{C}^3$.

(E2) $h_1^5 h_2 = h_3^3 + h_4^2$.

Then $Z = \dfrac{h_1^5 h_2}{h_3^3}$ is an almost Belyi function introduced by A. Kitaev.

We now focus our attention on Condition (E2). By putting $F = h_2$, $L = \dfrac{h_3}{h_1^{5/3}}$, $M = \dfrac{h_4}{h_1^{5/2}}$, we obtain the relation given in (A3:3).

This observation suggests that some of almost Belyi functions provide triplets $\{F, L, M\}$ with Conditions (A1)', (A2), (A3:3). Then it is worth checking to apply the arguments in the case of the discriminant $\Delta(A_3)$ to such triplets $\{F, L, M\}$ constructed by the use of almost Belyi functions. The motivation of the present study is to see if this attempt leads to a good conclusion or not. The purpose of this paper is to construct examples of the triplet $\{F, L, M\}$ and study some properties of them. In spite that the result obtained is not satisfactory, the author hopes the efforts developed in this paper will be useful for researches on free divisors, almost Belyi functions and the uniformization.

The construction of this paper is as follows. In section 2, we review the result by K. Saito [3] and in section 3, we will formulate the notion of the uniformizing model of type A_3 and confirm the argument for the discriminant of type A_3 hold for the case of the uniformizing model of type A_3. The definition of the almost Belyi function and its interpretation in terms of the use of free divisors are explained in section 4. In section 5, we will construct three examples of the triplets $\{F, L, M\}$ by using almost Belyi functions. The first example is related with the free divisor defined by the polynomial $F_{B,6}$ found in [5] and the almost Belyi function introduced in [9]. The second example is related with the discriminant of the real reflection group of type H_3. The third example is related with the almost Belyi function found by Kitaev [2].

2. The case of A_3. A prototype

We give a survey on the result of [3] in this section.

The discriminant $\Delta(A_3)$ of the polynomial of t defined by $f(t) = t^4 + x_1t^2 + x_2t + x_3$ coincides with the determinant of the matrix

$$M_{\Delta(A_3)} = \begin{pmatrix} 2x_1 & 3x_2 & 4x_3 \\ 3x_2 & -x_1^2 + 4x_3 & -\frac{1}{2}x_1x_2 \\ 4x_3 & -\frac{1}{2}x_1x_2 & \frac{1}{4}(-3x_2^2 + 8x_1x_3) \end{pmatrix}$$

up to a constant factor. It is easy to see that $\Delta(A_3)$ defines a free divisor. This is shown as follows. Let V_1, V_2, V_3 be vector fields defined by

$$^t(V_1, V_2, V_3) = M_{\Delta(A_3)}{}^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}).$$

Then by direct computation, we have

$$V_1\Delta(A_3) = 12\Delta(A_3), \quad V_2\Delta(A_3) = 0, \quad V_3\Delta(A_3) = 2x_1\Delta(A_3).$$

and these imply that $\Delta(A_3) = 0$ is free.

The system of differential equations

$$(1) \qquad \begin{cases} V_1u & = -u, \\ V_2V_2u & = 0, \\ V_2V_3u & = -\frac{1}{2}x_2u - \frac{1}{2}x_1V_2u, \\ V_3V_3u & = -x_3u - \frac{x_2}{4}V_2u \end{cases}$$

is introduced in [3]. This is an example of uniformizing systems of equations singular along $\Delta(A_3) = 0$. We will construct solutions of (1) following the idea by K. Saito. We first introduce polynomials

$$(2) \qquad L_0(x) = -\frac{1}{9}(12x_3 + x_1^2), \quad M_0(x) = \frac{1}{54}(2x_1^3 + 27x_2^2 - 72x_1x_3).$$

It is easy to see that

$$(3) \qquad -\frac{1}{108}\Delta(A_3) = L_0(x)^3 + M_0(x)^2,$$

(4) $V_1L_0 = 4L_0,\ V_1M_0 = 6M_0,\ V_2L_0 = V_2M_0 = 0,\ V_3'L_0 = M_0,\ V_3'M_0 = -\dfrac{3}{2}L_0^2.$

Here we put

$$V_3' = \frac{1}{2}\left(V_3 - \frac{1}{6}x_1V_1\right).$$

Let

(5) $$P(t) = t^3 + 3L_0(x)t + 2M_0(x)$$

be a cubic polynomial of t. Then the following formulas are easy to show:

(6) $$P\left(-\frac{2}{3}x_1\right) = x_2^2,$$

$$(V_1 + 2t\partial_t)P = 6P,$$

(7)
$$(V_3'^2 - \frac{1}{16}L_0(x))P(t)^{-1/2}$$
$$= \frac{1}{8}\frac{\partial}{\partial t}\left\{\left(L_0(x)t^4 - 3L_0(x)^2t^2 - 4L_0(x)M_0(x)t - 12(M_0(x)^2 + L_0(x)^3)\right)P(t)^{-3/2}\right\}.$$

We put

(8) $$\eta(x) = \frac{1}{2}\int_\infty^{\varphi(x)} P(t)^{-1/2}dt,$$

where $\varphi(x) = -\frac{2}{3}x_1$. Since $P(\varphi) = x_2^2$, we take the branch of $P(t)^{1/2}$ so that $P(\varphi)^{1/2} = x_2$. Then, by an argument explained in [3], we see that

$$V_1\eta = -\eta, \quad V_2\eta = -1.$$

To compute $V_3'^2\eta$, we first note that

$$V_3'\eta = \frac{1}{2}\int_\infty^{\varphi} V_3'(P^{-1/2})dt + \frac{(V_3'\varphi)}{2x_2}$$

and

$$V_3'^2\eta = \frac{1}{2}\left\{\int_\infty^{\varphi} V_3'^2(P^{-1/2})dt + (V_3'P(t)^{-1/2})|_{t=\varphi}\cdot(V_3'\varphi) + V_3'\left(\frac{(V_3'\varphi)}{x_2}\right)\right\}.$$

Then

$$\begin{aligned}
&\left(V_3'^2 - \frac{1}{16}L_0\right)\eta \\
&= \frac{1}{2}\left\{\int_\infty^\varphi \left(V_3'^2 - \frac{1}{16}L_0\right)P(t)^{-1/2}dt + (V_3'P(t)^{-1/2})|_{t=\varphi}\cdot(V_3'\varphi) + V_3'(\frac{(V_3'\varphi)}{x_2})\right\} \\
&= \frac{1}{16}\left\{\left(L_0(x)t^4 - 3L_0(x)^2t^2 - 4L_0(x)M_0(x)t - 12(M_0(x)^2 + L_0(x)^3)\right)P(t)^{-3/2}\right\}|_{t=\varphi} \\
&\quad + \frac{1}{2}\left\{(V_3'P(\varphi)^{-1/2})|_{t=\varphi}\cdot(V_3'\varphi) + V_3'(\frac{(V_3'\varphi)}{x_2})\right\} \\
&= \frac{1}{16x_2^3}(L_0(x)\varphi^4 - 3L_0(x)^2\varphi^2 - 4L_0(x)M_0(x)\varphi - 12(M_0(x)^2 + L_0(x)^3)) \\
&\quad + \frac{1}{2}\left\{(V_3'P(t)^{-1/2})|_{t=\varphi}\cdot(V_3'\varphi) + V_3'(\frac{(V_3'\varphi)}{x_2})\right\}.
\end{aligned}$$

It follows from direct computation that

$$\begin{aligned}
V_3'\varphi &= \frac{1}{2}\varphi^2 + L_0, \\
V_3'^2\varphi &= \frac{1}{2}\varphi^3 + L_0\varphi + M_0, \\
V_3'x_2 &= \frac{3}{4}\varphi x_2, \\
V_3'P(t) &= 3(M_0t - L_0^2).
\end{aligned}$$

As a consequence, we have

$$V_3'P(t)^{-1/2}|_{t=\varphi} = -\frac{3(M_0\varphi - L_0^2)}{2x_2^3}, \quad V_3'(\frac{V_3'\varphi}{x_2}) = \frac{\varphi^3 + 2L_0\varphi + 8M_0}{8x_2}.$$

Then

$$\begin{aligned}
&\frac{1}{16x_2^3}(L_0(x)\varphi^4 - 3L_0(x)^2\varphi^2 - 4L_0(x)M_0(x)\varphi - 12(M_0(x)^2 + L_0(x)^3)) \\
&+\frac{1}{2}(V_3'P(t)^{-1/2})|_{t=\varphi}\cdot(V_3'\varphi) \\
=&\frac{L_0(x)\varphi^4 - 3L_0(x)^2\varphi^2 - 4L_0(x)M_0(x)\varphi - 12(M_0(x)^2 + L_0(x)^3)}{16x_2^3} \\
&-\frac{3(M_0\varphi - L_0^2)(\varphi^2 + 2L_0)}{8x_2^3} \\
=&\frac{1}{16x_2^3}\{L_0(x)\varphi^4 - 3L_0(x)^2\varphi^2 - 4L_0(x)M_0(x)\varphi - 12(M_0(x)^2 + L_0(x)^3) \\
&\qquad -6(M_0\varphi - L_0^2)(\varphi^2 + 2L_0)\}
\end{aligned}$$

$$=\frac{P(\varphi)(L_0\varphi-6M_0)}{16x_2^3}$$

$$=\frac{L_0\varphi-6M_0}{16x_2},$$

which implies

$$\begin{aligned}&\frac{L_0(x)\varphi^4-3L_0(x)^2\varphi^2-4L_0(x)M_0(x)\varphi-12(M_0(x)^2+L_0(x)^3)}{16x_2^3}\\&+\frac{1}{2}\left\{(V_3'P(t)^{-1/2})|_{t=\varphi}\cdot(V_3'\varphi)+V_3'(\frac{(V_3'\varphi)}{x_2})\right\}\\=&\frac{x_2}{16}.\end{aligned}$$

Therefore we obtain

$$\left(V_3'^2-\frac{L_0}{16}\right)\eta=\frac{x_2}{16}.$$

Since $V_2\eta=-1$, we find that

$$V_3'^2\eta=-\frac{1}{16}x_2V_2\eta+\frac{1}{16}L_0(x)\eta.$$

As a consequence, it follows that the function $\eta(x)$ is a solution of

$$\text{(9)}\qquad \begin{cases}V_1u&=-u,\\V_2^2u&=0,\\V_3'V_2u&=0,\\V_3'^2u&=-\frac{1}{16}x_2V_2u+\frac{1}{16}L_0(x)u.\end{cases}$$

It is easy to show that (1) is same as (9).

If $u(x)$ is a solution of (9) such that $V_2u=0$, then u is a solution of the system

$$\text{(10)}\qquad V_1u=-u,\ V_2u=0,\ V_3'^2u=\frac{1}{16}L_0u.$$

One method to solve this system is to reduce it to Gaussian hypergeometric differential equation. In fact, putting $x=\frac{L_0^3+M_0^2}{L_0^3}$ and $u=L_0^{-1/4}f(x)$, we obtain

$$\left(V_3'^2-\frac{L_0}{16}\right)u=9L_0^{3/4}\left\{x\left(\vartheta+\frac{1}{12}\right)\left(\vartheta+\frac{5}{12}\right)-\vartheta^2\right\}f(x),$$

where $\vartheta = x\dfrac{d}{dx}$. As a consequence, we find that $u = L_0^{-1/4}F(\frac{1}{12}, \frac{5}{12}, 1; x)$ is a solution to (10). In this manner, we obtain three linearly independent solutions of (1).

The equation $P(-\frac{2}{3}x_1) = x_2^2$ is equivalent to

$$Y^2 = 4X^3 - g_2X - g_3 \tag{11}$$

under the correspondence

$$X = -\frac{2}{3}x_1,\ Y = 2x_2,\ g_2 = -12L_0,\ g_3 = -8M_0.$$

Noting this and taking appropriate linearly independent solutions u_1, u_2 of (10) which are expressed by hypergeometric functions, we define a map

$$(x_1, x_2, x_3) \to (u_1(x), u_2(x), \eta(x))$$

which is defined on the universal covering $\mathcal{U}$ of $\mathbf{C}^3 - \{\Delta = 0\}$ and its image domain coincides with

$$\mathcal{S} = \{(u_1, u_2, \eta) : \operatorname{Im}(u_1/u_2) > 0,\ mu_1 + nu_2 \neq \eta\ (\forall\, m, n \in \mathbf{Z})\}.$$

By the theory of elliptic functions, the inversion map for $(u_1, u_2, \eta) : \mathcal{U} \to \mathcal{S}$ is realized by

(12)
$$\begin{cases} -\dfrac{2}{3}x_1 = \wp(\eta, u_1, u_2) = \dfrac{1}{\eta^2} + \sum{}' \left\{ \dfrac{1}{(\eta - mu_1 - nu_2)^2} - \dfrac{1}{(mu_1 + nu_2)^2} \right\}, \\ 2x_2 = \wp'(\eta, u_1, u_2) = -2\left\{ \dfrac{1}{\eta^3} + \sum{}' \dfrac{1}{(\eta - mu_1 - nu_2)^3} \right\}, \\ -12L_0 = \dfrac{4}{3}(12x_3 + x_1^2) = 60 \sum{}' \dfrac{1}{(mu_1 + nu_2)^4}. \end{cases}$$

3. Uniformizing models of type A_3

In this section, we formulate an analogue of the result explained in the previous section.

First we take a weighted homogeneous polynomial $F(x_1, x_2, x_3)$ of three variables with weights $w(x_j) = d_j$ $(j = 1, 2, 3)$ such that d_1, d_2, d_3 are positive integers and assume that $d_1 \leq d_2 \leq d_3$. Then $E = d_1x_1\partial_{x_1} + d_2x_2\partial_{x_2} + d_3x_3\partial_{x_3}$ is a vector field such that $EF = k_0F$ for some positive integer k_0.

We assume that there are functions $L = L(x_1, x_2, x_3), \varphi = \varphi(x_1, x_2, x_3), \tau = \tau(x_1, x_2, x_3)$, vector fields V, W on $\mathbf{C}^3$ and a positive number k_1 with the following conditions:

(B1) $EL = 4k_1L,\ E\varphi = 2k_1\varphi,\ E\tau = 3k_1,\ [E, W] = k_1W,\ [E, V] = 2k_1V,$

(B2) $\det\left(\dfrac{\partial(\varphi, \tau, L)}{\partial(x_1, x_2, x_3)}\right)$ is not identically zero,

(B3) $4L^3 + (\tau^2 - \varphi^3 - 3L\varphi)^2 = c_0F$ for some non-zero number c_0.

(B4) $V\varphi = \dfrac{1}{2}\varphi^2 + L,\ V\tau = \dfrac{3}{4}\varphi\tau,\ VL = \dfrac{1}{2}(\tau^2 - \varphi^3 - 3L\varphi),$

(B5) $W\varphi = \tau,\ W\tau = \dfrac{3}{2}(\varphi^2 + L),\ WL = 0.$

Definition 3.1. *A uniformizing model of type A_3 is the data $\{F, L, M, E, V, W, \varphi, \tau\}$ consisting of a polynomial $F = F(x_1, x_2, x_3)$, functions L, M, φ, τ, vector fields E, W, V with the conditions (B1)-(B5).*

As easy consequences of Conditions (B1)-(B5), we have the following.

(C1) Put $M = \frac{1}{2}(\tau^2 - \varphi^3 - 3L\varphi)$ and $P(t) = t^3 + 3Lt + 2M$. Then $P(\varphi) = \tau^2$ and the discriminant of $P(t)$ coincides with $L^3 + M^2$ up to a constant factor.

(C2) $VL = M,\ VM = -\dfrac{3}{2}L,\ WM = 0,\ L^3 + M^2 = \dfrac{c_4}{4}F.$

(C3) $(V^2 - \frac{1}{16})P(t)^{-1/2} = \frac{1}{8}\frac{\partial}{\partial t}\left\{(Lt^4 - 3Lt^2 - 4LMt - 12(L^3 + M^2))P(t)^{-3/2}\right\}.$

(C4) We define vector fields E^*, W^*, V^* on the (φ, τ, L)-space by

$$\begin{aligned} E^* &= 2\varphi\partial_\varphi + 3\tau\partial_\tau + 4L\partial_L, \\ W^* &= \tau\partial_\varphi + \tfrac{3}{2}(\varphi^2 + L)\partial_\tau, \\ V^* &= (\frac{1}{2}\varphi^2 + L)\partial_\varphi + \frac{3}{4}\varphi\tau\partial_\tau + \frac{1}{2}(\tau^2 - \varphi^3 - 3L\varphi)\partial_L. \end{aligned}$$

Then there is a natural correspondence

$$(E, W, V) \longleftrightarrow (k_1E^*, W^*, V^*)$$

and $4L^3 + (\tau^2 - \varphi^3 - 3L\varphi)^2$ defines a free divisor on the (φ, τ, L)-space and E^*, W^*, V^* generate the Lie algebra of the logarithmic vector fields along the free divisor $4L^3 + (\tau^2 - \varphi^3 - 3L\varphi)^2 = 0$.

Remark 3.2. *An identification between the variables x_1, x_2, x_3 of §2 and φ, τ, L is given by $\varphi = -\dfrac{2}{3}x_1,\ \tau = x_2,\ L = -\dfrac{1}{9}(12x_3 + x_1^2)$.*

Noting the property (C4), we define a system of differential equations

$$(13)\qquad \begin{cases} Eu &= -k_1 u, \\ W^2 u &= 0, \\ VWu &= 0, \\ V^2 u &= (\dfrac{L}{16} + \dfrac{\tau}{8} W) u. \end{cases}$$

This system is an analogue to (9). This observation suggests that solutions to (13) can be constructed by an argument similar to that developed in the previous section, which we are going to do. We first introduce a function $\eta(x)$ of $x = (x_1, x_2, x_3)$ defined by

$$\eta(x) = \frac{1}{2} \int_\infty^\varphi P(t)^{-1/2} dt.$$

For a moment, we take a branch of $P(t)$ such that $P(\varphi)^{1/2} = \tau$. It is clear from the definition that $E\eta = -k_1 \eta$. On the other hand, (B5) shows that $W\eta = \dfrac{1}{2}$. This implies that $W^2\eta = VW\eta = 0$. Moreover using (B1)-(B5), we find that $V^2\eta = \dfrac{1}{16} L\eta + \dfrac{\tau}{16}$. This combined with $W\eta = 1/2$ implies that $V^2\eta = \left(\dfrac{L}{16} + \dfrac{\tau}{16} W \right) \eta$. As a consequence, $\eta(x)$ is a solution to (13).

It is clear that the argument of the previous section shows

$$(14)\qquad \begin{cases} \varphi &= \wp(\eta, u_1, u_2) \\ &= \dfrac{1}{\eta^2} + \sum{}' \left\{ \dfrac{1}{(\eta - mu_1 - nu_2)^2} - \dfrac{1}{(mu_1 + nu_2)^2} \right\}, \\ 2\tau &= \wp'(\eta, u_1, u_2) \left(= -2 \left\{ \dfrac{1}{\eta^3} + \sum{}' \dfrac{1}{(\eta - mu_1 - nu_2)^3} \right\} \right), \\ -12L &= 60 \sum{}' \dfrac{1}{(mu_1 + nu_2)^4}. \end{cases}$$

It is non-trivial to write down x_1, x_2, x_3 by φ, τ, L.

We propose a few basic problems on the uniformizing model of type A_3.

(D1) Find the uniformizing models of type A_3.

(D2) Solve the system of equations (14) for x_1, x_2, x_3.

(D3) Classify the uniformizing models of type A_3.

Remark 3.3. *In this paper, the problems (D1) and (D2) are treated but (D3)* *is not.*

4. Almost Belyi functions

An almost Belyi function introduced by Kitaev [2] is an algebraic covering $\varphi : C \longrightarrow \mathbf{P}^1$ that branches above $\{0, 1, \infty\}$ of $\mathbf{P}^1$ and has exactly one simple branching point outside the fibres $\{0, 1, \infty\}$ of $\mathbf{P}^1$. One of the reasons why Kitaev introduced almost Belyi functions is to construct algebraic solutions to Painlevé VI equation with the help of almost Belyi functions (cf. [2]). There is a deep relationship between some of almost Belyi functions and free divisors in three variables. We explain a typical case of such almost Belyi functions. See [8] for the interpretation of the almost Belyi functions in terms of free divisors. There are four weighted homogeneous polynomials h_1, h_2, h_3, h_4 of x_1, x_2, x_3 with the following conditions.

(E1) $h_2 = 0$ is a free divisor in $\mathbf{C}^3$,

(E2) $h_1^5 h_2 = h_3^3 + h_4^2$.

Then (E2) implies that $Z = \dfrac{h_1^5 h_2}{h_3^3}$ defines an almost Belyi function. We focus our attention on the equation of (E2), which is equivalent to

$$h_2 = \frac{h_3^3}{h_1^5} + \frac{h_4^2}{h_1^5}. \tag{15}$$

There is a similarity between the last equation of the property (C2) and (15) by putting

$$L = \frac{h_3}{h_1^{5/3}}, \quad M = \frac{h_4}{h_1^{5/2}}.$$

This suggests the possibility of constructing examples of uniformizing models of type A_3 with the help of almost Belyi functions. It is underlined here that L and M are not polynomials but algebraic functions of x_1, x_2, x_3. Even assuming Conditions (E1), (E2), we need to show the existence of the functions φ and τ to reach the goal. Therefore we focus our efforts on the discovery of φ, τ in each case of almost Belyi functions. The meaning of φ, τ is left to future research.

5. Examples of uniformizing models of type A_3

In this section, we focus our attention on the construction problem of uniformizing model of type A_3. We treat three examples of uniformizing models of type A_3 which are constructed with the help of almost Belyi functions. In the first example, the problems (D1), (D2) explained in §3 are solved. In the second and third examples, (D1) is solved but (D2) is still open.

5.1. Example 1: Free divisor defined by $F_{B,6}$

We start with the 3×3 matrix

$$M_{B,6} = \begin{pmatrix} x_1 & 2x_2 & 3x_3 \\ -\frac{4}{15}(5x_1^2 - 3x_2) & \frac{1}{15}(-4x_1x_2 + 15x_3) & \frac{2}{5}(9x_2^2 + 5x_1x_3) \\ \frac{1}{15}\begin{pmatrix} 10x_1^3 - 56x_1x_2 \\ +15x_3 \end{pmatrix} & \frac{1}{30}x_2(4x_1^2 + 25x_2) & \frac{1}{5}\begin{pmatrix} -9x_1x_2^2 - 5x_1^2x_3 \\ +10x_2x_3 \end{pmatrix} \end{pmatrix}.$$

The polynomial $f_0 = \det(M_{B,6})$ defines a free divisor same as $F_{B,6}$ introduced in [5]. Let V_1, V_2, V_3 be vector fields by using $M_{B,6}$ similar to the case of A_3. In this case, it is easy to show that

$$V_1 f_0 = 9f_0, \quad V_2 f_0 = 0, \quad V_3 f_0 = 0.$$

There is an almost Belyi function related with the polynomial f_0 which we are going to introduce. Let F_8, G_8, P_8 be the polynomials of x, s defined in [9, p.8], namely

$$\begin{aligned} F_8 &= 9(s+1)^2x^3 - 24s(s+3)x^2 + 8s(11s-1)x + 48s^2, \\ G_8 &= x^2 - 2sx - s, \\ P_8 &= 3(s+1)^2x^4 - 4s(s+3)x^3 + 12s(s-1)x^2 + 24s^2x + 8s^2. \end{aligned}$$

We also define (cf. [9])

$$\varphi_8(x,s) = \frac{(s+1)^2x^5F_8}{64sG_8^3}.$$

Then

$$\varphi_8(x,s) - 1 = \frac{P_8^2}{64sG_8^3},$$

which is equivalent to the identity

$$(s+1)^2x^5F_8 - 64sG_8^3 - P_8^2 = 0. \tag{16}$$

To connect F_8, G_8, P_8 with the polynomial f_0, we introduce rational functions $\tilde{x}, \tilde{s}$

of x_1, x_2, x_3 by

$$\tilde{x} = \frac{x_1 x_3}{3x_2^2}, \quad \tilde{s} = \frac{-x_1^2}{x_1^2 - 4x_2}$$

and polynomials h_1, h_2, h_3, h_4 by

$$\begin{aligned}
h_1 &= x_3, \\
h_2 &= 9x_1x_2^4 + 6x_1^2x_2^2x_3 - 2x_2^3x_3 + x_1^3x_3^2 - 6x_1x_2x_3^2 + x_3^3, \\
h_3 &= 9x_2^4 + 6x_1x_2^2x_3 + x_1^2x_3^2 - 4x_2x_3^2, \\
h_4 &= 27x_2^6 + 27x_1x_2^4x_3 + 9x_1^2x_2^2x_3^2 - 18x_2^3x_3^2 + x_1^3x_3^3 - 6x_1x_2x_3^3 + 2x_3^4.
\end{aligned}$$

Then it is straightforward to show that

$$\begin{aligned}
F_8(\tilde{x}, \tilde{s}) &= \frac{16x_1^3h_2}{3(x_1^2 - 4x_2)^2x_2^4}, \\
G_8(\tilde{x}, \tilde{s}) &= \frac{x_1^2h_3}{9(x_1^2 - 4x_2)x_2^4}, \\
P_8(\tilde{x}, \tilde{s}) &= \frac{8x_1^4h_4}{27(x_1^2 - 4x_2)^2x_2^6}.
\end{aligned}$$

We note that $f_0 = -3h_2$. Moreover, the identity equation (16) turns out to be the relation

$$4h_1^5h_2 + h_3^3 - h_4^2 = 0 \tag{17}$$

among h_1, h_2, h_3, h_4. Following the argument in the previous section, we put

$$L = -\frac{h_3}{h_1^{5/3}}, \quad M = \frac{h_4}{h_1^{5/2}}. \tag{18}$$

Then (17) turns out to be

$$-4h_2 + L^3 + M^2 = 0, \tag{19}$$

which is an analogue of (3) and (B3). We introduce vector fields

$$V = -\frac{1}{20x_3^{5/6}}(18x_2V_3 + (9x_1x_2 - 10x_3)V_2), \quad W = \frac{1}{2}x_3^{-5/12}(x_1V_2 + 2V_3). \tag{20}$$

It is easy to check that

$$VL = M, \quad VM = -\frac{3}{2}L^2, \quad WL = WM = 0.$$

We put

$$\begin{cases} P(t) = t^3 + 3Lt + 2M, \\ \varphi \quad = (3x_2^2 - 2x_1x_3)x_3^{-5/6}, \\ \tau \quad = (9x_1x_2 - 2x_3)x_3^{-1/4}. \end{cases} \tag{21}$$

Then it is straightforward to show that (B1)-(B5) hold for φ and τ.

Summarizing the above argument, we obtain the following.

THEOREM 5.1. *Put* $E = x_1\partial_{x_1} + 2x_2\partial_{x_2} + 3x_3\partial_{x_3}$. *Then* $\{F, L, M, E, V, W, \varphi, \tau\}$ *is a uniformizing model of type* A_3*, where* $F = \det(M_{B,6})$ *and* $L, M, V, W, \varphi, \tau$ *are defined by (18), (20) (21).*

As a consequence, we construct a uniformizing model of type A_3 from the polynomial $f_0 = \det(M_{B,6})$.

The problem (D2) is the next one to solve. It is equivalent to solve

$$\varphi = \wp(\eta, u_1, u_2), \quad 2\tau = \wp'(\eta, u_1, u_2), \quad -12L = g_2, \tag{22}$$

which is a system of algebraic equations for x_1, x_2, x_3 (cf. (14)).

In the rest of this subsection, we study (22) in detail. We first introduce y_1, y_2, y_3 by

$$y_1 = x_1/x_3^{5/24}, \quad y_2 = x_2/x_3^{5/12}, \quad y_3 = x_3^{3/8}. \tag{23}$$

These imply

$$x_1 = y_1y_3^{5/9}, \quad x_2 = y_2y_3^{10/9}, \quad x_3 = y_3y_3^{5/3} = y_3^{8/3}. \tag{24}$$

Then we have

$$\begin{cases} \varphi = 3y_2^2 - 2y_1y_3, \\ \tau = y_3(9y_1y_2 - 2y_3), \\ L = -(9y_2^4 + 6y_1y_2^2y_3 + y_1^2y_3^2 - 4y_2y_3^2). \end{cases} \tag{25}$$

Noting (22) and (25), we introduce the system of algebraic equations for y_1, y_2, y_3

by

$$
\begin{cases} 3y_2^2 - 2y_1y_3 - X = 0, \\ 2y_3(9y_1y_2 - 2y_3) - Y = 0, \\ 12(9y_2^4 + 6y_1y_2^2y_3 + y_1^2y_3^2 - 4y_2y_3^2) - g_2 = 0. \end{cases} \tag{26}
$$

Note that X, Y, g_2 are constants but these come from elliptic curve

$$
Y^2 = 4X^3 - g_2X - g_3.
$$

For a moment, g_3 is regarded as the constant defined by $g_3 = 4X^3 - g_2X - Y^2$. From the first equation of (26), it follows that

$$
y_1 = \frac{3y_2^2 - X}{2y_3}. \tag{27}
$$

Then the second and third equations of (26) are reduced to

$$
\begin{cases} Y + 9Xy_2 - 27y_2^3 + 4y_3^2 = 0, \\ g_2 - 3X^2 + 54Xy_2^2 - 243y_2^4 + 48y_2y_3^2 = 0. \end{cases} \tag{28}
$$

To solve (28), we recall some basic properties of elliptic functions. We first factorize

$$
4X^3 - g_2X - g_3 = 4(X - e_1)(X - e_2)(X - e_3).
$$

and introduce q_1, q_2, q_3 by $q_j = \sqrt{X - e_j}\quad (j = 1, 2, 3)$. (Usually q_j is denoted by σ_j/σ in the elliptic function theory.) Then

$$
Y^2 = 4(X - e_1)(X - e_2)(X - e_3) = 4q_1^2q_2^2q_3^2.
$$

Noting this, we assume that $Y = -2q_1q_2q_3$. Since $X = e_j + q_j^2$, and $e_1 + e_2 + e_3 = 0$, it follows that $X = \dfrac{q_1^2 + q_2^2 + q_3^2}{3}$. As a consequence

$$
g_2 = -\frac{4}{3}(q_1^2q_2^2 + q_1^2q_3^2 + q_2^2q_3^2 - q_1^4 - q_2^4 - q_3^4).
$$

Eliminating the terms of equations in (26) containing y_3^2, we obtain

$$
81y_2^4 - 54Xy_2^2 - 12Yy_2 + g_2 - 3X^2 = 0, \tag{29}
$$

which is regarded as an algebraic equation for y_2. Since

$$\begin{aligned}
&81y_2^4 - 54Xy_2^2 - 12Yy_2 + g_2 - 3X^2 \\
=&81\left(y_2^2 + \frac{2}{3}q_1y_2 - \frac{1}{9}(X+2e_1) + \frac{1}{9q_1}Y\right)\left(y_2^2 - \frac{2}{3}q_1y_2 - \frac{1}{9}(X+2e_1) - \frac{1}{9q_1}Y\right) \\
=&81\left(y_2 + \frac{q_1+q_2+q_3}{3}\right)\left(y_2 + \frac{q_1-q_2-q_3}{3}\right) \\
&\times\left(y_2 + \frac{-q_1+q_2-q_3}{3}\right)\left(y_2 + \frac{-q_1-q_2+q_3}{3}\right),
\end{aligned}$$

it is easy to solve (29). As a consequence,

$$y_2 = -\frac{q_1+q_2+q_3}{3} \tag{30}$$

is one of solutions to (29). Assuming that y_2 is given by (30), we have

$$\begin{cases} y_3^2 = -\dfrac{(q_2+q_3)(q_3+q_1)(q_1+q_2)}{2}, \\ y_1 = \dfrac{q_2q_3+q_3q_1+q_1q_2}{3y_3}. \end{cases} \tag{31}$$

Finally, taking

$$U = -\frac{(q_2+q_3)(q_3+q_1)(q_1+q_2)}{2},$$

we obtain

$$\begin{cases} x_1 = \dfrac{q_2q_3+q_3q_1+q_1q_2}{3}\cdot U^{-2/9}, \\ x_2 = -\dfrac{q_1+q_2+q_3}{3}\cdot U^{5/9}, \\ x_3 = U^{4/3}. \end{cases} \tag{32}$$

Concerning $h_2, h_3, h_4, \varphi, \tau$, we have

$$\begin{aligned}
h_2 =&-\frac{1}{432}(q_2^2-q_3^2)^2(q_3^2-q_1^2)^2(q_1^2-q_2^2)^2, \\
h_3 =&\frac{1}{9}(q_1^4+q_2^4+q_3^4-q_2^2q_3^2-q_3^2q_1^2-q_1^2q_2^2)U^{20/9}, \\
h_4 =&-\frac{1}{54}(-2q_1^2+q_2^2+q_3^2)(q_1^2-2q_2^2+q_3^2)(q_1^2+q_2^2-2q_3^2)U^{10/3},
\end{aligned}$$

$$\varphi = \frac{1}{3}(q_1^2 + q_2^2 + q_3^2),$$

$$\tau = -q_1 q_2 q_3.$$

5.2. Example 2: The discriminant of the real reflection group of type H_3

Let F_{12}, G_{12}, P_{12} be the polynomials of x, s introduced in [9, p.5]:

$$\begin{aligned}
F_{12} &= x^4 - 4(s+3)x^3 + (s^2+6s+14)x^2 + 2(s+6)x + 1,\\
G_{12} &= sx^3 - 4(s^2+3s-1)x^2 - 4(2s+11)x - 4,\\
P_{12} &= 2x^6 - 12(s+3)x^5 + 15(s^2+6s+10)x^4 + 2s(s^2+9s+15)x^3\\
&\quad +6(s^2+9s+25)x^2 + 6(s+6)x + 2.
\end{aligned}$$

We also introduce the map $\zeta_{12}(x,s) = \dfrac{27(s+4)^3}{4} \cdot \dfrac{x^5 G_{12}}{F_{12}^3}$ which coincides with $1/\varphi_{12}$ in [9]. It is easy to check that

$$4F_{12}^3 - 27(s+4)^3 x^5 G_{12} - P_{12}^2 = 0. \tag{33}$$

We introduce $\tilde{x}$, $\tilde{s}$ by $\tilde{x} = \dfrac{2x_2^2 + x_1 x_3}{(x_1^3 - 2x_2)x_2}$, $\tilde{s} = \dfrac{2(x_1^3 - 2x_2)}{x_2}$, which are rational functions of x_1, x_2, x_3. We write $\zeta_{12}(\tilde{x}, \tilde{s})$ by x_1, x_2, x_3. For this purpose, we introduce weighted homogeneous polynomials of type $(1, 3, 5)$ by

$$\begin{aligned}
h_{1a} &= x_1^3 - 2x_2,\\
h_{1b} &= 2x_2^2 + x_1 x_3,\\
h_1 &= h_{1a} h_{1b},\\
h_2 &= 50x_1^3x_2^4 - 108x_2^5 + 40x_1^4x_2^2x_3 - 90x_1x_2^3x_3 + 8x_1^5x_3^2 - 20x_1^2x_2x_3^2 - x_3^3,\\
h_3 &= 25x_1^8x_2^4 - 128x_1^5x_2^5 + 160x_1^2x_2^6 + 20x_1^9x_2^2x_3 - 100x_1^6x_2^3x_3 + 80x_1^3x_2^4x_3\\
&\quad + 96x_2^5x_3 + 4x_1^{10}x_3^2 - 20x_1^7x_2x_3^2 - 10x_1^4x_2^2x_3^2 + 80x_1x_2^3x_3^2 - 8x_1^5x_3^3\\
&\quad + 20x_1^2x_2x_3^3 + x_3^4,\\
h_4 &= 125x_1^{12}x_2^6 - 960x_1^9x_2^7 + 3120x_1^6x_2^8 - 5120x_1^3x_2^9 + 3456x_2^{10} + 150x_1^{13}x_2^4x_3\\
&\quad - 1134x_1^{10}x_2^5x_3 + 3960x_1^7x_2^6x_3 - 7440x_1^4x_2^7x_3 + 5760x_1x_2^8x_3 + 60x_1^{14}x_2^2x_3^2\\
&\quad - 450x_1^{11}x_2^3x_3^2 + 1875x_1^8x_2^4x_3^2 - 4440x_1^5x_2^5x_3^2 + 4080x_1^2x_2^6x_3^2 + 8x_1^{15}x_3^3\\
&\quad - 60x_1^{12}x_2x_3^3 + 390x_1^9x_2^2x_3^3 - 1280x_1^6x_2^3x_3^3 + 1320x_1^3x_2^4x_3^3 + 144x_2^5x_3^3 + 30x_1^{10}x_3^4
\end{aligned}$$

$$- 150x_1^7x_2x_3^4 + 135x_1^4x_2^2x_3^4 + 120x_1x_2^3x_3^4 - 12x_1^5x_3^5 + 30x_1^2x_2x_3^5 + x_3^6.$$

Then $\zeta_{12}(\tilde{x}, \tilde{s}) = -\dfrac{108h_1^5h_2}{h_3^3}$. It is easy to show that

$$108h_1^5h_2 + h_3^3 - h_4^2 = 0, \tag{34}$$

which is equivalent to the relation (33). As a consequence, we have

$$1 - \zeta_{12}(\tilde{x}, \tilde{s}) = \frac{h_4^2}{h_3^3}.$$

In this case, h_2 is regarded as the discriminant of the reflection group of type H_3. In particular $h_2 = 0$ is a free divisor in the (x_1, x_2, x_3)-space. To show this, we introduce a matrix M_{H_3} by

$$M_{H_3} = \begin{pmatrix} x_1 & 3x_2 & 5x_3 \\ \frac{2}{3}(4x_1^3 - 9x_2), & x_3 & -\frac{10}{3}x_1(3x_2^2 + 2x_1x_3) \\ \frac{1}{3}(-16x_1^5 + 40x_1^2x_2 + 3x_3) & 5x_1x_2^2 & \frac{10}{3}(6x_1^3x_2^2 - 9x_2^3 + 4x_1^4x_3 - 4x_1x_2x_3) \end{pmatrix}$$

and define vector fields V_j $(j = 1, 2, 3)$ by

$${}^t(V_1, V_2, V_3) = M_{H_3}\,{}^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}).$$

Then

$$\det(M_{H_3}) = 5h_2, \quad V_1h_2 = 15h_2, \quad V_2h_2 = 0, \quad V_3h_2 = 0,$$

which actually show that h_2 coincides with $\det(M_{H_3})$ up to a constant factor and that $h_2 = 0$ defines a free divisor. Putting $Z = \zeta_{12}(\tilde{x}, \tilde{s})$, we have

$$V_2Z = -10x_1 \cdot \frac{h_4}{h_1h_3} \cdot Z, \quad V_3Z = 10(2x_1^3 - x_2) \cdot \frac{h_4}{h_1h_3} \cdot Z.$$

Noting these equations, we define $V_3' = V_3 + \dfrac{2x_1^3 - x_2}{x_1}V_2$. Then $V_3'Z = 0$. Moreover, we have

$$V_3'h_1 = -4x_1x_2h_1, \quad V_3'h_3 = -\frac{20}{3}x_1x_2h_3, \quad V_3'h_4 = -10x_1x_2h_4,$$

In this case, we define

$$L = -\frac{h_3}{h_1^{5/3}}, \quad M = \cdot\frac{h_4}{h_1^{5/2}}. \tag{35}$$

Then it follows that

$$108h_2 = L^3 + M^2. \tag{36}$$

We put

$$V = -\frac{3}{20h_1^{5/6}}\left\{3x_2(2x_1^5 - 5x_1^2x_2 - x_3)V_2 + (3x_1^3x_2 - 8x_2^2 - x_1x_3)V_3\right\}. \tag{37}$$

Then $VL = M$, $VM = -\frac{3}{2}L^2$. Noting these, we introduce a polynomial of t by $P(t) = t^3 + 3Lt + 2M$. We are going to find functions $\varphi = \varphi(x_1, x_2, x_3)$ and $\tau = \tau(x_1, x_2, x_3)$ such that $P(\varphi) = \tau^2$. After a little tedious computation, we find that

$$\varphi = \frac{\varphi_0}{h_1^{5/6}}, \tag{38}$$

where

$$\varphi_0 = -7x_1^4x_2^2 + 16x_1x_2^3 - 4x_1^5x_3 + 10x_1^2x_2x_3 + x_3^2.$$

Then

$$P(\varphi) = \frac{108x_2^2h_{1a}^2}{h_1^{1/2}}.$$

Moreover we put

$$W = \frac{\sqrt{3}}{2}\cdot\frac{1}{h_1^{5/12}}\cdot((2x_1^3 - x_2)V_2 + x_1V_3). \tag{39}$$

Then

$$W\varphi = \frac{6\sqrt{3}x_2h_{1a}}{h_1^{1/4}}.$$

These identities imply that

$$P(\varphi) = (W\varphi)^2.$$

On the other hand, it is easy to show that

$$WL = WM = 0.$$

Comparing the argument above with that in the previous section, we define

$$\tau_0 = 6 \cdot 3^{1/2} x_2 h_{1a}$$

and

$$\tau = \frac{\tau_0}{h_1^{1/4}}. \tag{40}$$

Then

$$P(\varphi) = \tau^2, \quad W\varphi = \tau.$$

Moreover, by direct computation, we find that (B1)-(B5) hold.

THEOREM 5.2. *Put* $E = x_1\partial_{x_1} + 3x_2\partial_{x_2} + 5x_3\partial_{x_3}$. *Then* $\{F, L, M, E, V, W, \varphi, \tau\}$ *is a uniformizing model of type* A_3*, where* $F = h_2$ *and* $L, M, V, W, \varphi, \tau$ *are defined by (35), (37), (39), (38), (40).*

The proof of this theorem is a consequence of the argument above.

We have thus constructed a uniformizing model of type A_3 from the discriminant of the reflection group of type H_3. The problem to be done is to solve

$$\varphi = y_1, \quad 2\tau = y_2, \quad -12L = y_3,$$

which is a system of algebraic equations for x_1, x_2, x_3, where y_1, y_2, y_3 are constants (cf. (22)). This problem is not solved in the present paper and left to the future study.

5.3. Example 3: The almost Belyi function constructed by A. V. Kitaev

We treat one of the almost Belyi functions constructed by Kitaev [2]. We define a function z of z_1 and s by

$$z = -\frac{2^5 3^3 (s^2-5)^5}{(s+3)^9(s-2)^6} \cdot \frac{(z_1-a)^5(z_1-1)^2 z_1(z_1-t)}{(z_1^4 + c_3 z_1^3 + c_2 z_1^2 + c_1 z_1 + c_0)^3}, \tag{41}$$

where

$$
\begin{aligned}
a &= \frac{(s-1)(s^2-5)}{2^3(s-2)^2},\\
t &= \frac{2s^3(s^2-5)}{(s-2)^2(s+3)^3},\\
c_0 &= \frac{s^2(s^2-5)^4}{2^4(s+3)^4(s-2)^6},\\
c_1 &= -\frac{(-5+s^2)^3(45-75s+65s^2-35s^3+8s^4)}{4(-2+s)^6(3+s)^4},\\
c_2 &= \frac{5(-1+s)(-5+s^2)^2(-9-3s+2s^2+2s^3)}{2(-2+s)^4(3+s)^4},\\
c_3 &= -\frac{2(s^2-5)(2s^3+5s^2-15)}{(s+3)^3(s-2)^2}.
\end{aligned}
$$

By definition, z is an almost Belyi function.

We introduce a free divisor defined by the matrix $M_{E_{13}}$ below:

$$
M_{E_{13}} = \begin{pmatrix} x_1 & 2x_2 & 5x_3 \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix},
$$

where

$$
\begin{aligned}
m_{21} &= -2(36x_1^4 + 31x_1^2x_2 + 6x_2^2),\\
m_{22} &= 36x_1^3x_2 + 8x_1x_2^2 + 3x_3,\\
m_{23} &= 20x_1(9x_1^2 + 4x_2)x_3,\\
m_{31} &= -432x_1^5 - 396x_1^3x_2 - 88x_1x_2^2 + 3x_3,\\
m_{32} &= -4x_2(-54x_1^4 - 15x_1^2x_2 + 2x_2^2),\\
m_{33} &= 20(54x_1^4 + 27x_1^2x_2 + 2x_2^2)x_3.
\end{aligned}
$$

Let V_1, V_2, V_3 be the vector fields defined by $M_{E_{13}}$ as before. Putting $f_0 = \det(M_{E_{13}})$, we find that

$$
V_1 f_0 = 15 f_0, \quad V_2 f_0 = V_3 f_0 = 0.
$$

This shows that $f_0 = 0$ is a free divisor.

Remark 5.3. *We remark on the polynomial f_0 introduced in this subsection. If $x_1 = 0$, then f_0 turns out to be $x_3(x_3^2 - 32x_2^5)$ up to a constant factor. This implies that if $x_1 = 0$, the curve defined by $f_0 = 0$ has the E_{13}-singularity in the sense of Arnol'd. Moreover, regarding x_1 as a parameter, $f_0 = 0$ defines a family of curves in the (x_2, x_3)-plane which is a deformation of the E_{13}-singularity. The polynomial f_0 coincides with the polynomial introduced in* [1, §7.1].

To rewrite the almost Belyi function z defined above in terms of the coordinate

(x_1, x_2, x_3), we introduce polynomials $h_{1a}, h_{1b}, h_1, h_2, h_3, h_4$ by

$$
\begin{aligned}
h_{1a} &= 2x_1^2 + x_2,\\
h_{1b} &= x_2^3 + x_1x_3,\\
h_1 &= h_{1a}h_{1b},\\
h_2 &= -200x_1^4x_2^3 - 160x_1^2x_2^4 - 32x_2^5 - 216x_1^5x_3 - 180x_1^3x_2x_3 - 40x_1x_2^2x_3 + x_3^2,\\
h_3 &= 400x_1^4x_2^8 + 320x_1^2x_2^9 + 64x_2^{10} + 1024x_1^5x_2^5x_3 + 800x_1^3x_2^6x_3 + 160x_1x_2^7x_3\\
&\quad +640x_1^6x_2^2x_3^2 + 320x_1^4x_2^3x_3^2\\
&\quad -40x_1^2x_2^4x_3^2 - 32x_2^5x_3^2 - 192x_1^5x_3^3 - 160x_1^3x_2x_3^3 - 40x_1x_2^2x_3^3 + x_3^4,\\
h_4 &= 8000x_1^6x_2^{12} + 9600x_1^4x_2^{13} + 3840x_1^2x_2^{14} + 512x_2^{15} + 30720x_1^7x_2^9x_3\\
&\quad +36288x_1^5x_2^{10}x_3 + 14400x_1^3x_2^{11}x_3 + 1920x_1x_2^{12}x_3 + 49920x_1^8x_2^6x_3^2\\
&\quad +63360x_1^6x_2^7x_3^2 + 30000x_1^4x_2^8x_3^2 + 6240x_1^2x_2^9x_3^2 + 480x_2^{10}x_3^2 + 40960x_1^9x_2^3x_3^3\\
&\quad +59520x_1^7x_2^4x_3^3 + 35520x_1^5x_2^5x_3^3 + 10240x_1^3x_2^6x_3^3 + 1200x_1x_2^7x_3^3 + 13824x_1^{10}x_3^4\\
&\quad +23040x_1^8x_2x_3^4 + 16320x_1^6x_2^2x_3^4 + 5280x_1^4x_2^3x_3^4 + 540x_1^2x_2^4x_3^4 - 48x_2^5x_3^4\\
&\quad -288x_1^5x_3^5 - 240x_1^3x_2x_3^5 - 60x_1x_2^2x_3^5 + x_3^6.
\end{aligned}
$$

Note that x_3h_2 coincides with f_0 up to a constant factor. By direct computation, we find that

$$2^{10}\cdot 3^3h_1^5x_3^2h_2 - h_3^3 + h_4^2 = 0. \tag{42}$$

Then $Z = \dfrac{2^{10}\cdot 3^3h_1^5x_3^2h_2}{h_3^3}$ is the almost Belyi function essentially the same as z introduced in (41). In particular, (42) shows that $Z-1 = -\dfrac{h_4^2}{h_3^3}$. This reflects an important property of the almost Belyi function.

We define vector field V, W by

$$
\begin{cases}
V = \dfrac{1}{10x_3^{1/6}h_1^{5/6}}\{(-36x_1^3x_2^2 - 14x_1x_2^3 - 2x_1^2x_3 + x_2x_3)V_2\\
\qquad\qquad\qquad +(6x_1^2x_2^2 + 2x_2^3 + x_1x_3)V_3\},\\
W = \dfrac{3^{1/2}}{15}(x_3h_1^5)^{-1/12}\{x_1V_3 - (6x_1^2 + x_2)V_2\}
\end{cases} \tag{43}
$$

and functions L, M by

$$L = -\frac{h_3}{x_3^{1/3}h_1^{5/3}},\quad M = \frac{h_4}{x_3^{1/2}h_1^{5/2}}. \tag{44}$$

It is easy to see that

$$VL = M, \quad VM = -\frac{3}{2}L^2, \quad WL = WM = 0.$$

We put

$$(45) \quad \begin{cases} P(t) = t^3 + 3Lt + 2M, \\ \varphi \quad = \dfrac{20x_1^2x_2^4 + 8x_2^5 - 80x_1^3x_2x_3 - 20x_1x_2^2x_3 + x_3^2}{x_3^{1/6}(h_0h_2)^{5/6}}, \\ \tau \quad = 2^3 \cdot 3^{3/2} \cdot x_3^{3/4} \cdot \dfrac{-20x_1^4x_2^3 - 10x_1^2x_2^4 - x_2^5 + 4x_1^5x_3}{(h_0h_2)^{5/4}}. \end{cases}$$

Then

$$P(\varphi) = \tau^2, \quad W\varphi = \tau.$$

Theorem 5.4. *Put* $E = x_1\partial_{x_1} + 2x_2\partial_{x_2} + 5x_3\partial_{x_3}$. *Then* $\{F, L, M, E, V, W, \varphi, \tau\}$ *is a uniformizing model of type* A_3, *where* $F = \det(M_{E_{13}})$ *and* $L, M, V, W, \varphi, \tau$ *are defined by (44), (43), (45).*

The proof of this theorem is a consequence of the argument above.

We have thus constructed a uniformizing model of type A_3 from the polynomial defining the free divisor $f_0 = 0$.

The problem to be done is to solve

$$\varphi = y_1, \quad 2\tau = y_2, \quad -12L = y_3,$$

which is a system of algebraic equations for x_1, x_2, x_3, where y_1, y_2, y_3 are constants (cf. (22)). This problem is not solved in the present paper and left to the future study.

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