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Multivariate Bernoulli polynomials

Dedicated to T. Oshima for his 70th birthday

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Abstract. We introduce a multivariate analogue of Bernoulli polynomials and give their fundamental properties : difference and differential relations, symmetry, explicit formula, inversion formula, multiplication theorem, and binomial type formula. Further, we consider a multivariate analogue of the multiple Bernoulli polynomials and give their fundamental properties.

1. Introduction

The Bernoulli numbers B_m are defined by the generating function

$$\frac{u}{e^u - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} u^m, \quad |u| < 2\pi, \tag{1}$$

and the Bernoulli polynomials $B_m(z)$ by means of

$$\frac{u}{e^u - 1} e^{zu} = \sum_{m=0}^{\infty} \frac{B_m(z)}{m!} u^m, \quad |u| < 2\pi. \tag{2}$$

Bernoulli polynomial $B_m(z)$ has the following fundamental properties (see for example Chapter 1 Section 13 [1]).

$$B_m(0) = B_m, \tag{3}$$

$$B_m(z+1) - B_m(z) = mz^{m-1} \quad (m \geq 0), \tag{4}$$

$$B'_m(z) = mB_{m-1}(z), \tag{5}$$

$$B_m(1-z) = (-1)^m B_m(z), \tag{6}$$

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$$B_m(z) = \sum_{n=0}^{m} \binom{m}{n} B_n z^{m-n}, \tag{7}$$

$$\begin{aligned} z^m &= \frac{1}{m+1} \sum_{n=0}^{m} \binom{m+1}{n} B_n(z) \\ &= \sum_{n=0}^{m} \frac{1}{m-n+1} \binom{m}{n} B_n(z), \end{aligned} \tag{8}$$

$$\sum_{i=0}^{N-1} B_m \left(z + \frac{i}{N} \right) = N^{1-m} B_m(Nz), \tag{9}$$

$$B_m(z+1) = \sum_{n=0}^{m} \binom{m}{n} B_n\left(z\right). \tag{10}$$

Let us describe proofs of (3) - (10).

(3) It follows from the definition of Bernoulli numbers (1) and polynomials (2).

(4) By the generating function of Bernoulli polynomials (2), the index law and the definition of exponential function e^{zu}, we have

$$\begin{aligned} \sum_{m \geq 0} (B_m(z+1) - B_m(z)) \frac{u^m}{m!} &= \frac{u}{e^u - 1} (e^{(z+1)u} - e^{zu}) \\ &= \frac{u}{e^u - 1} (e^u e^{zu} - e^{zu}) \\ &= ue^{zu} \\ &= \sum_{m \geq 0} z^m u \frac{u^m}{m!} \\ &= \sum_{m \geq 0} z^m \frac{u^{m+1}}{(m+1)!} (m+1) \\ &= \sum_{m \geq 0} m z^{m-1} \frac{u^m}{m!}. \end{aligned}$$

Similarly, we obtain the following.

(5)

$$\begin{aligned} \sum_{m \geq 0} B'_m(z) \frac{u^m}{m!} &= \frac{u}{e^u - 1} \partial_z e^{zu} \\ &= \frac{u}{e^u - 1} e^{zu} u \end{aligned}$$

$$= \sum_{m\geq 0} B_m(z) u \frac{u^m}{m!}$$
$$= \sum_{m\geq 0} B_m(z) \frac{u^{m+1}}{(m+1)!}(m+1)$$
$$= \sum_{m\geq 0} m B_{m-1}(z) \frac{u^m}{m!}.$$

(6)

$$\sum_{m\geq 0} B_m(1-z)\frac{u^m}{m!} = \frac{u}{e^u-1}e^{(1-z)u}$$
$$= \frac{ue^u}{e^u-1}e^{-zu}$$
$$= \frac{-u}{e^{-u}-1}e^{-zu}$$
$$= \sum_{m\geq 0} B_m(z)\frac{(-u)^m}{m!}$$
$$= \sum_{m\geq 0} (-1)^m B_m(z)\frac{u^m}{m!}.$$

(7)

$$\sum_{m\geq 0} B_m(z)\frac{u^m}{m!} = \frac{u}{e^u-1}e^{zu}$$
$$= \sum_{N\geq 0} B_N \frac{u^N}{N!} \sum_{n\geq 0} z^n \frac{u^n}{n!}$$
$$= \sum_{N\geq 0} B_N \sum_{n\geq 0} z^n \frac{u^N}{N!}\frac{u^n}{n!}$$
$$= \sum_{m\geq 0}\sum_{N=0}^{m} B_N z^{m-N}\binom{m}{N}\frac{u^m}{m!}.$$

(8)

$$\sum_{m\geq 0} z^m \frac{u^m}{m!} = e^{zu}$$

$$\begin{aligned}
&= \frac{e^u - 1}{u} \sum_{n \geq 0} B_n(z) \frac{u^n}{n!} \\
&= \sum_{N \geq 0} \frac{1}{N+1} \frac{u^N}{N!} \sum_{n \geq 0} B_n(z) \frac{u^n}{n!} \\
&= \sum_{N \geq 0} \sum_{n \geq 0} \frac{1}{N+1} B_n(z) \frac{u^N}{N!} \frac{u^n}{n!} \\
&= \sum_{N \geq 0} \sum_{n \geq 0} \frac{1}{N+1} B_n(z) \sum_{m \geq 0} \binom{m}{N} \frac{u^m}{m!} \\
&= \sum_{m \geq 0} \sum_{n=0}^{m} \frac{1}{m-n+1} \binom{m}{n} B_n(z) \frac{u^m}{m!}.
\end{aligned}$$

(9)

$$\begin{aligned}
\sum_{m \geq 0} \sum_{i=0}^{N-1} B_m\left(z + \frac{i}{N}\right) \frac{u^m}{m!} &= \sum_{i=0}^{N-1} \frac{u}{e^u - 1} e^{\left(z + \frac{i}{N}\right)u} \\
&= \frac{u}{e^u - 1} e^{zu} \sum_{i=0}^{N-1} e^{\frac{i}{N}u} \\
&= \frac{u}{e^u - 1} e^{zu} \frac{e^u - 1}{e^{\frac{u}{N}} - 1} \\
&= N \frac{\frac{u}{N}}{e^{\frac{u}{N}} - 1} e^{Nz\frac{u}{N}} \\
&= N \sum_{m \geq 0} B_m(Nz) \frac{1}{m!} \left(\frac{u}{N}\right)^m \\
&= \sum_{m \geq 0} N^{1-m} B_m(Nz) \frac{u^m}{m!}.
\end{aligned}$$

(10)

$$\begin{aligned}
\sum_{m \geq 0} B_m(z+1) \frac{u^m}{m!} &= \frac{u}{e^u - 1} e^{(z+1)u} \\
&= e^u \frac{u}{e^u - 1} e^{zu} \\
&= \sum_{n \geq 0} B_n(z) e^u \frac{u^n}{n!}
\end{aligned}$$

$$= \sum_{n\geq 0} B_n(z) \sum_{m\geq 0} \binom{m}{n} \frac{u^m}{m!}$$
$$= \sum_{m\geq 0} \sum_{n\geq 0} \binom{m}{n} B_n(z) \frac{u^m}{m!}.$$

We remark that in the above proofs we only use the following formulas which are trivial results in the one variable case.

Pieri type formulas For any nonnegative integer $m \in \mathbb{Z}$,

$$e^u \frac{u^m}{m!} = \sum_{n\geq 0} \binom{n}{m} \frac{u^n}{n!}. \tag{11}$$

Since

$$e^u = \sum_{N\geq 0} \frac{1}{N!} u^N$$

and comparing the terms of degree $N+m$ in (11), we have

$$\frac{u^N}{N!} \frac{u^m}{m!} = \binom{m+N}{m} \frac{u^{m+N}}{(m+N)!}. \tag{12}$$

In particular, the $N = 1$ case of (12) is the following :

$$u \frac{u^m}{m!} = \binom{m+1}{m} \frac{u^{m+1}}{(m+1)!} = (m+1) \frac{u^{m+1}}{(m+1)!}. \tag{13}$$

Properties of e^{zu}

$$\partial_z e^{zu} = e^{zu} u. \tag{14}$$

In particular, we obtain the index law of e^{zu}

$$e^{(1+z)u} = e^{\partial_z} e^{zu} = e^{zu} e^u. \tag{15}$$

Other formula (trivial!) For any nonnegative integer $N \in \mathbb{Z}$,

$$u^N = N! \frac{u^N}{N!}. \tag{16}$$

On the other hand, a multivariate analogue of the formulas (11) - (16) has been studied (see Section 2), which is non-trivial results unlike the one variable case.

Therefore if we give a good multivariate analogue of Bernoulli polynomials which can be applied a multivariate analogue of (11) - (16), then we drive a multivariate analogue of (3) - (10).

In this article, we introduce a multivariate analogue of Bernoulli polynomials $B_m(z)$ by Jack polynomials and others, which we call "*multivariate Bernoulli polynomials*". We also provide a multivariate analogue of (3) - (10) based on a multivariate analogue of (11) - (16). Further, we consider a multiple analogue of our multivariate Bernoulli polynomials and give their fundamental properties.

The content of this article is as follows. In Section 2, we introduce a multivariate analysis which is a natural generalization of special functions for matrix arguments. In particular, we explain a multivariate analogue of (11) - (16). Section 3 is the main part of this article. In this section, we introduce multivariate Bernoulli polynomials by a generating function which is a natural multivariate analogue of (2), and give their fundamental properties. We also investigate a multivariate analogue of the multiple Bernoulli polynomials which is a multiple analogue of our multivariate Bernoulli polynomials in Section 4.

2. Preliminaries

Refer to [3], [4], [5], [6], [9], [10] for the details in this section. Let $r \in \mathbb{Z}_{\geq 1}$, $d \in \mathbb{C}$ and

$$\begin{aligned}
\mathcal{P} &:= \{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r \mid m_1 \geq \cdots \geq m_r \geq 0\}, \\
\delta &:= (r-1, r-2, \ldots, 2, 1, 0) \in \mathcal{P}, \\
|\mathbf{z}| &:= z_1 + \cdots + z_r, \\
E_k(\mathbf{z}) &:= \sum_{j=1}^{r} z_j^k \partial_{z_j} \quad (k \in \mathbb{Z}_{\geq 0}), \\
D_k(\mathbf{z}) &:= \sum_{j=1}^{r} z_j^k \partial_{z_j}^2 + d \sum_{1 \leq j \neq l \leq r} \frac{z_j^k}{z_j - z_l} \partial_{z_j} \quad (k \in \mathbb{Z}_{\geq 0}).
\end{aligned}$$

For any partition $\mathbf{m} = (m_1, \ldots, m_r) \in \mathcal{P}$ and $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$, put

$$m_{\mathbf{m}}(\mathbf{z}) := \sum_{\mathbf{n} \in \mathfrak{S}_r \mathbf{m}} z^{\mathbf{n}},$$

where $\mathfrak{S}_r$ is the symmetric group in r letters and $\mathbf{z}^{\mathbf{n}} := z_1^{n_1} \cdots z_r^{n_r}$. We define **Jack polynomials** $P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)$ by the following two conditions.

$$(1)\, D_2(\mathbf{z}) P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) \sum_{j=1}^{r} m_j \left(m_j - 1 - d(r-j)\right),$$

$$(2)\, P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = m_{\mathbf{m}}(\mathbf{z}) + \sum_{\mathbf{k}<\mathbf{m}} c_{\mathbf{mk}} m_{\mathbf{k}}(\mathbf{z}).$$

Similarly, **the shifted (or interpolation) Jack polynomials** $P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)$ are defined by the following two conditions.

$$(1)^{\mathrm{ip}}\, P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right) = 0, \quad \text{unless } \mathbf{k} \subset \mathbf{m} \in \mathcal{P}$$

$$(2)^{\mathrm{ip}}\, P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) + (\text{lower terms}).$$

Here, $<$ is the dominance partial ordering which is defined by

$$\mathbf{k} < \mathbf{m} \quad \Leftrightarrow \quad \mathbf{k} \neq \mathbf{m}, \quad \sum_{l=1}^{i} k_l \leq \sum_{l=1}^{i} m_l \quad i = 1, \ldots, r.$$

Further, we put

$$\Phi^{(d)}_{\mathbf{m}}(\mathbf{z}) := \frac{P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)}{P_{\mathbf{m}}\left(\mathbf{1}; \frac{d}{2}\right)} \text{ (normalized Jack polynomials)},$$

$$\Psi^{(d)}_{\mathbf{m}}(\mathbf{z}) := \frac{P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right)}$$

and

$$\binom{\mathbf{z}}{\mathbf{k}}^{(d)} := \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{z} + \frac{d}{2}\delta; \frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{k} + \frac{d}{2}\delta; \frac{d}{2}\right)} \text{ (generalized (or Jack) binomial coefficients)},$$

$${}_0\mathcal{F}_0{}^{(d)}\left(; \mathbf{z}, \mathbf{u}\right) := \sum_{\mathbf{m}\in\mathcal{P}} \Psi^{(d)}_{\mathbf{m}}(\mathbf{z}) \Phi^{(d)}_{\mathbf{m}}(\mathbf{u}) = \sum_{\mathbf{m}\in\mathcal{P}} \Phi^{(d)}_{\mathbf{m}}(\mathbf{z}) \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).$$

Special values From [6] VI (6.14), (10.20) and [4] (4.8), we have

(17)
$$P_{\mathbf{m}}\left(\mathbf{1};\frac{d}{2}\right)=\prod_{(i,j)\in\mathbf{m}}\frac{j-1+\frac{d}{2}(r-i+1)}{m_i-j+\frac{d}{2}(m'_j-i+1)}=\prod_{1\le i<j\le r}\frac{\left(\frac{d}{2}(j-i+1)\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i)\right)_{m_i-m_j}}.$$

Further, by [4] (7.4) and (7.5)

$$P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)=\prod_{(i,j)\in\mathbf{m}}\left(m_i-j+1+\frac{d}{2}(m'_j-i)\right)$$
$$=\prod_{j=1}^{r}\left(\frac{d}{2}(r-j)+1\right)_{m_j}\prod_{1\le i<j\le r}\frac{\left(\frac{d}{2}(j-i-1)+1\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i)+1\right)_{m_i-m_j}}. \tag{18}$$

Although these multivariate special functions are very complicated, we write down these functions explicitly in $r=1$, $r=2$ and $d=2$.

The $r=1$ **case** For non positive integer m and $z\in\mathbb{C}$,

$$P_m\left(z;\frac{d}{2}\right)=z^m,\quad P^{\mathrm{ip}}_m\left(z;\frac{d}{2}\right)=\begin{cases}z(z-1)\cdots(z-m+1) & (m\neq 0)\\ 1 & (m=0)\end{cases}.$$

Further,

$$P_m\left(1;\frac{d}{2}\right)=1,\quad P^{\mathrm{ip}}_m\left(m;\frac{d}{2}\right)=m!,\quad \Phi^{(d)}_m(z)=z^m,\quad \Psi^{(d)}_m(z)=\frac{z^m}{m!},$$
$$\binom{z}{k}^{(d)}=\begin{cases}\frac{z(z-1)\cdots(z-k+1)}{k!} & (k\neq 0)\\ 1 & (k=0)\end{cases},$$
$${}_0\mathcal{F}_0{}^{(d)}\left(;z,u\right)=\sum_{m\ge 0}\frac{z^m}{m!}u^m=\sum_{m\ge 0}z^m\frac{u^m}{m!}=e^{zu}.$$

The $r=2$ **case** (see [4] 10.3, [10] 3.2.1) For any partition $\mathbf{m}=(m_1,m_2)\in\mathcal{P}$ and $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$,

$$P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right)=z_1^{m_1}z_2^{m_2}{}_2F_1\left(\begin{matrix}-m_1+m_2,\frac{d}{2}\\1-m_1+m_2-\frac{d}{2}\end{matrix};\frac{z_2}{z_1}\right)$$
$$P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right)=(-1)^{m_1+m_2}(-z_1)_{m_2}(-z_2)_{m_1}{}_3F_2\left(\begin{matrix}-m_1+m_2,\frac{d}{2},-m_1+1-\frac{d}{2}+z_1\\1-m_1+m_2-\frac{d}{2},-m_1+1+z_2\end{matrix};1\right).$$

Further,

$$
\begin{aligned}
P_{\mathbf{m}}\left(\mathbf{1};\frac{d}{2}\right) &= \frac{(d)_{m_1-m_2}}{\left(\frac{d}{2}\right)_{m_1-m_2}}\\
P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right) &= \frac{\left(\frac{d}{2}+1\right)_{m_1} m_2!(m_1-m_2)!}{\left(\frac{d}{2}+1\right)_{m_1-m_2}},\\
\Phi^{(d)}_{\mathbf{m}}(\mathbf{z}) &= \frac{\left(\frac{d}{2}\right)_{m_1-m_2}}{(d)_{m_1-m_2}} z_1^{m_1} z_2^{m_2}\,{}_2F_1\left(\begin{matrix}-m_1+m_2,\frac{d}{2}\\ 1-m_1+m_2-\frac{d}{2}\end{matrix};\frac{z_2}{z_1}\right),\\
\Psi^{(d)}_{\mathbf{m}}(\mathbf{z}) &= \frac{\left(\frac{d}{2}+1\right)_{m_1-m_2}}{\left(\frac{d}{2}+1\right)_{m_1}(m_1-m_2)!m_2!} z_1^{m_1} z_2^{m_2}\,{}_2F_1\left(\begin{matrix}-m_1+m_2,\frac{d}{2}\\ 1-m_1+m_2-\frac{d}{2}\end{matrix};\frac{z_2}{z_1}\right),\\
\binom{\mathbf{z}}{\mathbf{k}}^{(d)} &= \frac{\left(\frac{d}{2}+1\right)_{k_1-k_2}}{\left(\frac{d}{2}+1\right)_{k_1}(k_1-k_2)!k_2!}(-1)^{k_1+k_2}\left(-z_1-\frac{d}{2}\right)_{k_2}(-z_2)_{k_1}\\
&\quad\cdot {}_3F_2\left(\begin{matrix}-k_1+k_2,\frac{d}{2},-k_1+1+z_1\\ 1-k_1+m_2-\frac{d}{2},-k_1+1+z_2\end{matrix};1\right),\\
{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) &= e^{z_1u_1+z_2u_2}\,{}_1F_1\left(\begin{matrix}\frac{d}{2}\\ d\end{matrix};-(z_1-z_2)(u_1-u_2)\right).
\end{aligned}
$$

The $d=2$ **case** In this case, $P_{\mathbf{m}}(\mathbf{z};1)$ and $P^{\mathrm{ip}}_{\mathbf{m}}(\mathbf{z};1)$ are Schur polynomials and shifted Schur polynomials respectively [8].

$$
\begin{aligned}
P_{\mathbf{m}}(\mathbf{z};1) = s_{\mathbf{m}}(\mathbf{z}) &= \frac{\det\left(z_i^{m_j+r-j}\right)_{1\le i,j\le r}}{\Delta(\mathbf{z})},\\
P^{\mathrm{ip}}_{\mathbf{m}}(\mathbf{z};1) &= \frac{\det\left(P^{\mathrm{ip}}_{m_j+r-j}(z_i+r-i;1)\right)_{1\le i,j\le r}}{\Delta(\mathbf{z})}
\end{aligned}
$$

where $\Delta(\mathbf{z}) := \prod_{1\le i<j\le r}(u_i-u_j)$. Further,

$$
\begin{aligned}
P_{\mathbf{m}}(\mathbf{1};1) = s_{\mathbf{m}}(\mathbf{1}) &= \prod_{1\le i<j\le r}\frac{(j-i)_{m_i-m_j}}{(j-i+1)_{m_i-m_j}},\\
P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m}+\frac{d}{2}\delta;1\right) &= \prod_{j=1}^{r}(r-j+1)_{m_j}\prod_{1\le i<j\le r}\frac{(j-i)_{m_i-m_j}}{(j-i+1)_{m_i-m_j}},\\
\Phi^{(2)}_{\mathbf{m}}(\mathbf{z}) &= \prod_{1\le i<j\le r}\frac{(j-i+1)_{m_i-m_j}}{(j-i)_{m_i-m_j}} s_{\mathbf{m}}(\mathbf{z}),
\end{aligned}
$$

$$\Psi^{(2)}_{\mathbf{m}}(\mathbf{z}) = \prod_{j=1}^{r} \frac{1}{(r-j+1)_{m_j}} \prod_{1\le i<j\le r} \frac{(j-i+1)_{m_i-m_j}}{(j-i)_{m_i-m_j}} s_{\mathbf{m}}(\mathbf{z}),$$
$$\binom{\mathbf{z}}{\mathbf{k}}^{(2)} = \frac{1}{\Delta(\mathbf{z})} \det\left(\binom{z_i+r-i}{k_j+r-j}\right)_{1\le i,j\le r},$$
$${}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) = \frac{\det\left(e^{z_i u_j}\right)_{1\le i,j\le r}}{\Delta(\mathbf{z})\Delta(\mathbf{u})}.$$

	notation	special value at $\mathbf{z}=\mathbf{1}$
Faraut-Korányi	$\Phi^{(d)}_{\mathbf{m}}(\mathbf{z})$	1
Stanley	$J^{\left(\frac{2}{d}\right)}_{\mathbf{m}}(\mathbf{z})$	$\left(\frac{2}{d}\right)^{\lvert\mathbf{m}\rvert}\prod_{(i,j)\in\mathbf{m}}\left(j-1+\frac{d}{2}(r-i+1)\right)$ ([9] Thm. 5.4)
Macdonald	$P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right)$	$\prod_{(i,j)\in\mathbf{m}}\frac{j-1+\frac{d}{2}(r-i+1)}{m_i-j+\frac{d}{2}(m'_j-i+1)}$ ([6] VI (10.20))
Kaneko	$C^{\left(\frac{2}{d}\right)}_{\mathbf{m}}(\mathbf{z})$	$\lvert\mathbf{m}\rvert!\prod_{(i,j)\in\mathbf{m}}\frac{\left(j-1+\frac{d}{2}(r-i+1)\right)}{\left(m_i-j+\frac{d}{2}(m'_j-i+1)\right)\left(m_i-j+1+\frac{d}{2}(m'_j-i)\right)}$ ([3] (18))
S	$\Psi^{(d)}_{\mathbf{m}}(\mathbf{z})$	$\prod_{(i,j)\in\mathbf{m}}\frac{\left(j-1+\frac{d}{2}(r-i+1)\right)}{\left(m_i-j+\frac{d}{2}(m'_j-i+1)\right)\left(m_i-j+1+\frac{d}{2}(m'_j-i)\right)}$

Table 1. Notations and normalizations of Jack polynomials

Remark 2.1. *We remark normalization of various Jack polynomials. First, we list some notations of Jack polynomials and their special values at* $\mathbf{z}=\mathbf{1}$ *(see Table 1). In this article, our notations are based on* [2]*. In particular,*

$$\Psi^{(d)}_{\mathbf{m}}(\mathbf{z}) = d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi^{(d)}_{\mathbf{m}}(\mathbf{z}),$$

where

$$n := r + \frac{d}{2}r(r-1),$$
$$(\alpha)_m := \begin{cases} \alpha(\alpha+1)\cdots(\alpha+m-1) & (m\in\mathbb{Z}_{>0}) \\ 1 & (m=0) \end{cases},$$
$$(\alpha)_{\mathbf{m}} := \prod_{j=1}^{r}\left(\alpha-\frac{d}{2}(j-1)\right)_{m_j},$$
$$d_{\mathbf{m}} := \prod_{1\le i<j\le r} \frac{m_i-m_j+\frac{d}{2}(j-i)}{\frac{d}{2}(j-i)} \frac{\left(\frac{d}{2}(j-i+1)\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i-1)+1\right)_{m_i-m_j}} \quad \text{([2], p315).}$$

From special values of Jack polynomials $P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)$ *and interpolation Jack polynomials* $P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{z} + \frac{d}{2}\delta; \frac{d}{2}\right)$ *(17) and (18), we have*

$$d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} = \frac{P_{\mathbf{m}}\left(\mathbf{1}; \frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right)}.$$

Next, we remark the relationship between Stanley style $J^{\left(\frac{2}{d}\right)}_{\mathbf{m}}(\mathbf{z})$ *and Macdonald style* $P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)$

$$J^{\left(\frac{2}{d}\right)}_{\mathbf{m}}(\mathbf{z}) = \left(\frac{2}{d}\right)^{|\mathbf{m}|} \prod_{(i,j)\in\mathbf{m}} \left(m_i - j + \frac{d}{2}(m'_j - i + 1)\right) P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) \qquad \text{([6] VI (10.22))}$$

where a partition $\mathbf{m}$ *is identified with its diagram :*

$$\mathbf{m} = \{s = (i,j) \mid 1 \leq i \leq r, 1 \leq j \leq m_i\}.$$

Hence we have

$$\begin{aligned}\Psi^{(d)}_{\mathbf{m}}(\mathbf{z}) &= \frac{P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right)} \\ &= \left(\frac{d}{2}\right)^{|\mathbf{m}|} \prod_{(i,j)\in\mathbf{m}} \frac{1}{\left(m_i - j + \frac{d}{2}(m'_j - i + 1)\right)} \frac{1}{P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right)} J^{\left(\frac{2}{d}\right)}_{\mathbf{m}}(\mathbf{z}).\end{aligned}$$

The relationship between $C^{\left(\frac{2}{d}\right)}_{\mathbf{m}}(\mathbf{1})$ *(Kaneko style) and* $\Psi^{(d)}_{\mathbf{m}}(\mathbf{1})$ *(our style)*

$$\begin{aligned}C^{\left(\frac{2}{d}\right)}_{\mathbf{m}}(\mathbf{1}) &= |\mathbf{m}|! \prod_{(i,j)\in\mathbf{m}} \frac{\left(j - 1 + \frac{d}{2}(r - i + 1)\right)}{\left(m_i - j + \frac{d}{2}(m'_j - i + 1)\right)\left(m_i - j + 1 + \frac{d}{2}(m'_j - i)\right)} \\ &= |\mathbf{m}|! \frac{P_{\mathbf{m}}\left(\mathbf{1}; \frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right)}.\end{aligned}$$

follows from (17), (18) and [3] *(18). Thus, we have*

$$\Psi^{(d)}_{\mathbf{m}}(\mathbf{z}) = \frac{1}{|\mathbf{m}|!} C^{\left(\frac{2}{d}\right)}_{\mathbf{m}}(\mathbf{z}).$$

To summarize the above results, we obtain

$$\Psi_{\mathbf{m}}^{(d)}(\mathbf{z}) = d_{\mathbf{m}} \frac{1}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}^{(d)}(\mathbf{z}) = \frac{1}{P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)} P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right) = \frac{1}{|\mathbf{m}|!} C_{\mathbf{m}}^{\left(\frac{2}{d}\right)}(\mathbf{z}). \tag{19}$$

Under the following, we provide all necessary formulas to prove our main results.
Pieri type formulas for Jack polynomials For any partition $\mathbf{m} \in \mathcal{P}$,

$$e^{|\mathbf{u}|}\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) = \sum_{\mathbf{n}\in\mathcal{P}} \binom{\mathbf{n}}{\mathbf{m}}^{(d)} \Psi_{\mathbf{n}}^{(d)}(\mathbf{u}) \quad \text{([5] Section 14)}. \tag{20}$$

Since

$$e^{|\mathbf{u}|} = \sum_{N\geq 0} \frac{1}{N!}|\mathbf{u}|^N$$

and comparing the terms of degree $N+|\mathbf{m}|$ in (20), we have

$$\frac{|\mathbf{u}|^N}{N!}\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) = \sum_{|\mathbf{n}|-|\mathbf{m}|=N} \binom{\mathbf{n}}{\mathbf{m}}^{(d)} \Psi_{\mathbf{n}}^{(d)}(\mathbf{u}). \tag{21}$$

From [5] Section 14,

$$\binom{\mathbf{m}^i}{\mathbf{m}}^{(d)} = \left(m_i + 1 + \frac{d}{2}(r-i)\right) h_{-,i}^{(d)}(\mathbf{m}^i),$$

where $\epsilon_i := (0,\dots,0,\overset{\overset{i}{\vee}}{1},0,\dots,0) \in \mathbb{Z}^r$, $\mathbf{m}^i := \mathbf{m}+\epsilon_i$ and

$$h_{\pm,i}^{(d)}(\mathbf{m}) := \prod_{1\leq k\neq i\leq r} \frac{m_i - m_k - \frac{d}{2}(i-k) \pm \frac{d}{2}}{m_i - m_k - \frac{d}{2}(i-k)}.$$

We remark that if $\mathbf{m}^i \notin \mathcal{P}$ then $h_{-,i}^{(d)}(\mathbf{m}^i) = 0$. In particular, the $N=1$ case of (21) is the following :

(22)

$$|\mathbf{u}|\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) = \sum_{i=1}^{r} \binom{\mathbf{m}^i}{\mathbf{m}}^{(d)} \Psi_{\mathbf{m}^i}^{(d)}(\mathbf{u}) = \sum_{i=1}^{r} \Psi_{\mathbf{m}^i}^{(d)}(\mathbf{u}) \left(m_i + 1 + \frac{d}{2}(r-i)\right) h_{-,i}^{(d)}(\mathbf{m}^i).$$

Properties of ${}_0\mathcal{F}_0{}^{(d)}$ By [5] Section 14, we have

$$E_0(\mathbf{z}){}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)={}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)|\mathbf{u}|. \tag{23}$$

In particular, we obtain the index law of ${}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)$

$${}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{1}+\mathbf{z},\mathbf{u}\right)=e^{E_0(\mathbf{z})}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)={}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)e^{|\mathbf{u}|}. \tag{24}$$

Other formula

$$|\mathbf{u}|^N=N!\sum_{|\mathbf{m}|=N}\Psi_{\mathbf{m}}^{(d)}(\mathbf{u})\quad \text{([9] Prop. 2.3 or [3] (17)).} \tag{25}$$

To summarize the above results, we obtain the following dictionary.

$$P_m\left(1;\frac{d}{2}\right)=1 \quad\Rightarrow\quad P_{\mathbf{m}}\left(\mathbf{1};\frac{d}{2}\right)=\prod_{1\le i<j\le r}\frac{\left(\frac{d}{2}(j-i+1)\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i)\right)_{m_i-m_j}},$$

$$P_m^{\mathrm{ip}}\left(m;\frac{d}{2}\right)=m! \quad\Rightarrow\quad P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)=\prod_{j=1}^{r}\left(\frac{d}{2}(r-j)+1\right)_{m_j}$$

$$\cdot\prod_{1\le i<j\le r}\frac{\left(\frac{d}{2}(j-i-1)+1\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i)+1\right)_{m_i-m_j}},$$

$$\Phi_m^{(d)}(z):=z^m \quad\Rightarrow\quad \Phi_{\mathbf{m}}^{(d)}(\mathbf{z}):=\frac{P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right)}{P_{\mathbf{m}}\left(\mathbf{1};\frac{d}{2}\right)},$$

$$\Psi_m^{(d)}(z):=\frac{z^m}{m!} \quad\Rightarrow\quad \Psi_{\mathbf{m}}^{(d)}(\mathbf{z}):=\frac{P_{\mathbf{m}}\left(\mathbf{1};\frac{d}{2}\right)\Phi_{\mathbf{m}}^{(d)}(\mathbf{z})}{P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)}=\frac{P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right)}{P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)},$$

$$\binom{m}{k}:=\frac{P_k^{\mathrm{ip}}\left(m;\frac{d}{2}\right)}{P_k^{\mathrm{ip}}\left(k;\frac{d}{2}\right)} \quad\Rightarrow\quad \binom{\mathbf{m}}{\mathbf{k}}^{(d)}:=\frac{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)},$$

$$e^{zu}=\sum_{m=0}^{\infty}\frac{1}{m!}z^mu^m \quad\Rightarrow\quad {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right):=\sum_{\mathbf{m}\in\mathcal{P}}\Psi_{\mathbf{k}}^{(d)}(\mathbf{z})\Phi_{\mathbf{k}}^{(d)}(\mathbf{u})$$

$$e^u\frac{u^m}{m!}=\sum_{n=0}^{\infty}\binom{n}{m}\frac{u^n}{n!} \quad\Rightarrow\quad e^{|\mathbf{u}|}\Psi_{\mathbf{m}}^{(d)}(\mathbf{u})=\sum_{\mathbf{n}\in\mathcal{P}}\binom{\mathbf{n}}{\mathbf{m}}^{(d)}\Psi_{\mathbf{n}}^{(d)}(\mathbf{u}),$$

$$\frac{u^N}{N!}\frac{u^m}{m!}=\binom{N+m}{m}\frac{u^{N+m}}{(N+m)!} \quad\Rightarrow\quad \frac{|\mathbf{u}|^N}{N!}\Psi_{\mathbf{m}}^{(d)}(\mathbf{u})=\sum_{|\mathbf{n}|-|\mathbf{m}|=N}\binom{\mathbf{n}}{\mathbf{m}}^{(d)}\Psi_{\mathbf{n}}^{(d)}(\mathbf{u}),$$

$$u\frac{u^m}{m!} = \frac{u^{m+1}}{(m+1)!}(m+1) \quad \Rightarrow \quad |\mathbf{u}|\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) = \sum_{i=1}^{r} \Psi_{\mathbf{m}^i}^{(d)}(\mathbf{u}) \cdot \left(m_i + 1 + \frac{d}{2}(r-i)\right) h_{-,i}^{(d)}(\mathbf{m}^i),$$

$$\partial_z e^{zu} = e^{zu}u \quad \Rightarrow \quad E_0(\mathbf{z}) {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) = {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)|\mathbf{u}|,$$

$$e^{(1+z)u} = e^u e^{zu} \quad \Rightarrow \quad {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{1}+\mathbf{z},\mathbf{u}\right) = e^{|\mathbf{u}|} {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right),$$

$$u^N = N!\frac{u^N}{N!} \quad \Rightarrow \quad |\mathbf{u}|^N = N! \sum_{|\mathbf{m}|=N} \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).$$

3. Multivariate Bernoulli polynomials

We define multivariate Bernoulli polynomials $B_{\mathbf{m}}^{(d)}(\mathbf{z})$ or $B_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right)$ by the following generating function.

$$\frac{u}{e^u-1}e^{zu} = \sum_{m=0}^{\infty} B_m(z)\Psi_m(u) \quad (|u| < 2\pi)$$

$$\Downarrow$$

$$\frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1} {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) = \sum_{\mathbf{m}\in\mathcal{P}} B_{\mathbf{m}}^{(d)}(\mathbf{z})\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) \quad (|u_1+\cdots+u_r| < 2\pi). \tag{26}$$

Remark 3.1. *Originally, we consider the following type generating function and multivariate analogue of Bernoulli polynomials.*

$$\prod_{j=1}^{r} \frac{u_j}{e^{u_j}-1} {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) = \sum_{\mathbf{m}\in\mathcal{P}} \widetilde{B}_{\mathbf{m}}^{(d)}(\mathbf{z})\Psi_{\mathbf{m}}^{(d)}(\mathbf{u})$$

In the $d=2$ case, this type generating function has the determinant expression

$$\prod_{j=1}^{r} \frac{u_j}{e^{u_j}-1} {}_0\mathcal{F}_0{}^{(2)}\left(;\mathbf{z},\mathbf{u}\right) = \frac{\det\left(\frac{u_j e^{z_i u_j}}{e^{u_j}-1}\right)_{1\le i,j\le r}}{\Delta(\mathbf{z})\Delta(\mathbf{u})}$$

and $\widetilde{B}_{\mathbf{m}}^{(2)}(\mathbf{z})$ has the Jacobi-Trudi type formula

$$\widetilde{B}_{\mathbf{m}}^{(2)}(\mathbf{z}) = \frac{\det\left(B_{m_i+r-i}(z_j)\right)}{\Delta(\mathbf{z})}.$$

However, for this multivariate analogue of Bernoulli polynomials, we can not find *an analogue of the formulas (3) - (10). Therefore, we investigate the above type (26) multivariate Bernoulli polynomials.*

THEOREM 3.2.

$$B_{\mathbf{m}}^{(d)}(\mathbf{0}) = B_{|\mathbf{m}|}, \tag{27}$$

$$B_{\mathbf{m}}^{(d)}(\mathbf{z}+\mathbf{1}) - B_{\mathbf{m}}^{(d)}(\mathbf{z}) = \sum_{i=1}^{r} \Phi_{\mathbf{m}_i}^{(d)}(\mathbf{z}) \left(m_i + \frac{d}{2}(r-i) \right) h_{-,i}^{(d)}(\mathbf{m}), \tag{28}$$

$$E_0(\mathbf{z}) B_{\mathbf{m}}^{(d)}(\mathbf{z}) = \sum_{i=1}^{r} B_{\mathbf{m}_i}^{(d)}(\mathbf{z}) \left(m_i + \frac{d}{2}(r-i) \right) h_{-,i}^{(d)}(\mathbf{m}), \tag{29}$$

$$B_{\mathbf{m}}^{(d)}(\mathbf{1}-\mathbf{z}) = (-1)^{|\mathbf{m}|} B_{\mathbf{m}}^{(d)}(\mathbf{z}), \tag{30}$$

$$B_{\mathbf{m}}^{(d)}(\mathbf{z}) = \sum_{N=0}^{|\mathbf{m}|} B_N \sum_{\mathbf{n}\subset\mathbf{m}, |\mathbf{m}|-|\mathbf{n}|=N} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} \Phi_{\mathbf{n}}^{(d)}(\mathbf{z}), \tag{31}$$

$$\Phi_{\mathbf{m}}^{(d)}(\mathbf{z}) = \sum_{\mathbf{n}\subset\mathbf{m}} \frac{1}{|\mathbf{m}|-|\mathbf{n}|+1} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} B_{\mathbf{n}}^{(d)}(\mathbf{z}), \tag{32}$$

$$\sum_{i=0}^{N-1} B_{\mathbf{m}}^{(d)}\left(\mathbf{z} + \frac{i}{N}\mathbf{1}\right) = N^{1-|\mathbf{m}|} B_{\mathbf{m}}^{(d)}(N\mathbf{z}), \tag{33}$$

$$B_{\mathbf{m}}^{(d)}(\mathbf{z}+\mathbf{1}) = \sum_{\mathbf{n}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} B_{\mathbf{n}}^{(d)}(\mathbf{z}). \tag{34}$$

PROOF. (27) By the definition of the multivariate Bernoulli polynomials and Bernoulli numbers, we have

$$\sum_{\mathbf{m}\in\mathcal{P}} B_{\mathbf{m}}^{(d)}(\mathbf{0}) \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) = \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1} = \sum_{N=0}^{\infty} \frac{B_N}{N!} |\mathbf{u}|^N.$$

On the other hand, by (25)

$$\sum_{N=0}^{\infty} \frac{B_N}{N!} |\mathbf{u}|^N = \sum_{N=0}^{\infty} B_N \sum_{|\mathbf{m}|=N} \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) = \sum_{\mathbf{m}\in\mathcal{P}} B_{|\mathbf{m}|} \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).$$

(28) By (24) and (22), we have

$$\sum_{\mathbf{m}\in\mathcal{P}} \left(B_{\mathbf{m}}^{(d)}(\mathbf{z}+\mathbf{1}) - B_{\mathbf{m}}^{(d)}(\mathbf{z}) \right) \Psi_{\mathbf{m}}^{(d)}(\mathbf{z})$$

$$
\begin{aligned}
&= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}\left({}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z}+\mathbf{1},\mathbf{u}\right) - {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\right) \\
&= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}\left(e^{|\mathbf{u}|}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) - {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\right) \\
&= |\mathbf{u}|{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} \Phi^{(d)}_{\mathbf{m}}(\mathbf{z})|\mathbf{u}|\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} \Phi^{(d)}_{\mathbf{m}}(\mathbf{z})\sum_{i=1}^{r}\Psi^{(d)}_{\mathbf{m}^i}(\mathbf{u})\left(m_i+1+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m}^i) \\
&= \sum_{\mathbf{m}\in\mathcal{P}}\sum_{i=1}^{r} \Phi^{(d)}_{\mathbf{m}_i}(\mathbf{z})\left(m_i+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}
$$

(29) By (23) and (22),

$$
\begin{aligned}
\sum_{\mathbf{m}\in\mathcal{P}} E_0(\mathbf{z})B^{(d)}_{\mathbf{m}}\left(\mathbf{z}\right)\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) &= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}E_0(\mathbf{z}){}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \\
&= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)|\mathbf{u}| \\
&= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{\mathbf{m}}\left(\mathbf{z}\right)|\mathbf{u}|\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{\mathbf{m}}\left(\mathbf{z}\right) \\
&\quad \cdot \sum_{i=1}^{r}\Psi^{(d)}_{\mathbf{m}^i}(\mathbf{u})\left(m_i+1+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m}^i) \\
&= \sum_{\mathbf{m}\in\mathcal{P}}\sum_{i=1}^{r} B^{(d)}_{\mathbf{m}_i}(\mathbf{z})\left(m_i+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}
$$

(30) By (24),

$$
\begin{aligned}
\sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{\mathbf{m}}\left(\mathbf{1}-\mathbf{z}\right)\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) &= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{1}-\mathbf{z},\mathbf{u}\right) \\
&= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},-\mathbf{u}\right)e^{|\mathbf{u}|} \\
&= \frac{|-\mathbf{u}|}{e^{|-\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},-\mathbf{u}\right)
\end{aligned}
$$

$$= \sum_{\mathbf{m}\in\mathcal{P}} B_{\mathbf{m}}^{(d)}(\mathbf{z})\,\Psi_{\mathbf{m}}^{(d)}(-\mathbf{u})$$
$$= \sum_{\mathbf{m}\in\mathcal{P}} (-1)^{|\mathbf{m}|} B_{\mathbf{m}}^{(d)}(\mathbf{z})\,\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).$$

(31) By (21),

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathcal{P}} B_{\mathbf{m}}^{(d)}(\mathbf{z})\,\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) &= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}\,{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\\
&= \sum_{N=0}^{\infty} \frac{B_N}{N!}|\mathbf{u}|^N \sum_{\mathbf{n}\in\mathcal{P}} \Phi_{\mathbf{n}}^{(d)}(\mathbf{z})\Psi_{\mathbf{n}}^{(d)}(\mathbf{u})\\
&= \sum_{N=0}^{\infty} B_N \sum_{\mathbf{n}\in\mathcal{P}} \Phi_{\mathbf{n}}^{(d)}(\mathbf{z})\frac{|\mathbf{u}|^N}{N!}\Psi_{\mathbf{n}}^{(d)}(\mathbf{u})\\
&= \sum_{N=0}^{\infty} B_N \sum_{\mathbf{n}\in\mathcal{P}} \Phi_{\mathbf{n}}^{(d)}(\mathbf{z}) \sum_{|\mathbf{m}|-|\mathbf{n}|=N} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} \Psi_{\mathbf{m}}^{(d)}(\mathbf{u})\\
&= \sum_{\mathbf{m}\in\mathcal{P}} \sum_{N=0}^{|\mathbf{m}|} B_N \sum_{\mathbf{n}\subset\mathbf{m},|\mathbf{m}|-|\mathbf{n}|=N} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} \Phi_{\mathbf{n}}^{(d)}(\mathbf{z})\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).
\end{aligned}$$

(32) By (21),

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathcal{P}} \Phi_{\mathbf{m}}^{(d)}(\mathbf{z})\,\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) &= {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\\
&= \frac{e^{|\mathbf{u}|}-1}{|\mathbf{u}|} \sum_{\mathbf{n}\in\mathcal{P}} B_{\mathbf{n}}^{(d)}(\mathbf{z})\,\Psi_{\mathbf{n}}^{(d)}(\mathbf{u})\\
&= \sum_{N=0}^{\infty} \frac{1}{N+1}\frac{1}{N!}|\mathbf{u}|^N \sum_{\mathbf{n}\in\mathcal{P}} B_{\mathbf{n}}^{(d)}(\mathbf{z})\,\Psi_{\mathbf{n}}^{(d)}(\mathbf{u})\\
&= \sum_{N=0}^{\infty} \frac{1}{N+1} \sum_{\mathbf{n}\in\mathcal{P}} B_{\mathbf{n}}^{(d)}(\mathbf{z})\,\frac{|\mathbf{u}|^N}{N!}\Psi_{\mathbf{n}}^{(d)}(\mathbf{u})\\
&= \sum_{N=0}^{\infty} \frac{1}{N+1} \sum_{\mathbf{n}\in\mathcal{P}} B_{\mathbf{n}}^{(d)}(\mathbf{z}) \sum_{|\mathbf{m}|-|\mathbf{n}|=N} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} \Psi_{\mathbf{m}}^{(d)}(\mathbf{u})\\
&= \sum_{\mathbf{m}\in\mathcal{P}} \sum_{\mathbf{n}\subset\mathbf{m}} \frac{1}{|\mathbf{m}|-|\mathbf{n}|+1} B_{\mathbf{n}}^{(d)}(\mathbf{z}) \binom{\mathbf{m}}{\mathbf{n}}^{(d)} \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).
\end{aligned}$$

(33) By (24) and the summation of a geometric series,

$$\begin{aligned}\sum_{\mathbf{m}\in\mathcal{P}}\sum_{i=0}^{N-1} B_{\mathbf{m}}^{(d)}\left(\mathbf{z}+\frac{i}{N}\mathbf{1}\right)\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) &= \sum_{i=0}^{N-1}\frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z}+\frac{i}{N}\mathbf{1},\mathbf{u}\right)\\ &= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\sum_{i=0}^{N-1}e^{\frac{i}{N}|\mathbf{u}|}\\ &= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\frac{e^{|\mathbf{u}|}-1}{e^{\frac{|\mathbf{u}|}{N}}-1}\\ &= N\frac{\frac{|\mathbf{u}|}{N}}{e^{\frac{|\mathbf{u}|}{N}}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;N\mathbf{z},\frac{\mathbf{u}}{N}\right)\\ &= N\sum_{\mathbf{m}\in\mathcal{P}} B_{\mathbf{m}}^{(d)}\left(N\mathbf{z}\right)\Psi_{\mathbf{m}}^{(d)}\left(\frac{\mathbf{u}}{N}\right)\\ &= \sum_{\mathbf{m}\in\mathcal{P}} N^{1-|\mathbf{m}|}B_{\mathbf{m}}^{(d)}\left(N\mathbf{z}\right)\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).\end{aligned}$$

(34) By (24)

$$\begin{aligned}\sum_{\mathbf{m}\in\mathcal{P}} B_{\mathbf{m}}^{(d)}\left(\mathbf{z}+\mathbf{1}\right)\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) &= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z}+\mathbf{1},\mathbf{u}\right)\\ &= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}e^{|\mathbf{u}|}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\\ &= \sum_{\mathbf{n}\in\mathcal{P}} B_{\mathbf{n}}^{(d)}\left(\mathbf{z}\right)e^{|\mathbf{u}|}\Psi_{\mathbf{n}}^{(d)}(\mathbf{u})\\ &= \sum_{\mathbf{n}\in\mathcal{P}} B_{\mathbf{n}}^{(d)}\left(\mathbf{z}\right)\sum_{\mathbf{m}\in\mathcal{P}}\binom{\mathbf{m}}{\mathbf{n}}^{(d)}\Psi_{\mathbf{m}}^{(d)}(\mathbf{u})\\ &= \sum_{\mathbf{m}\in\mathcal{P}}\sum_{\mathbf{n}\subset\mathbf{m}}\binom{\mathbf{m}}{\mathbf{n}}^{(d)} B_{\mathbf{n}}^{(d)}\left(\mathbf{z}\right)\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).\end{aligned}$$

□

4. A multivariate analogue of the multiple Bernoulli polynomials

For n-tuple complex numbers

$$\boldsymbol{\omega} := (\omega_1,\ldots,\omega_n),\quad \omega_j\in\mathbb{C}\setminus\{0\},$$

we define the multiple Bernoulli polynomials $B_{n,m}\left(z \mid \boldsymbol{\omega}\right)$ with a generating function

$$e^{zu}\prod_{j=1}^{n}\frac{u}{e^{\omega_j u}-1}=\sum_{m\geq 0}B_{n,m}\left(z \mid \boldsymbol{\omega}\right)\Psi_m(u)\quad (|\omega_j u|<2\pi, j=1,\ldots,n). \tag{35}$$

Let

$$\begin{aligned}\widehat{\boldsymbol{\omega}}(j) &:= (\omega_1,\cdots,\omega_{j-1},\omega_{j+1},\cdots,\omega_r)\in\mathbb{C}^{r-1}\\ &= (\omega_1,\cdots,\widehat{\omega}_j,\cdots,\omega_r),\\ \boldsymbol{\omega}^-[j] &:= (\omega_1,\cdots,-\omega_j,\cdots,\omega_r)\in\mathbb{C}^{r}.\end{aligned}$$

For $B_{n,m}\left(z \mid \boldsymbol{\omega}\right)$, the following formulas are well-known (see [7] (12)–(17)).

$$B_{n,m}(cz \mid c\boldsymbol{\omega}) = c^{m-n}B_{n,m}(z \mid \boldsymbol{\omega})\ \ (c\in\mathbb{C}^*), \tag{36}$$

$$B_{n,m}(|\boldsymbol{\omega}|-z \mid \boldsymbol{\omega}) = (-1)^m B_{n,m}(z \mid \boldsymbol{\omega}), \tag{37}$$

$$B_{n,m}(z+\omega_j \mid \boldsymbol{\omega}) - B_{n,m}(z \mid \boldsymbol{\omega}) = mB_{n-1,m-1}(z \mid \widehat{\boldsymbol{\omega}}(j)), \tag{38}$$

$$B_{n,m}(z \mid \boldsymbol{\omega}^-[j]) = -B_{n,m}(z+\omega_j \mid \boldsymbol{\omega}), \tag{39}$$

$$B_{n,m}(z \mid \boldsymbol{\omega}) + B_{n,m}(z \mid \boldsymbol{\omega}^-[j]) = -mB_{n-1,m-1}(z \mid \widehat{\boldsymbol{\omega}}(j)), \tag{40}$$

$$\frac{d}{dz}B_{n,m}(z \mid \boldsymbol{\omega}) = mB_{n,m-1}(z \mid \boldsymbol{\omega}). \tag{41}$$

We also introduce a multivariate analogue of the multiple Bernoulli polynomials by

$${}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\prod_{i=1}^{n}\frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1} = \sum_{\mathbf{m}\in\mathcal{P}}B^{(d)}_{n,\mathbf{m}}\left(\mathbf{z} \mid \boldsymbol{\omega}\right)\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) \tag{42}$$

and obtain a multivariate analogue of the above formulas (36)–(41) easily.

Theorem 4.1.

$$B^{(d)}_{n,\mathbf{m}}(c\mathbf{z} \mid c\boldsymbol{\omega}) = c^{|\mathbf{m}|-n}B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega})\ \ (c\in\mathbb{C}^*), \tag{43}$$

$$B^{(d)}_{n,\mathbf{m}}(|\boldsymbol{\omega}|\mathbf{1}-\mathbf{z} \mid \boldsymbol{\omega}) = (-1)^{|\mathbf{m}|}B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}), \tag{44}$$

(45)

$$B^{(d)}_{n,\mathbf{m}}(\mathbf{z}+\omega_j\mathbf{1} \mid \boldsymbol{\omega}) - B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}) = \sum_{i=1}^{r}B^{(d)}_{n-1,\mathbf{m}_i}(z \mid \widehat{\boldsymbol{\omega}}(j))\left(m_i+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m}),$$

(46) $$B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}^-[j]) = - B^{(d)}_{n,\mathbf{m}}(\mathbf{z} + \omega_j \mathbf{1} \mid \boldsymbol{\omega}),$$

(47)
$$B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}) + B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}^-[j]) = - \sum_{i=1}^{r} B^{(d)}_{n-1,\mathbf{m}_i}(z \mid \widehat{\boldsymbol{\omega}}(j)) \left(m_i + \frac{d}{2}(r-i) \right) h^{(d)}_{-,i}(\mathbf{m}),$$

(48) $$E_0(\mathbf{z}) B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}) = \sum_{i=1}^{r} B^{(d)}_{n,\mathbf{m}_i}(z \mid \boldsymbol{\omega}) \left(m_i + \frac{d}{2}(r-i) \right) h^{(d)}_{-,i}(\mathbf{m}).$$

PROOF. (43) From the generating function of the multiple multivariate Bernoulli polynomials and homogeneity of Jack polynomials, we have

$$\begin{aligned}
\sum_{\mathbf{m} \in \mathcal{P}} B^{(d)}_{n,\mathbf{m}}(c\mathbf{z} \mid c\boldsymbol{\omega}) \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) &= {}_0\mathcal{F}_0{}^{(d)} \left(; c\mathbf{z}, \mathbf{u}\right) \prod_{i=1}^{n} \frac{|\mathbf{u}|}{e^{c\omega_i |\mathbf{u}|} - 1} \\
&= c^{-n} {}_0\mathcal{F}_0{}^{(d)} \left(; \mathbf{z}, c\mathbf{u}\right) \prod_{i=1}^{n} \frac{|c\mathbf{u}|}{e^{\omega_i |c\mathbf{u}|} - 1} \\
&= c^{-n} {}_0\mathcal{F}_0{}^{(d)} \left(; \mathbf{z}, c\mathbf{u}\right) \prod_{i=1}^{n} \frac{|c\mathbf{u}|}{e^{\omega_i |c\mathbf{u}|} - 1} \\
&= c^{-n} \sum_{\mathbf{m} \in \mathcal{P}} B^{(d)}_{n,\mathbf{m}} \left(\mathbf{z} \mid \boldsymbol{\omega}\right) \Psi^{(d)}_{\mathbf{m}}(c\mathbf{u}) \\
&= \sum_{\mathbf{m} \in \mathcal{P}} c^{|\mathbf{m}|-n} B^{(d)}_{n,\mathbf{m}} \left(\mathbf{z} \mid \boldsymbol{\omega}\right) \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}$$

(44) By (24),

$$\begin{aligned}
\sum_{\mathbf{m} \in \mathcal{P}} B^{(d)}_{n,\mathbf{m}}(|\boldsymbol{\omega}|\mathbf{1} - \mathbf{z} \mid \boldsymbol{\omega}) \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) &= {}_0\mathcal{F}_0{}^{(d)} \left(; |\boldsymbol{\omega}|\mathbf{1} - \mathbf{z}, \mathbf{u}\right) \prod_{i=1}^{n} \frac{|\mathbf{u}|}{e^{\omega_i |\mathbf{u}|} - 1} \\
&= e^{|\boldsymbol{\omega}||\mathbf{u}|} {}_0\mathcal{F}_0{}^{(d)} \left(; \mathbf{z}, -\mathbf{u}\right) \prod_{i=1}^{n} \frac{|\mathbf{u}|}{e^{\omega_i |\mathbf{u}|} - 1} \\
&= {}_0\mathcal{F}_0{}^{(d)} \left(; \mathbf{z}, -\mathbf{u}\right) \prod_{i=1}^{n} \frac{|\mathbf{u}| e^{\omega_i |\mathbf{u}|}}{e^{\omega_i |\mathbf{u}|} - 1} \\
&= {}_0\mathcal{F}_0{}^{(d)} \left(; \mathbf{z}, -\mathbf{u}\right) \prod_{i=1}^{n} \frac{|-\mathbf{u}|}{e^{-\omega_i |\mathbf{u}|} - 1} \\
&= \sum_{\mathbf{m} \in \mathcal{P}} (-1)^{|\mathbf{m}|} B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}) \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}$$

(45) By (24), (22)

$$\begin{aligned}
&\sum_{\mathbf{m}\in\mathcal{P}} (B^{(d)}_{n,\mathbf{m}}(\mathbf{z}+\omega_j\mathbf{1}\mid\boldsymbol{\omega}) - B^{(d)}_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega}))\Psi^{(d)}_{\mathbf{m}}(\mathbf{u})\\
&= \left({}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z}+\omega_j\mathbf{1},\mathbf{u}\right) - {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\right)\prod_{i=1}^{n}\frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1}\\
&= (e^{\omega_j|\mathbf{u}|}-1){}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\prod_{i=1}^{n}\frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1}\\
&= |\mathbf{u}|{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\prod_{1\le i\ne j\le n}\frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1}\\
&= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{n-1,\mathbf{m}}(z\mid\widehat{\boldsymbol{\omega}}(j))|\mathbf{u}|\Psi^{(d)}_{\mathbf{m}}(\mathbf{u})\\
&= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{n-1,\mathbf{m}}(z\mid\widehat{\boldsymbol{\omega}}(j))\sum_{i=1}^{r}\Psi^{(d)}_{\mathbf{m}^i}(\mathbf{u})\left(m_i+1+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m}^i)\\
&= \sum_{\mathbf{m}\in\mathcal{P}}\sum_{i=1}^{r} B^{(d)}_{n-1,\mathbf{m}_i}(z\mid\widehat{\boldsymbol{\omega}}(j))\left(m_i+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}$$

(46) By (24)

$$\begin{aligned}
&\sum_{\mathbf{m}\in\mathcal{P}} B_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega}^-[j])\Psi^{(d)}_{\mathbf{m}}(\mathbf{u})\\
&= {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\frac{|\mathbf{u}|}{e^{-\omega_j|\mathbf{u}|}-1}\prod_{1\le i\ne j\le n}\frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1}\\
&= -e^{\omega_j|\mathbf{u}|}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\frac{|\mathbf{u}|}{e^{\omega_j|\mathbf{u}|}-1}\prod_{1\le i\ne j\le n}\frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1}\\
&= -{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z}+\omega_j\mathbf{1},\mathbf{u}\right)\prod_{i=1}^{n}\frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1}\\
&= \sum_{\mathbf{m}\in\mathcal{P}} -B_{n,\mathbf{m}}(\mathbf{z}+\omega_j\mathbf{1}\mid\boldsymbol{\omega})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}$$

(47) By (46) and (45), we have

$$\begin{aligned}
&B_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega}) + B_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega}^-[j])\\
&\quad= B_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega}) - B_{n,\mathbf{m}}(\mathbf{z}+\omega_j\mathbf{1}\mid\boldsymbol{\omega})
\end{aligned}$$

$$= -\sum_{i=1}^{r} B^{(d)}_{n-1,\mathbf{m}_i}(z \mid \widehat{\boldsymbol{\omega}}(j)) \left(m_i + \frac{d}{2}(r-i)\right) h^{(d)}_{-,i}(\mathbf{m}).$$

(48) By (23) and (22), we have

$$\begin{aligned}\sum_{\mathbf{m}\in\mathcal{P}} E_0(\mathbf{z}) B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}) \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) &= E_0(\mathbf{z}) {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \prod_{i=1}^{n} \frac{|\mathbf{u}|}{e^{c\omega_i|\mathbf{u}|}-1} \\ &= |\mathbf{u}| {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \prod_{i=1}^{n} \frac{|\mathbf{u}|}{e^{c\omega_i|\mathbf{u}|}-1} \\ &= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}) |\mathbf{u}| \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) \\ &= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}) \\ &\quad \cdot \sum_{i=1}^{r} \Psi^{(d)}_{\mathbf{m}^i}(\mathbf{u}) \left(m_i + 1 + \frac{d}{2}(r-i)\right) h^{(d)}_{-,i}(\mathbf{m}^i) \\ &= \sum_{i=1}^{r} B^{(d)}_{n,\mathbf{m}_i}(z \mid \boldsymbol{\omega}) \left(m_i + \frac{d}{2}(r-i)\right) h^{(d)}_{-,i}(\mathbf{m}) \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).\end{aligned}$$

□

5. Concluding remarks

Since our multivariate Bernoulli polynomials have various properties which are regarded as a natural generalization of (3) - (10), our multivariate Bernoulli polynomials are regarded as a good multivariate analogue of Bernoulli polynomials. Therefore we desire to find a multivariate zeta function whose some special values are written by our multivariate Bernoulli polynomials $B^{(d)}_{\mathbf{m}}(\mathbf{z})$.

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References

[1] Erdélyi, A. et al. : *Higher transcendental functions. Vol. 1*, McGraw-Hill New York, 1953.
[2] J. Faraut and A. Korányi : *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.
[3] J. Kaneko : *Selberg integrals and hypergeometric functions associated with Jack polynomials*, SIAM J. Math. Anal., **24** (1993), 1086–1110.

[4] T. H. Koornwinder : *Okounkov's BC-type interpolation Macdonald polynomials and their* $q = 1$ *limit*, Sém. Lothar. Combin, **72** (2014/15), 27pp.
[5] M. Lassalle : *Coefficients binomiaux généralisés et polynômes de Macdonald*, J. Funct. Anal., **158** (1998), 289–324.
[6] I. G. Macdonald : *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1995.
[7] A. Narukawa : *The modular properties and the integral representations of the multiple elliptic gamma functions*, Adv. Math. **189**-2 (2004), 247–267.
[8] A. Okounkov and G. Olshanski : *Shifted Jack polynomials, binomial formula, and applications*, Math. Res. Letters, **4** (1997), 69–78.
[9] R. Stanley : *Some combinatorial properties of Jack symmetric functions*, Adv. Math., **77**-1 (1989) 76–115.
[10] N. Ja. Vilenkin and A. U. Klimyk: *Representation of Lie Groups and Special Functions – Recent Advances–*, Kluwer Academic Publishers, 1995.

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