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A revisit to periodic continuants

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Abstract. We give a simple proof of some explicit formulas of periodic continuants by Chebyshev polynomials of the second kind given by Rózsa. We also mention an application of these explicit formulas to q-analogue of rationals and continued fractions introduced by Morier-Genoud and Ovsienko.

1. Introduction

Let $\mathbb{Z}_{\geq p}$ be a set of integers greater than or equal to p. For an integer p and infinite complex sequences $\mathbf{a}_p := (a_m)_{m\in\mathbb{Z}_{\geq p}}$, $\mathbf{b}_p := (b_m)_{m\in\mathbb{Z}_{\geq p}}$ and $\mathbf{c}_p := (c_m)_{m\in\mathbb{Z}_{\geq p}}$, we put

$$\boldsymbol{\alpha}_p := (\mathbf{a}_p, \mathbf{b}_p, \mathbf{c}_p).$$

We define the (extended) continuant polynomials $K_n(\boldsymbol{\alpha}_p)$ by

$$K_{-1}(\boldsymbol{\alpha}_p) := 0, \quad K_0(\boldsymbol{\alpha}_p) := 1, \quad K_1(\boldsymbol{\alpha}_p) := a_p, \quad K_n(\boldsymbol{\alpha}_p) := \det T_n(\boldsymbol{\alpha}_p),$$

where $T_n(\boldsymbol{\alpha}_p)$ is the following $n \times n$ tridiagonal matrix:

$$T_n(\boldsymbol{\alpha}_p) = \begin{pmatrix} a_p & b_p & 0 & \cdots & 0 & 0 \\ c_p & a_{p+1} & b_{p+1} & \cdots & 0 & 0 \\ 0 & c_{p+1} & a_{p+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{p+n-2} & b_{p+n-2} \\ 0 & 0 & 0 & \cdots & c_{p+n-2} & a_{p+n-1} \end{pmatrix}.$$

There has been many research on continuant polynomials in relation to continued fractions and orthogonal polynomials. For continuant polynomials, several properties have been well known since Euler, especially regarding some explicit for-

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mulas [3]. P. Rózsa [2] considered for a positive integer l the following l-periodicity condition for the sequences $\mathbf{a}_p, \mathbf{b}_p$ and $\mathbf{c}_p$

$$a_{m+l} = a_m, \quad b_{m+l} = b_m, \quad c_{m+l} = c_m,$$

and proposed an explicit formula of l-periodic continuant $K_n(\boldsymbol{\alpha}_p)$ by the Chebyshev polynomials of the second kind.

In this article, we give another proof of this explicit formula. While Rózsa's proof is based on direct calculations of the determinant of the definition for continuant polynomials, our proof uses only 2×2 matrices. We also mention some examples of this explicit formula related to q-deformed rationals and q-continued fractions introduced by Morier-Genoud and Ovsienko [1]. Since Morier-Genoud and Ovsienko related their q-deformations of rationals and continued fractions to the Jones polynomials of rational knots and the F-polynomials of a cluster algebra with coefficients respectively, our explicit formulas for the periodic continuants are useful to give explicit expressions by the Chebyshev polynomials of some examples of the Jones polynomials and the F-polynomials.

2. Preliminaries

Throughout the paper, we denote the ring of rational integers by $\mathbb{Z}$. We set the Gauss hypergeometric function

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) := \sum_{m \geq 0} \frac{(a)_m (b)_m}{m! (c)_m} x^m, \quad (a)_m := \begin{cases} a(a+1)\cdots(a+m-1) & (m \neq 0) \\ 1 & (m = 0) \end{cases}.$$

Chebyshev polynomial of the second kind is defined by

$$U_n(x) := (n+1)\,{}_2F_1\left(\begin{matrix} -n, n+2 \\ \frac{3}{2} \end{matrix}; \frac{1-x}{2}\right) = \sum_{k=0}^{n} \binom{n}{k} \frac{(n+1)_{k+1}}{\left(\frac{3}{2}\right)_k} \left(\frac{1-x}{2}\right)^k,$$

$$\binom{n}{k} := \begin{cases} \frac{n(n-1)\cdots(n-k+1)}{k!} & (k \neq 0) \\ 1 & (k = 0) \end{cases}.$$

It should be remarked that by the definition of $U_n(x)$ we have

$$U_0(x) = 1, \quad U_{-1}(x) = 0, \quad U_{-2}(x) = -1.$$

The generating function for the $U_n(x)$ is

$$\frac{1}{1-2xu+u^2} = \sum_{n\geq 0} U_n(x)u^n. \tag{1}$$

Let $h_n(x,y)$ denote the bivariate complete homogeneous symmetric polynomials of degree n

$$h_n(x,y) := \sum_{\substack{i+j=n\\ 0\leq i,j\leq n}} x^i y^j = \frac{x^{n+1}-y^{n+1}}{x-y}.$$

The generating function for the $h_n(x,y)$ is

$$\frac{1}{(1-xu)(1-yu)} = \sum_{n\geq 0} h_n(x,y)u^n. \tag{2}$$

By (1) and (2) we have

$$h_n(x,y) = \begin{cases} (xy)^{\frac{n}{2}} U_n\left(\frac{x+y}{2\sqrt{xy}}\right) & (xy\neq 0) \\ (x+y)^n & (xy=0) \end{cases}. \tag{3}$$

Lemma 2.1. *Let A be a complex matrix*

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and E_2 be the 2×2 identity matrix. For any positive integer m, we have

$$\begin{aligned} A^m &= h_{m-1}(\rho_+,\rho_-)A - h_{m-2}(\rho_+,\rho_-)(\det A)E_2 \\ &= \begin{cases} (\det A)^{\frac{m-1}{2}} U_{m-1}\left(\frac{\operatorname{tr} A}{2\sqrt{\det A}}\right) A \\ \quad -(\det A)^{\frac{m}{2}} U_{m-2}\left(\frac{\operatorname{tr} A}{2\sqrt{\det A}}\right) E_2 & (\det A\neq 0) \\ (\operatorname{tr} A)^{m-1}A & (\det A = 0) \end{cases}. \end{aligned} \tag{4}$$

Here ρ_+ and ρ_- are the roots of the characteristic polynomial $\det(\lambda E_2 - A)$.

Proof. We consider Euclidean division for λ^m and $\det(\lambda E_2 - A) = (\lambda-\rho_+)(\lambda-\rho_-)$. By the Euclidean theorem for division of polynomials, there exist unique polynomial $q(\lambda)$ and two constants c_1, c_0 such that

$$\lambda^m = q(\lambda)(\lambda - \rho_+)(\lambda - \rho_-) + c_1\lambda + c_0. \tag{5}$$

By substituting $\rho_\pm$ for λ in (5), we have

$$\rho_\pm^m = c_1\rho_\pm + c_0.$$

Hence we obtain

$$\begin{aligned} c_1 &= \frac{\rho_+^m - \rho_-^m}{\rho_+ - \rho_-} = h_{m-1}(\rho_+, \rho_-), \\ c_0 &= -\frac{\rho_+^m\rho_- - \rho_+\rho_-^m}{\rho_+ - \rho_-} = -h_{m-2}(\rho_+, \rho_-) \det A. \end{aligned}$$

We remark that these expressions hold even if the case of $\rho_+ = \rho_-$. □

Example 2.2 *(Power of a quaternion). Let a, b, c, d be real numbers. We define 2×2 matrices Q, I, J and K by*

$$Q = Q(a,b,c,d) = aE_2 + bI + cJ + dK = \begin{pmatrix} a + b\sqrt{-1} & c + d\sqrt{-1} \\ -c + d\sqrt{-1} & a - b\sqrt{-1} \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$I := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix},$$

which is a matrix realization of a quaternion:

$$q = a+bi+cj+dk \in \mathbb{H} := \{a+bi+cj+dk \mid a,b,c,d \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}.$$

By substituting

$$\det Q = a^2 + b^2 + c^2 + d^2 =: |Q|^2, \quad \mathrm{tr}\ Q = 2a$$

and $A = Q$ in (4), we have

$$\begin{aligned} Q^n &= |Q|^{n-1}U_{n-1}\left(\frac{a}{|Q|}\right)Q - |Q|^n U_{n-2}\left(\frac{a}{|Q|}\right)E_2 \\ &= |Q|^n\left\{\frac{a}{|Q|}U_{n-1}\left(\frac{a}{|Q|}\right) - U_{n-2}\left(\frac{a}{|Q|}\right)\right\}E_2 \\ &\quad + |Q|^{n-1}U_{n-1}\left(\frac{a}{|Q|}\right)(bI + cJ + dK) \end{aligned}$$

$$= \frac{|Q|^n}{2}\left\{U_n\left(\frac{a}{|Q|}\right) - U_{n-2}\left(\frac{a}{|Q|}\right)\right\}E_2 + |Q|^{n-1}U_{n-1}\left(\frac{a}{|Q|}\right)(bI + cJ + dK).$$

The last equality follows from the Pieri's formula for $U_n(x)$:

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x).$$

If we put

$$Q(a,b,c,d)^n = A_n(a,b,c,d)E_2 + B_n(a,b,c,d)I + C_n(a,b,c,d)J + D_n(a,b,c,d)K,$$

then we obtain

$$\begin{aligned} A_n(a,b,c,d) &= \frac{|Q|^n}{2}\left\{U_n\left(\frac{a}{|Q|}\right) - U_{n-2}\left(\frac{a}{|Q|}\right)\right\} = |q|^n T_n\left(\frac{a}{|q|}\right), \\ B_n(a,b,c,d) &= b|Q|^{n-1}U_{n-1}\left(\frac{a}{|Q|}\right), \\ C_n(a,b,c,d) &= c|Q|^{n-1}U_{n-1}\left(\frac{a}{|Q|}\right), \\ D_n(a,b,c,d) &= d|Q|^{n-1}U_{n-1}\left(\frac{a}{|Q|}\right), \end{aligned}$$

where $T_n(x)$ is Chebyshev polynomials of the first kind defined by

$$T_n(x) := {}_2F_1\left(\begin{matrix}-n, n \\ \frac{1}{2}\end{matrix}; \frac{1-x}{2}\right) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(n)_k}{\left(\frac{1}{2}\right)_k}\left(\frac{1-x}{2}\right)^k, T_n(\cos\theta) = \cos n\theta.$$

The most right hand side of $A_n(a,b,c,d)$ follows from

$$\frac{1}{2}\{U_{n+1}(x) - U_{n-1}(x)\} = \frac{1}{2}\left\{\frac{\sin(n+1)\theta}{\sin\theta} - \frac{\sin(n-1)\theta}{\sin\theta}\right\} = \cos n\theta = T_n(x).$$

Lemma 2.3 (Fundamental properties of continuant polynomials). 1)

$$\begin{aligned} &K_{-1}(\boldsymbol{\alpha}_p) := 0, \quad K_0(\boldsymbol{\alpha}_p) := 1, \\ &K_n(\boldsymbol{\alpha}_p) = a_pK_{n-1}(\boldsymbol{\alpha}_{p+1}) - b_pc_pK_{n-2}(\boldsymbol{\alpha}_{p+2}). \end{aligned} \tag{6}$$

2) *Let*

$$L(\alpha,\beta) := \begin{pmatrix} \alpha\ \beta \\ 1\ 0 \end{pmatrix},$$
$$A_n(\boldsymbol{\alpha}_p) := L(a_p, -b_p c_p) L(a_{p+1}, -b_{p+1}c_{p+1}) \cdots L(a_{p+n-1}, -b_{p+n-1}c_{p+n-1}).$$

We have

$$A_n(\boldsymbol{\alpha}_p) = \begin{pmatrix} K_n(\boldsymbol{\alpha}_p) & -b_{p+n-1}c_{p+n-1}K_{n-1}(\boldsymbol{\alpha}_p) \\ K_{n-1}(\boldsymbol{\alpha}_{p+1}) & -b_{p+n-1}c_{p+n-1}K_{n-2}(\boldsymbol{\alpha}_{p+1}) \end{pmatrix}. \tag{7}$$

Especially

$$\mathrm{tr}\ A_n(\boldsymbol{\alpha}_p) = K_n(\boldsymbol{\alpha}_p) - b_{p+n-1}c_{p+n-1}K_{n-2}(\boldsymbol{\alpha}_{p+1}), \tag{8}$$
$$\det\ A_n(\boldsymbol{\alpha}_p) = \prod_{j=1}^{n} b_{p+j-1}c_{p+j-1}.$$

3) *Put*

$$\mathbf{k}_{n+1}(\boldsymbol{\alpha}_p) := \begin{pmatrix} K_{n+1}(\boldsymbol{\alpha}_p) \\ K_n(\boldsymbol{\alpha}_{p+1}) \end{pmatrix}.$$

For any integer m such that $n \geq m$, we have

$$\mathbf{k}_{n+1}(\boldsymbol{\alpha}_p) = A_m(\boldsymbol{\alpha}_p)\mathbf{k}_{n+1-m}(\boldsymbol{\alpha}_{p+m}). \tag{9}$$

4) *If for any integer n $c_n = -1$, then we have*

$$a_p + \mathop{\mathrm{K}}_{i=1}^{n-1} \frac{b_{p+i-1}}{a_{p+i}} = \frac{K_n(\boldsymbol{\alpha}_p)}{K_{n-1}(\boldsymbol{\alpha}_{p+1})}, \tag{10}$$

where

$$a_p + \mathop{\mathrm{K}}_{i=1}^{n-1} \frac{b_{p+i-1}}{a_{p+i}} := a_p + \cfrac{b_p}{a_{p+1} + \cfrac{b_{p+1}}{a_{p+2} + \cfrac{b_{p+2}}{\ddots\ \cfrac{}{a_{p+n-2} + \frac{b_{p+n-2}}{a_{p+n-1}}}}}}.$$

PROOF. 1) It follows from the definition of $K_n(\boldsymbol{\alpha}_p)$.

2) When $n = 1$, (7) holds. Assume the result true for n. From induction on n and (6), we have

$$\begin{aligned} A_{n+1}(\boldsymbol{\alpha}_p) &= L(a_p, -b_p c_p) A_n(\boldsymbol{\alpha}_{p+1}) \\ &= \begin{pmatrix} a_p & -b_p c_p \\ 1 & 0 \end{pmatrix} \begin{pmatrix} K_n(\boldsymbol{\alpha}_{p+1}) & -b_{p+n} c_{p+n} K_{n-1}(\boldsymbol{\alpha}_{p+1}) \\ K_{n-1}(\boldsymbol{\alpha}_{p+2}) & -b_{p+n} c_{p+n} K_{n-2}(\boldsymbol{\alpha}_{p+2}) \end{pmatrix} \\ &= \begin{pmatrix} K_{n+1}(\boldsymbol{\alpha}_{p+1}) & -b_{p+n} c_{p+n} K_n(\boldsymbol{\alpha}_{p+1}) \\ K_n(\boldsymbol{\alpha}_{p+1}) & -b_{p+n} c_{p+n} K_{n-1}(\boldsymbol{\alpha}_{p+1}) \end{pmatrix}. \end{aligned}$$

Here, the third equality follows from (6).
3) By the definition of $K_n(\boldsymbol{\alpha}_p)$ and (6),

$$\begin{aligned} \mathbf{k}_{n+1}(\boldsymbol{\alpha}_p) &= \begin{pmatrix} K_{n+1}(\boldsymbol{\alpha}_p) \\ K_n(\boldsymbol{\alpha}_{p+1}) \end{pmatrix} \\ &= \begin{pmatrix} a_p K_n(\boldsymbol{\alpha}_{p+1}) - b_p c_p K_{n-1}(\boldsymbol{\alpha}_{p+2}) \\ K_n(\boldsymbol{\alpha}_{p+1}) \end{pmatrix} \\ &= L(a_p, -b_p c_p) \mathbf{k}_n(\boldsymbol{\alpha}_{p+1}). \end{aligned}$$

Hence

$$\begin{aligned} &\mathbf{k}_{n+1}(\boldsymbol{\alpha}_p) \\ &\quad = L(a_p, -b_p c_p) L(a_{p+1}, -b_{p+1} c_{p+1}) \cdots L(a_{p+m-1}, -b_{p+m-1} c_{p+m-1}) \mathbf{k}_{n+1-m}(\boldsymbol{\alpha}_{p+m}) \\ &\quad = A_m(\boldsymbol{\alpha}_p) \mathbf{k}_{n+1-m}(\boldsymbol{\alpha}_{p+m}). \end{aligned}$$

4) It follows from (6) and induction on n. □

3. Main results

Under the following we assume

$$a_{p+l} = a_p, \quad b_{p+l} = b_p, \quad c_{p+l} = c_p \quad (p \in \mathbb{Z}),$$

that is to say

$$\boldsymbol{\alpha}_{p+l} = \boldsymbol{\alpha}_p. \tag{11}$$

THEOREM 3.1. *For any positive integer m, we obtain*

(12)

$$K_{lm}(\boldsymbol{\alpha}_p) = \begin{cases} (\det\ A_l(\boldsymbol{\alpha}_p))^{\frac{m-1}{2}} U_{m-1}\left(\frac{\mathrm{tr}\ A_l(\boldsymbol{\alpha}_p)}{2\sqrt{\det\ A_l(\boldsymbol{\alpha}_p)}}\right) K_l(\boldsymbol{\alpha}_p) \\ \quad -(\det\ A_l(\boldsymbol{\alpha}_p))^{\frac{m}{2}} U_{m-2}\left(\frac{\mathrm{tr}\ A_l(\boldsymbol{\alpha}_p)}{2\sqrt{\det\ A_l(\boldsymbol{\alpha}_p)}}\right) & (\det\ A_l(\boldsymbol{\alpha}_p) \neq 0) \\ (\mathrm{tr}\ A_l(\boldsymbol{\alpha}_p))^{m-1} K_l(\boldsymbol{\alpha}_p) & (\det\ A_l(\boldsymbol{\alpha}_p) = 0) \end{cases},$$

(13)

$$K_{lm-1}(\boldsymbol{\alpha}_{p+1}) = \begin{cases} (\det\ A_l(\boldsymbol{\alpha}_p))^{\frac{m-1}{2}} U_{m-1}\left(\frac{\mathrm{tr}\ A_l(\boldsymbol{\alpha}_p)}{2\sqrt{\det\ A_l(\boldsymbol{\alpha}_p)}}\right) K_{l-1}(\boldsymbol{\alpha}_{p+1}) & (\det\ A_l(\boldsymbol{\alpha}_p) \neq 0) \\ (\mathrm{tr}\ A_l(\boldsymbol{\alpha}_p))^{m-1} K_{l-1}(\boldsymbol{\alpha}_{p+1}) & (\det\ A_l(\boldsymbol{\alpha}_p) = 0) \end{cases}.$$

PROOF. By (9) and periodicity (11)

$$\mathbf{k}_{lm}(\boldsymbol{\alpha}_p) = A_l(\boldsymbol{\alpha}_p)\mathbf{k}_{l(m-1)}(\boldsymbol{\alpha}_{p+l}) = A_l(\boldsymbol{\alpha}_p)\mathbf{k}_{l(m-1)}(\boldsymbol{\alpha}_p).$$

Then we have

$$\mathbf{k}_{lm}(\boldsymbol{\alpha}_p) = A_l(\boldsymbol{\alpha}_p)^m \mathbf{k}_0(\boldsymbol{\alpha}_p).$$

When

$$\det\ A_l(\boldsymbol{\alpha}_p) = \prod_{j=1}^{l} b_{p+j-1}c_{p+j-1} \neq 0,$$

from (4) we have

$$\begin{aligned} A_l(\boldsymbol{\alpha}_p)^m = (\det\ A_l(\boldsymbol{\alpha}_p))^{\frac{m-1}{2}} U_{m-1}\left(\frac{\mathrm{tr}\ A_l(\boldsymbol{\alpha}_p)}{2\sqrt{\det\ A_l(\boldsymbol{\alpha}_p)}}\right) A_l(\boldsymbol{\alpha}_p) \\ - (\det\ A_l(\boldsymbol{\alpha}_p))^{\frac{m}{2}} U_{m-2}\left(\frac{\mathrm{tr}\ A_l(\boldsymbol{\alpha}_p)}{2\sqrt{\det\ A_l(\boldsymbol{\alpha}_p)}}\right) E_2. \end{aligned}$$

If $\det\ A_l(\boldsymbol{\alpha}_p) = 0$, then

$$A_l(\boldsymbol{\alpha}_p)^m = (\mathrm{tr}\ A_l(\boldsymbol{\alpha}_p))^{m-1} A_l(\boldsymbol{\alpha}_p).$$

Finally, by (7)

$$A_l(\boldsymbol{\alpha}_p) = \begin{pmatrix} K_l(\boldsymbol{\alpha}_p) & -b_{p+l-1}c_{p+l-1}K_{l-1}(\boldsymbol{\alpha}_p) \\ K_{l-1}(\boldsymbol{\alpha}_{p+1}) & -b_{p+l-1}c_{p+l-1}K_{l-2}(\boldsymbol{\alpha}_{p+1}) \end{pmatrix}$$
$$= \begin{pmatrix} K_l(\boldsymbol{\alpha}_p) & -b_{p-1}c_{p-1}K_{l-1}(\boldsymbol{\alpha}_p) \\ K_{l-1}(\boldsymbol{\alpha}_{p+1}) & -b_{p-1}c_{p-1}K_{l-2}(\boldsymbol{\alpha}_{p+1}) \end{pmatrix}.$$

By comparing the entries of the vector $\mathbf{k}_{lm}(\boldsymbol{\alpha}_p)$, we obtain the conclusion. □

Our main result follows from this theorem immediately.

THEOREM 3.2. *For any positive integer m and $j = -1, 0, 1, \ldots, l-2$, we have*

(14)

$$K_{lm+j}(\boldsymbol{\alpha}_{p-j})$$
$$= K_j(\boldsymbol{\alpha}_{p-j})K_{lm}(\boldsymbol{\alpha}_p) - b_{p-1}c_{p-1}K_{j-1}(\boldsymbol{\alpha}_{p-j})K_{lm-1}(\boldsymbol{\alpha}_{p+1})$$
$$= \begin{cases} (\det A_l(\boldsymbol{\alpha}_p))^{\frac{m-1}{2}} U_{m-1}\left(\frac{\mathrm{tr}\ A_l(\boldsymbol{\alpha}_p)}{2\sqrt{\det A_l(\boldsymbol{\alpha}_p)}}\right) \\ \quad \cdot\{K_j(\boldsymbol{\alpha}_{p-j})K_l(\boldsymbol{\alpha}_p) - b_{p-1}c_{p-1}K_{j-1}(\boldsymbol{\alpha}_{p-j})K_{l-1}(\boldsymbol{\alpha}_{p+1})\} \\ \quad -(\det A_l(\boldsymbol{\alpha}_p))^{\frac{m}{2}} U_{m-2}\left(\frac{\mathrm{tr}\ A_l(\boldsymbol{\alpha}_p)}{2\sqrt{\det A_l(\boldsymbol{\alpha}_p)}}\right) K_j(\boldsymbol{\alpha}_{p-j}) & (\det A_l(\boldsymbol{\alpha}_p) \neq 0) \\ (\mathrm{tr}\ A_l(\boldsymbol{\alpha}_p))^{m-1} \\ \quad \cdot\{K_j(\boldsymbol{\alpha}_{p-j})K_l(\boldsymbol{\alpha}_p) - b_{p-1}c_{p-1}K_{j-1}(\boldsymbol{\alpha}_{p-j})K_{l-1}(\boldsymbol{\alpha}_{p+1})\} & (\det A_l(\boldsymbol{\alpha}_p) = 0) \end{cases} .$$

Here we define $K_{-2}(\boldsymbol{\alpha}_{p+1})$ by

$$-b_{p-1}c_{p-1}K_{-2}(\boldsymbol{\alpha}_{p+1}) := K_0(\boldsymbol{\alpha}_{p-1}) = 1.$$

PROOF. From (9) and (7), we have

(15)
$$\mathbf{k}_{lm+j}(\boldsymbol{\alpha}_{p-j})$$
$$= A_j(\boldsymbol{\alpha}_{p-j})\mathbf{k}_{lm}(\boldsymbol{\alpha}_p)$$
$$= \begin{pmatrix} K_j(\boldsymbol{\alpha}_{p-j}) & -b_{p-1}c_{p-1}K_{j-1}(\boldsymbol{\alpha}_{p-j}) \\ K_{j-1}(\boldsymbol{\alpha}_{p-j+1}) & -b_{p-1}c_{p-1}K_{j-2}(\boldsymbol{\alpha}_{p-j+1}) \end{pmatrix} \begin{pmatrix} K_{lm}(\boldsymbol{\alpha}_p) \\ K_{lm-1}(\boldsymbol{\alpha}_{p+1}) \end{pmatrix}.$$

By (12), (13) and comparing the entires of (15), we obtain our main result (14). □

4. Examples

In this section, we give the examples of (12) for $l = 1, 2, 3$ explicitly.

4.1. $l = 1$

In this subsection, we put

$$a := a_p = a_{p+1}, \quad b := b_p = b_{p+1}, \quad c := c_p = c_{p+1}.$$

In the case of $l = 1$, since

$$\mathrm{tr}\ A_1(\boldsymbol{\alpha}_p) = a, \quad \det\ A_1(\boldsymbol{\alpha}_p) = bc,$$

we have the following well-known result:

$$K_m(\boldsymbol{\alpha}_p) = \begin{cases} (bc)^{\frac{m-1}{2}} U_{m-1}\left(\frac{a}{2\sqrt{bc}}\right) & (bc \neq 0) \\ a^{m-1} & (bc = 0) \end{cases}. \tag{16}$$

4.2. $l = 2$

In this subsection, we put

$$\begin{aligned} &a_1 := a_{2m+1} = a_{2m+3}, \quad b_1 := b_{2m+1} = b_{2m+3}, \quad c_1 := c_{2m+1} = c_{2m+3}, \\ &a_2 := a_{2m} = a_{2m+2}, \quad b_2 := b_{2m} = b_{2m+2}, \quad c_2 := c_{2m} = c_{2m+2}. \end{aligned}$$

Since

$$A_2(\boldsymbol{\alpha}_p) = \begin{pmatrix} a_1a_2 - b_pc_p & -a_pb_{p+1}c_{p+1} \\ a_{p+1} & -b_{p+1}c_{p+1} \end{pmatrix}$$

and

$$\mathrm{tr}\ A_2(\boldsymbol{\alpha}_p) = a_1a_2 - b_1c_1 - b_2c_2, \quad \det\ A_1(\boldsymbol{\alpha}_p) = b_1c_1b_2c_2,$$

our main result (14) is

$$\begin{aligned} &K_{2m}(\boldsymbol{\alpha}_p) \\ &= \begin{cases} (b_1c_1b_2c_2)^{\frac{m-1}{2}} U_{m-1}\left(\frac{a_1a_2-b_1c_1-b_2c_2}{2\sqrt{b_1c_1b_2c_2}}\right)(a_1a_2 - b_pc_p) & \\ \quad -(b_1c_1b_2c_2)^{\frac{m}{2}} U_{m-2}\left(\frac{a_1a_2-b_1c_1-b_2c_2}{2\sqrt{b_1c_1b_2c_2}}\right) & (b_1c_1b_2c_2 \neq 0) \\ (a_1a_2 - b_1c_1 - b_2c_2)^{m-1}(a_1a_2 - b_pc_p) & (b_1c_1b_2c_2 = 0) \end{cases}, \end{aligned} \tag{17}$$

(18) $K_{2m-1}(\boldsymbol{\alpha}_{p+1})$

$$= \begin{cases} (b_1c_1b_2c_2)^{\frac{m-1}{2}} U_{m-1}\left(\frac{a_1a_2-b_1c_1-b_2c_2}{2\sqrt{b_1c_1b_2c_2}}\right) a_{p+1} & (b_1c_1b_2c_2 \neq 0) \\ (a_1a_2 - b_1c_1 - b_2c_2)^{m-1} a_{p+1} & (b_1c_1b_2c_2 = 0) \end{cases}.$$

Example 4.1 *(A q-analogue of Fibonacci numbers). Morier-Genoud and Ovsienko* [1] *introduced the following notion of q-deformed rational numbers and continued fractions, motivated by Jones polynomials of rational knots or F-polynomials of a cluster algebra with coefficients. For a positive rational number* $\frac{r}{s}$ *and its (regular) continued fraction*

$$\frac{r}{s} = a_1 + \operatorname*{K}_{i=1}^{2n-1} \frac{1}{a_{i+1}}, \quad a_1, \ldots, a_{2n} > 0,$$

their q-analogue are defined by

$$\left[\frac{r}{s}\right]_q := [a_1]_q + \operatorname*{K}_{i=1}^{2n-1} \frac{q^{(-1)^{i-1}a_i}}{[a_{i+1}]_{q^{(-1)^i}}},$$

where $q \neq 0$ *is a complex parameter and*

$$[a]_q := \frac{1-q^a}{1-q}.$$

Our formulas (20) and (21) are useful to write down these q-analogue explicitly. We consider the case of

$$a_p = [a]_{q^{(-1)^{p-1}}}, \quad b_p = q^{(-1)^{p-1}a}, \quad c_p = -1, \tag{19}$$

where a is a positive integer. By substituting (19) in (17) and (18), we have

$$\begin{aligned} K_{2m}(\boldsymbol{\alpha}_p) = U_{m-1}&\left(\frac{[a]_q[a]_{q^{-1}} + q^a + q^{-a}}{2}\right)([a]_q[a]_{q^{-1}} + q^{(-1)^{p-1}a}) \\ &- U_{m-2}\left(\frac{[a]_q[a]_{q^{-1}} + q^a + q^{-a}}{2}\right), \end{aligned} \tag{20}$$

$$K_{2m-1}(\boldsymbol{\alpha}_{p+1}) = U_{m-1}\left(\frac{[a]_q[a]_{q^{-1}} + q^a + q^{-a}}{2}\right)[a]_{q^{(-1)^p}}. \tag{21}$$

From (10), we derive an explicit formula

(22)

$$[a]_q + \underset{i=1}{\overset{2n-1}{\mathrm{K}}} \frac{q^{(-1)^{i-1}a}}{[a]_{q^{(-1)^i}}}$$
$$= \frac{U_{m-1}\left(\frac{[a]_q[a]_{q^{-1}}+q^a+q^{-a}}{2}\right)([a]_q[a]_{q^{-1}} + q^{(-1)^{p-1}a}) - U_{m-2}\left(\frac{[a]_q[a]_{q^{-1}}+q^a+q^{-a}}{2}\right)}{U_{m-1}\left(\frac{[a]_q[a]_{q^{-1}}+q^a+q^{-a}}{2}\right)[a]_{q^{(-1)^p}}}.$$

In particular, The case of $a = 1$ is a q-analogue of Fibonacci numbers defined by

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n.$$

As is well known, the continued fraction of $\frac{F_{2n+1}}{F_{2n}}$ is given by

$$\frac{F_{2n+1}}{F_{2n}} = 1 + \underset{i=1}{\overset{2n-1}{\mathrm{K}}} \frac{1}{1}.$$

Thus, a q-analogue of this rational number and continued fraction expansion are equal to

$$\left[\frac{F_{2n+1}}{F_{2n}}\right]_q = 1 + \underset{i=1}{\overset{2n-1}{\mathrm{K}}} \frac{q^{(-1)^{i-1}}}{1}.$$

We put

$$a_1 = a_2 = 1, \quad b_i = q^{(-1)^{i-1}}, \quad c_1 = c_2 = -1$$

and

$$F_{2m+2}(q) := K_{2m+1}(\boldsymbol{\alpha}_2), \quad F_{2m+1}(q) := K_{2m}(\boldsymbol{\alpha}_1).$$

From Lemma 2.3 (4)*, we have*

$$\left[\frac{F_{2n+1}}{F_{2n}}\right]_q = \frac{F_{2n+1}(q)}{F_{2n}(q)}.$$

This sequence $\{F_n(q)\}$ *satisfies*

$$F_1(q) = 1, \quad F_2(q) = 1, \quad F_3(q) = 1 + q$$

and

$$F_{2m}(q) = F_{2m-1}(q) + q^{-1}F_{2m-2}(q), \quad F_{2m+1}(q) = F_{2m}(q) + qF_{2m-1}(q).$$

From (20) and (21), $\{F_n(q)\}$ *has the following explicit expression:*

$$F_{2m+2}(q) = U_{m-1}\left(\frac{1+q+q^{-1}}{2}\right)(1+q^{-1}) - U_{m-2}\left(\frac{1+q+q^{-1}}{2}\right), \tag{23}$$

$$F_{2m+1}(q) = U_{m-1}\left(\frac{1+q+q^{-1}}{2}\right). \tag{24}$$

4.3. $l = 3$

Put

$$\begin{aligned} a_1 &:= a_{3m+1} = a_{3m+4}, \quad b_1 := b_{3m+1} = b_{3m+4}, \quad c_1 := c_{3m+1} = c_{3m+4}, \\ a_2 &:= a_{3m+2} = a_{3m+5}, \quad b_2 := b_{3m+2} = b_{3m+5}, \quad c_2 := c_{3m+2} = c_{3m+5}, \\ a_3 &:= a_{3m} = a_{3m+3}, \quad b_3 := b_{3m} = b_{3m+3}, \quad c_3 := c_{3m} = c_{3m+3}. \end{aligned}$$

From (7) and (8), we have

$$A_3(\boldsymbol{\alpha}_p) = \begin{pmatrix} a_1a_2a_3 - a_{p+2}b_pc_p - a_pb_{p+1}c_{p+1} & -a_pa_{p+1}b_{p+2}c_{p+2} + b_pc_pb_{p+2}c_{p+2} \\ a_{p+1}a_{p+2} - b_{p+1}c_{p+1} & -a_{p+1}b_{p+2}c_{p+2} \end{pmatrix}$$

and

$$\mathrm{tr}\ A_3(\boldsymbol{\alpha}_p) = a_1a_2a_3 - a_1b_2c_2 - a_2b_3c_3 - a_3b_1c_1, \quad \det\ A_1(\boldsymbol{\alpha}_p) = \prod_{j=1}^{3} b_jc_j.$$

Then (14) can be written as

(25)

$$K_{3m+1}(\boldsymbol{\alpha}_{p-1})$$
$$= a_{p-1}K_{3m}(\boldsymbol{\alpha}_p) - b_{p-1}c_{p-1}K_{3m-1}(\boldsymbol{\alpha}_{p+1})$$
$$= \begin{cases} \prod_{j=1}^{3}(b_jc_j)^{\frac{m-1}{2}} U_{m-1}\left(\frac{a_1a_2a_3 - a_1b_2c_2 - a_2b_3c_3 - a_3b_1c_1}{2\sqrt{b_1c_1b_2c_2b_3c_3}}\right) \\ \quad \cdot\{a_{p-1}(a_1a_2a_3 - a_{p+2}b_pc_p - a_pb_{p+1}c_{p+1}) \\ \quad\quad - b_{p-1}c_{p-1}(a_{p+1}a_{p+2} - b_{p+1}c_{p+1})\} \\ \quad - \prod_{j=1}^{3}(b_jc_j)^{\frac{m}{2}} U_{m-2}\left(\frac{a_1a_2a_3 - a_1b_2c_2 - a_2b_3c_3 - a_3b_1c_1}{2\sqrt{b_1c_1b_2c_2b_3c_3}}\right) a_{p-1} & (\prod_{j=1}^{3} b_jc_j \neq 0), \\ (a_1a_2a_3 - a_1b_2c_2 - a_2b_3c_3 - a_3b_1c_1)^{m-1} \\ \quad \cdot\{a_{p-1}(a_1a_2a_3 - a_{p+2}b_pc_p - a_pb_{p+1}c_{p+1}) \\ \quad\quad - b_{p-1}c_{p-1}(a_{p+1}a_{p+2} - b_{p+1}c_{p+1})\} & (\prod_{j=1}^{3} b_jc_j = 0) \end{cases}$$

(26)

$$K_{3m}(\boldsymbol{\alpha}_p)$$
$$= \begin{cases} \prod_{j=1}^{3}(b_jc_j)^{\frac{m-1}{2}} U_{m-1}\left(\frac{a_1a_2a_3 - a_1b_2c_2 - a_2b_3c_3 - a_3b_1c_1}{2\sqrt{b_1c_1b_2c_2b_3c_3}}\right) \\ \quad \cdot(a_1a_2a_3 - a_{p+2}b_pc_p - a_pb_{p+1}c_{p+1}) \\ \quad - \prod_{j=1}^{3}(b_jc_j)^{\frac{m}{2}} U_{m-2}\left(\frac{a_1a_2a_3 - a_1b_2c_2 - a_2b_3c_3 - a_3b_1c_1}{2\sqrt{b_1c_1b_2c_2b_3c_3}}\right) & (\prod_{j=1}^{3} b_jc_j \neq 0), \\ (a_1a_2a_3 - a_1b_2c_2 - a_2b_3c_3 - a_3b_1c_1)^{m-1} \\ \quad \cdot(a_1a_2a_3 - a_{p+2}b_pc_p - a_pb_{p+1}c_{p+1}) & (\prod_{j=1}^{3} b_jc_j = 0) \end{cases}$$

(27)

$$K_{3m-1}(\boldsymbol{\alpha}_{p+1})$$
$$= \begin{cases} \prod_{j=1}^{3}(b_jc_j)^{\frac{m-1}{2}} U_{m-1}\left(\frac{a_1a_2a_3 - a_1b_2c_2 - a_2b_3c_3 - a_3b_1c_1}{2\sqrt{b_1c_1b_2c_2b_3c_3}}\right) \\ \quad \cdot(a_{p+1}a_{p+2} - b_{p+1}c_{p+1}) & (\prod_{j=1}^{3} b_jc_j \neq 0) \\ (a_1a_2a_3 - a_1b_2c_2 - a_2b_3c_3 - a_3b_1c_1)^{m-1} \\ \quad \cdot(a_{p+1}a_{p+2} - b_{p+1}c_{p+1}) & (\prod_{j=1}^{3} b_jc_j = 0) \end{cases}.$$

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References

[1] S. Morier-Genoud and V. Ovsienko, *q-deformed rationals and q-continued fractions*, Forum Math. Sigma **8** (2020), Paper No. e13, pp55.
[2] P. Rózsa, *On periodic continuants*, Linear Algebra Appl. **2** (1969), 267–274.
[3] M. Thomas, *A treatise on the theory of determinants*, Dover, 1960.

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