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Rotation numbers of regular closed curves on oriented aspherical surfaces

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Abstract. Whitney's rotation number classifies regular closed curves on the euclidean plane up to regular homotopy and, when the self-intersections of the curve are transverse double points, there is a combinatorial formula for the rotation number obtained by algebraically counting the self-intersections of the curve. In this paper, I generalize these results to the case of curves on oriented surfaces with a complete euclidean or hyperbolic metric.

1. Introduction

Whitney-Graustein theorem says that two regular closed curves on the euclidean plane $\mathbb{E}^2$ are regularly homotopic if and only if they have the same 'rotation number' ([5]), which is defined as follows. A regular (not necessarily closed) curve $\gamma : [a, b] \to \mathbb{E}^2$ induces a continuous function $\theta : [a, b] \to \mathbb{R}$, called the *angle function* of γ, which gives the angle of the tangent vector $\dot{\gamma}(t)$ measured from the positive direction of the x-axis, and the *euclidean rotation number* i_γ of γ is defined by

$$i_\gamma = (\theta(b) - \theta(a))/2\pi \in \mathbb{R}.$$

It is independent of the choice of θ. When γ is a regular closed curve, i_γ is an integer. Note that an analogous 'euclidean rotation number' of an oriented polygonal closed curve on $\mathbb{E}^2$ can be defined to be the integer given by the formula

$$\text{(the sum of the external angles at the vertices)}/2\pi \in \mathbb{Z}\ .$$

On the other hand, take any geodesic triangle with the positive orientation on the standard unit 2-sphere S^2. Then the formula above does not give an integer because the sum of the external angles is $2\pi -$ (the area of the triangle), and, if we slightly move the vertices, then its quotient by 2π changes continuously. This suggests

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that the 'rotation number' of a regular closed curve on S^2 cannot be defined by directly using the standard metric on S^2. Instead, assuming that the curve does not pass through the north pole P, we project the curve to the euclidean plane $\mathbb{E}^2$ by the stereographic projection and take the euclidean rotation number of the image. When a regular homotopy passes P, this number changes by ± 2; so it is well-defined only in $\mathbb{Z}/2\mathbb{Z}$. It is well-known that this rotation number gives the regular homotopy classification of regular closed curves on S^2.

In this paper, we consider regular closed curves on an oriented surface M which has a complete euclidean metric or a complete hyperbolic metric, define an integer valued *rotation number* $W(\gamma)$, and prove the following:

THEOREM 1.1. *Suppose M is an oriented surface with a complete euclidean or hyperbolic metric. For regular closed curves γ, γ' on M, the followings are equivalent.*
(1) *γ and γ' are regularly homotopic.*
(2) *γ and γ' are freely homotopic and have the same rotation number.*

Because of Gauss-Bonnet theorem, the given riemannian metric does not give us the desired rotation number when the metric is hyperbolic. The idea is to use a lift $\widetilde{\gamma}$ to the universal cover $\widetilde{M}$ of the given curve γ. In the euclidean case, $\widetilde{M}$ is $\mathbb{E}^2$ itself. In the hyperbolic case, $\widetilde{M}$ is the hyperbolic plane $\mathbb{H}^2$ and, if we use the upper-half plane model or the Poincaré disk model, it is a subset of $\mathbb{E}^2$. When γ is null-homotopic, then $\widetilde{\gamma}$ is also a regular closed curve on $\mathbb{E}^2$, and we define $W(\gamma)$ by $i_{\widetilde{\gamma}} \in \mathbb{Z}$. When γ is not null-homotopic, we define $W(\gamma)$ to be

$$i_{\widetilde{\gamma}} - (i_{\widetilde{\delta}} + \chi_\delta/2\pi) \in \mathbb{Z},$$

where $\widetilde{\delta}$ is the unique geodesic in $\widetilde{M}$ connecting the end points of $\widetilde{\gamma}$, and χ_δ is the external angle of the geodesic δ in M obtained by projecting $\widetilde{\delta}$ to M. See §2 for the details of the definition of the rotation number and the proof of Theorem 1.1.

In §3, we give some sample calculations of rotation numbers. In particular, we show that the rotation numbers of closed geodesics and closed horocycles are 0.

In §4, we give a Whitney-type formula for rotation numbers of non-null-homotopic regular closed curves that are 'generic' in the sense that the only singularities are transverse double points. Suppose γ is a non-null-homotopic regular closed curve on a complete euclidean or hyperbolic surface. Let $D(\gamma)$ be the set of double points of γ. For each $d \in D(\gamma)$, we define its sign $\mathrm{sgn}(\gamma, d)$ as an element of $\{-1, 0, 1\}$, and prove the following:

THEOREM 1.2. $W(\gamma) = \sum_{d \in D(\gamma)} \operatorname{sgn}(\gamma, d)$.

This implies that $W(\gamma)$ does not depend on the choice of the complete euclidean or hyperbolic metric.

Remark 1.3. *The combinatorial formula above was already given by Tanio and Kobayashi in* [4] *when M is the torus.*

Remark 1.4. *There is another generalization of Whitney's rotation number by Reinhart* [3] *and Chillingworth* [1]*. But they need to fix a vector field X on M to define their invariant as an element of $\mathbb{Z}/|\chi(M)|\mathbb{Z}$, where $\chi(M)$ is the euler characteristic of M, and could not give a regular homotopy classification when $\chi(M) < 0$.*

Remark 1.5. *In this paper, we allow reparametrizations of curves as long as they do not change the orientation so that we can talk about regular homotopies between two curves defined on different intervals.*

2. The Invariant $W(\gamma)$

Let M be an oriented surface with a complete euclidean/hyperbolic metric, and let $p_M : \widetilde{M} \to M$ be the universal cover of M. We may assume that $\widetilde{M}$ is either $\mathbb{E}^2$ itself or a subset of $\mathbb{E}^2$ (the upper half-plane or the interior of the unit disk). In any case, for an arbitrary compact subset C of $\widetilde{M}$, there is a strong deformation retraction of $\mathbb{E}^2$ into $\widetilde{M}$ fixing C. This implies that two regular curves with compact images in $\widetilde{M}$ are regularly homotopic in $\widetilde{M}$ if and only if they are regularly homotopic in $\mathbb{E}^2$.

Suppose $\gamma : [a, b] \to M$ is a regular closed curve with base point p. We define $W(\gamma)$. Let $\widetilde{\gamma} : [a, b] \to \widetilde{M} \subset \mathbb{E}^2$ be a lift of γ.

Case 1. γ is null-homotopic in M.
In this case, $\widetilde{\gamma}$ is a regular closed curve in $\mathbb{E}^2$, and its euclidean rotation number $i_{\widetilde{\gamma}}$ is an integer.

PROPOSITION 2.1. *If $\overline{\gamma}$ is another lift of γ, then $i_{\widetilde{\gamma}} = i_{\overline{\gamma}}$.*

PROOF. When the metric of M is euclidean, there is a euclidean isometry T of $\mathbb{E}^2$ such that $\overline{\gamma} = T \circ \widetilde{\gamma}$; therefore, the claim is obvious.

Next, we consider the hyperbolic case. Suppose $i_{\widetilde{\gamma}} = k$. If $k = 0$, then $\widetilde{\gamma}$ is regularly homotopic to a small figure-eight in $\mathbb{E}^2$, and this induces a regular homotopy of $\overline{\gamma}$ to another figure-eight. Therefore, $i_{\overline{\gamma}} = 0$. If $k \neq 0$, then $\widetilde{\gamma}$ is regularly homotopic to a curve which goes round a small circle in $\mathbb{E}^2$ k times, and this induces a regular homotopy of $\overline{\gamma}$ to a curve which goes round a simple closed curve k times. Therefore, $i_{\overline{\gamma}} = k$. □

DEFINITION 2.2. *When γ is null-homotopic, $W(\gamma) := i_{\widetilde{\gamma}} \in \mathbb{Z}$.*

Case 2. γ is not null-homotopic in M.
In this case, $\widetilde{\gamma}$ is not a closed curve. Then $\widetilde{p} = \widetilde{\gamma}(a)$ and $\widetilde{\gamma}(b)$ are distinct points. We first pick up the unique geodesic $\widetilde{\delta} : [a', b'] \to \widetilde{M}$ connecting these two points in $\widetilde{M}$ with respect to the riemannian metric induced from that of M. This depends only on the homotopy class ξ of γ in $\pi_1(M, p)$ and on the choice of $\widetilde{p}$. The geodesic $p_M \circ \widetilde{\delta}$ on M will be denoted δ. This δ is the shortest geodesic based at p that represents ξ in $\pi_1(M, p)$, and can be regarded as the simplest closed curve on M based at p that represents ξ. So we would like to compare γ with δ, or more precisely, $\widetilde{\gamma}$ with $\widetilde{\delta}$. We need the following preparation to do so.

PROPOSITION 2.3. *Two regular arcs γ_1 and γ_2 on $\mathbb{E}^2$ with the same end points and the same end directions are regularly homotopic fixing the end points and the end directions if and only if $i_{\gamma_1} = i_{\gamma_2}$.*

PROOF. According to Gromov's h-principle, γ_1 and γ_2 are regularly homotopic fixing the end points and the end directions if and only if the maps

$$(\gamma_i, \dot{\gamma}_i/|\dot{\gamma}_i|) : [a, b] \to \mathbb{E}^2 \times S^1 \qquad (i = 1, 2)$$

are homotopic rel ∂. The first components are homotopic rel ∂ because the target is contractible, and the second components are homotopic rel ∂ if and only if the angle functions are homotopic rel ∂, and it happens if and only if $i_{\gamma_1} = i_{\gamma_2}$. □

PROPOSITION 2.4. *The difference $i_{\widetilde{\gamma}} - i_{\widetilde{\delta}}$ is independent of the choice of the lift $\widetilde{\gamma}$.*

PROOF. Let us denote the angle functions of $\widetilde{\gamma}$ and $\widetilde{\delta}$ by $\theta_{\widetilde{\gamma}}$ and $\theta_{\widetilde{\delta}}$, respectively, and denote the differences of the values of these angle functions at the initial point and at the terminal point by α and β, respectively:

$$\alpha = \theta_{\widetilde{\gamma}}(a) - \theta_{\widetilde{\delta}}(a'), \qquad \beta = \theta_{\widetilde{\gamma}}(b) - \theta_{\widetilde{\delta}}(b') .$$

Then we have the identity $i_{\widetilde{\gamma}} - i_{\widetilde{\delta}} = \frac{\beta - \alpha}{2\pi}$. Now, make local twists of the arc $\widetilde{\delta}$ by the angle α around the initial point, and by the angle β around the terminal point to obtain a new curve $\widetilde{\varepsilon}$, which has the same end directions with $\widetilde{\gamma}$ (Figure

1). Since these twists actually define a regular homotopy from $\widetilde{\delta}$ to $\widetilde{\varepsilon}$, we obtain a homotopy between $\theta_{\widetilde{\delta}}$ and the angle function $\theta_{\widetilde{\varepsilon}}$ of $\widetilde{\varepsilon}$ that changes only near the end points, and we may assume that $\theta_{\widetilde{\varepsilon}}$ satisfies the identities

$$\theta_{\widetilde{\varepsilon}}(a') = \theta_{\widetilde{\delta}}(a') + \alpha = \theta_{\widetilde{\gamma}}(a), \qquad \theta_{\widetilde{\varepsilon}}(b') = \theta_{\widetilde{\delta}}(b') + \beta = \theta_{\widetilde{\gamma}}(b)\ .$$

So we have the equality $i_{\widetilde{\gamma}} = i_{\widetilde{\varepsilon}}$, and $\widetilde{\gamma}$ is regularly homotopic to $\widetilde{\varepsilon}$ fixing the end points and the end directions by Proposition 2.3. Set $\varepsilon = p_M \circ \widetilde{\varepsilon}$.

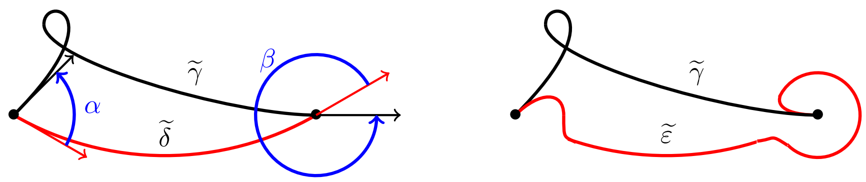

Figure 1. Local twists of $\widetilde{\delta}$ around the end points.

Suppose $\overline{\gamma}$ is another lift of γ, and let $\overline{p}$ be the initial point of $\overline{\gamma}$. Let $\overline{\delta}$ (resp. $\overline{\varepsilon}$) be the lift of δ (resp. ε) whose initial point is $\overline{p}$. The deck transformation T of p_M that sends $\widetilde{p}$ to $\overline{p}$ sends the curves $\widetilde{\gamma}$, $\widetilde{\delta}$, and $\widetilde{\varepsilon}$ to the curves $\overline{\gamma}$, $\overline{\delta}$, and $\overline{\varepsilon}$, respectively, and sends the regular homotopy between $\widetilde{\gamma}$ and $\widetilde{\varepsilon}$ to a regular homotopy between $\overline{\gamma}$ and $\overline{\varepsilon}$. Therefore, the equality $i_{\overline{\gamma}} = i_{\overline{\varepsilon}}$ holds, and we have identities

$$i_{\widetilde{\gamma}} - i_{\widetilde{\delta}} = i_{\widetilde{\varepsilon}} - i_{\widetilde{\delta}}, \qquad i_{\overline{\gamma}} - i_{\overline{\delta}} = i_{\overline{\varepsilon}} - i_{\overline{\delta}}.$$

Let us compare the right hand sides of these. Recall that $\widetilde{\varepsilon}$ was obtained by twists around the end points. The deck transformation T induces a regular homotopy between $\overline{\delta}$ and $\overline{\varepsilon}$. Since T is conformal, the changes of the end directions at the end points of $\overline{\delta}$ are exactly the same as the changes of the end directions at the end points of $\widetilde{\delta}$; therefore, we have $i_{\widetilde{\varepsilon}} - i_{\widetilde{\delta}} = i_{\overline{\varepsilon}} - i_{\overline{\delta}}$, and this completes the proof. $\square$

Note that δ may have a corner at the base point p; so, we need to take the external angle of δ at the vertex p into consideration.

Definition 2.5. *The* **external angle** χ_δ *of the geodesic* δ *based at* p *is the angle of the initial direction measured from the terminal direction such that* $-\pi < \chi_\delta < \pi$.

Remark 2.6. *The external angle of a geodesic which starts and ends at* p *cannot be equal to* $\pm\pi$*, because a point and a direction together determine a geodesic which* *passes the given point and has the given direction there.*

Definition 2.7. *When γ is not null-homotopic, $W(\gamma) := i_{\widetilde{\gamma}} - i_{\widetilde{\delta}} - \dfrac{\chi_\delta}{2\pi} \in \mathbb{R}$.*

Proposition 2.8. *$W(\gamma)$ is an integer.*

Proof. We only need to consider the case when γ is not null-homotopic. Let α and β be as in the proof of Proposition 2.4; then, $W(\gamma) = \dfrac{(\beta-\alpha)-\chi_\delta}{2\pi}$. The relation $\beta - \alpha \equiv \chi_\delta \mod 2\pi$ can be read off on M as shown in Figure 2. So, $W(\gamma)$ is an integer. □

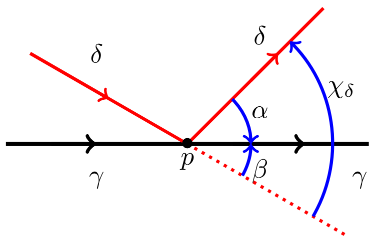

Figure 2. Relation of χ_δ with α and β modulo 2π.

Proof of Theorem 1.1 (1)⇒(2). When we change γ by regular homotopy, the relevant angle functions *etc.* and hence the rotation number $W(\gamma)$ changes continuously. Since $W(\gamma)$ is an integer, its value does not change.

Proof of Theorem 1.1 (2)⇒(1). Suppose that γ, γ' are homotopic and have the same rotation number.

Step 1. Perform a finger move on γ along an arc which connects the base points of γ and γ', so that they have the same base point p.

Step 2. If they represent different elements of $\pi_1(M,p)$, then they are conjugate by a certain element $[\xi] \in \pi_1(M,p)$. Perform another finger move on γ along ξ so that they represent the same element of $\pi_1(M,p)$.

Step 3. Locally twist γ around p so that they have the same direction at p.

Step 4. Take lifts $\widetilde{\gamma}$ and $\widetilde{\gamma'}$ of them whose initial points are the same. Then $i_{\widetilde{\gamma}}$ and $i_{\widetilde{\gamma'}}$ are the same. So, by Proposition 2.3, the lifts are regularly homotopic in $\mathbb{E}^2$ fixing the end points and the end directions.

Step 5. If the geometry of M is hyperbolic, squeeze $\mathbb{E}^2$ into $\mathbb{H}^2 = \widetilde{M}$ by an isotopy fixing the two lifts, and obtain a regular homotopy in $\mathbb{H}^2$.

Step 6. Compose the regular homotopy with p_M to obtain a desired regular homotopy between γ and γ' in M. This finishes the proof.

3. Examples

In this section, we calculate the rotation numbers for important examples. We consider only non-null-homotopic regular closed curves.

PROPOSITION 3.1. *If γ is a closed geodesic on a complete euclidean or hyperbolic surface, then $W(\gamma) = 0$.*

PROOF. We have $\widetilde{\gamma} = \widetilde{\delta}$ and $\chi_\delta = 0$. Therefore, $W(\gamma) = 0$. □

PROPOSITION 3.2. *If M is a a horocycle on a complete hyperbolic surface, then $W(\gamma) = 0$.*

PROOF. Without loss of generality, we may assume that $\widetilde{\gamma}$ is a horizontal line segment connecting $\widetilde{p}$ and $\widehat{p}$. The geodesic $\widetilde{\delta}$ connecting these points is a circular arc. See Figure 3. $\widehat{\gamma}$ and $\widehat{\delta}$ are the lifts of γ and δ, respectively, whose common initial point is $\widehat{p}$. From the figure, we can see that $i_{\widetilde{\gamma}} = 0$ and $i_{\widetilde{\delta}} = -\dfrac{\chi_\delta}{2\pi}$. Therefore,

$$W(\gamma) = 0 - \left(-\frac{\chi_\delta}{2\pi}\right) - \frac{\chi_\delta}{2\pi} = 0.$$

□

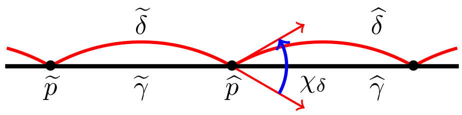

Figure 3. Horocycle case.

Suppose γ is a non-null-homotopic regular closed curve. Then γ is homotopic to either a closed geodesic or a closed horocycle. Let $\gamma_{(0)}$ denote such a curve. And, for a non-zero integer k, we define a curve $\gamma_{(k)}$ to be the curve obtained by

- adding k positive kinks to $\gamma_{(0)}$, if $k > 0$, and
- adding $|k|$ negative kinks to $\gamma_{(0)}$, if $k < 0$.

The following is obvious.

Proposition 3.3. $W(\gamma_{(k)}) = k$.

Corollary 3.4. *If the rotation number $W(\gamma)$ of a non-null-homotopic regular closed curve γ is equal to k, then γ is regularly homotopic to $\gamma_{(k)}$. In other words, $\gamma_{(k)}$'s are the standard representatives of regular closed curves homotopic to γ.*

4. Whitney-type Formula for Generic Curves

Let M be an oriented complete euclidean or hyperbolic surface, and consider a generic curve γ on M. If γ is null-homotopic, then $\widetilde{\gamma}$ is a regular closed curve on $\widetilde{M} \subset \mathbb{E}^2$; so, we can apply the classical Whitney's formula to this.

So, in this section, we consider the case when γ is not null-homotopic. We will define a combinatorial invariant $T(\gamma)$ and show that $W(\gamma) = T(\gamma)$. First, consider all the lifts of γ to $\widetilde{M}$ and compose all these lifts to obtain a curve $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$, which will be called a '*cover*' of γ, and it is given the orientation induced from the orientation of γ. Covers with different parametrizations are regarded to be the same cover. Since γ is generic, any cover $\widetilde{\gamma}$ is also generic. Also note that any cover can be obtained by composing a given cover and an appropriate deck transformation of $\widetilde{M}$.

As in §1, $D(\gamma)$ denotes the set of all the double points of γ. Let $d \in D(\gamma)$ be any element. At d, the curve γ splits into two closed loops γ_1 and γ_2 based at d. Since γ is non-null-homotopic, at least one of γ_1 and γ_2 must be non-null-homotopic. The followings are obviously equivalent:

- Either γ_1 or γ_2 is null-homotopic.
- There exists a cover $\widetilde{\gamma}$ of γ and a double point $\widetilde{d}$ of $\widetilde{\gamma}$ such that $p_M(\widetilde{d}) = d$.
- For any cover $\widetilde{\gamma}$ of γ, there exists a double point $\widetilde{d}$ of $\widetilde{\gamma}$ such that $p_M(\widetilde{d}) = d$.

We define $D_{\pm}(\gamma)$ to be the subset of $D(\gamma)$ consisting of those double points satisfying the conditions above, and define $D_0(\gamma)$ to be its complement. In other words, double points in $D_{\pm}(\gamma)$ correspond to self-intersections of the same cover, and the double points in $D_0(\gamma)$ correspond to mutual intersections of distinct covers.

For each point $d \in D(\gamma)$, we define $\operatorname{sgn}(\gamma, d) \in \{-1, 0, +1\}$ as follows: For a double point $d \in D_0(\gamma)$, we define $\operatorname{sgn}(\gamma, d)$ to be 0. Before we define $\operatorname{sgn}(\gamma, d)$ for $d \in D_{\pm}(\gamma)$, we first define $\operatorname{sgn}(\widetilde{\gamma}, \widetilde{d})$ of a double point $\widetilde{d}$ of a cover $\widetilde{\gamma}$ of γ. Let $t_1 < t_2$ be the real numbers such that $\widetilde{\gamma}(t_1) = \widetilde{\gamma}(t_2) = \widetilde{d}$. Choose a small positive number r. We define $\operatorname{sgn}(\widetilde{\gamma}, \widetilde{d})$ to be $+1$ (resp. -1) if the arc $\widetilde{\gamma}((t_2 - r, t_2 + r))$ (drawn horizontally in Figure 4) crosses the arc $\widetilde{\gamma}((t_1 - r, t_1 + r))$ (drawn vertically in

Figure 4) from left to right (resp. from right to left). For a double point $d \in D_{\pm}(\gamma)$, take any cover $\widetilde{\gamma}$ of γ and any double point $\widetilde{d}$ of $\widetilde{\gamma}$ such that $p_M(\widetilde{d}) = d$, and set $\mathrm{sgn}(\gamma, d) = \mathrm{sgn}(\widetilde{\gamma}, \widetilde{d})$. This does not depend on the choice of $\widetilde{\gamma}$ or $\widetilde{d}$.

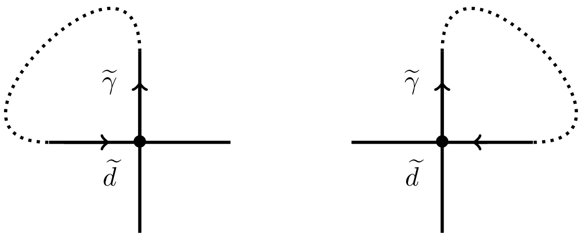

Figure 4. Sign convention for a double point of $\widetilde{\gamma}$.

Definition 4.1. *For a non-null-homotopic generic regular closed curve γ, we define $T(\gamma)$ by*

$$T(\gamma) = \sum_{d \in D_{\pm}(\gamma)} \mathrm{sgn}(\gamma, d) = \sum_{d \in D(\gamma)} \mathrm{sgn}(\gamma, d).$$

Remark 4.2. *In* [4], *Tanio and Kobayashi defined an invariant $t(\gamma)$ which is very close to the $T(\gamma)$ above. They set* $\mathrm{sgn}(\gamma, d) = \pm 1$ *when one of the loops obtained by splitting γ at d is null-homologous. So, if the fundamental group of a surface is abelian, $t(\gamma)$ and $T(\gamma)$ are the same for generic curves on that surface.*

Example 4.3. *Figure 5 shows a developed picture of a flat cylinder and a regular closed curve γ on it. This curve has two covers as shown in Figure 6. The curve γ has two self-intersections d and e, and* $\mathrm{sgn}(\gamma, d) = 0$, $\mathrm{sgn}(\gamma, e) = 1$. *Therefore,* $T(\gamma) = 1$.

Proposition 4.4. *$T(\gamma)$ is a regular homotopy invariant for generic regular closed curves.*

Proof. If we deform γ by a regular homotopy, then the regular homotopy lifts to regular homotopies of the covers. According to [2], births/deaths of double points

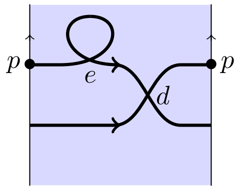

Figure 5. A curve γ on a cylinder.

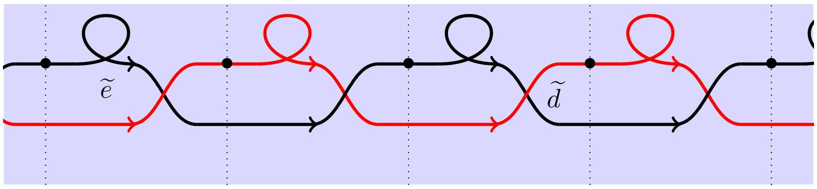

Figure 6. Two covers of γ.

always occur in pairs. If a birth/death of two double points d, e of γ occurs, then corresponding births/deaths occur for self- and/or mutual intersections of the covers, and each birth/death pair in $\widetilde{M}$ must be one of the following:
(1) mutual intersections of distinct covers,
(2) double points of the same cover with opposite signs.
In case (1), d and e are both points in $D_0(\gamma)$, and, in case (2), they are both points in $D_{\pm}(\gamma)$. $\square$

Proposition 4.5. *The definition of $T(\gamma)$ extends to non-null-homotopic non-generic regular closed curves.*

Proof. A non-generic regular closed curve γ can be approximated by a generic sine-like curve γ' along γ, and we set $T(\gamma) = T(\gamma')$. Since this approximation is given by a regular homotopy, Proposition 4.4 insures that this defines a well-defined regular homotopy invariant. $\square$

Proposition 4.6. *For any non-null-homotopic regular closed curve γ and any integer k, $T(\gamma_{(k)}) = k$.*

Proof. Let us consider the case when $k = 0$. If $\gamma_{(0)}$ is generic, all of the covers of $\gamma_{(0)}$ are either geodesic or horocycles, and they do not have self-intersections; therefore, $T(\gamma_{(0)}) = 0$. If $\gamma_{(0)}$ is not generic, we approximate it by a sine-like curve along it. Its covers are also sine-like curves along geodesics or horocycles, and they do not have self-intersections, either. Therefore, $T(\gamma_{(0)}) = 0$.

When $k \neq 0$, $\gamma_{(k)}$ is obtained by adding $|k|$ appropriate kinks to $\gamma_{(0)}$; therefore, $T(\gamma_{(k)}) = k$ is obvious. $\square$

So we have proved the following:

THEOREM 4.7 (THEOREM 1.2). *For any non-null-homotopic regular closed curve γ on an oriented complete euclidean or hyperbolic surface, the equality $W(\gamma) = T(\gamma)$ holds.*

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