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Introduction to variations of Hodge structure over Sasakian manifolds

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Abstract. We define basic variations of Hodge structures over manifolds with transverse holomorphic foliations. We prove some important properties of basic variations of Hodge structures over compact Sasakian manifolds.

1. Introduction

A real Hodge structure of weight n on a real vector space V is a bigrading

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

on the complexification $V_{\mathbb{C}} = V \otimes \mathbb{C}$ such that

$$\overline{V^{p,q}} = V^{q,p}.$$

It is a well-known fact that the cohomology of a compact Kähler manifold admits a canonical real Hodge structure. Hence, corresponding to a holomorphic family of compact Kähler manifolds, we obtain a family of real Hodge structures. In [Gri2], Griffiths discovered the significant property of such family so-called Griffiths ’ transversality. This leads to define variations of Hodge structures over complex manifolds as flat bundles with certain structures. Now there are many applications of variations of Hodge structures. In particular, considering variations of Hodge structures over compact Kähler manifolds are very fruitful.

A Sasakian manifold is viewed as an important odd-dimensional analogue of a Kähler manifold ([Sa]). The purpose of this paper is to give analogies of variations of Hodge structures on Sasakian manifolds.

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2. Variations of Hodge structures over complex manifolds

Let M be a complex manifold.

Definition 2.1. *A* polarized complex variation of Hodge structure *of weight w over M is $(E=\bigoplus_{p+q=w}E^{p,q},D,h)$ so that*

1. *E is a C^{∞}-complex vector bundle with a decomposition $\bigoplus_{p+q=w}E^{p,q}$ in a direct sum of C^{∞}-subbundles,*
2. *D is a flat connection satisfying the Griffiths transversality condition*

$$D: A^{0}(M,E^{p,q})$$
$$\to A^{0,1}(M,E^{p+1,q-1})\oplus A^{1,0}(M,E^{p,q})\oplus A^{0,1}(M,E^{p,q})\oplus A^{1,0}(M,E^{p-1,q+1})$$

   and

3. *h is a parallel Hermitian form so that the decomposition $\bigoplus_{p+q=w}E^{p,q}$ is orthogonal and h is positive on $E^{p,q}$ for even p and negative for odd p.*

Let $G=\mathrm{Aut}(E_x,h_x)$ and $V=\Pi_{p+q=w}\mathrm{Aut}(E_x^{p,q},h_x)$. By the definition, the monodromy representation $\rho:\pi_1(M,x)\to GL(E_x)$ corresponding to the flat bundle (E,D) satisfies $\rho(\pi_1(M,x))\subset G\cong U(s,t)$ where $s=\sum_{p\,even}\dim E^{p,q}$ and $t=\sum_{p\,odd}\dim E^{p,q}$. We notice that a polarized complex variation of Hodge structure over M with a monodromy representation $\rho:\pi_1(M,x)\to G$ is equivalent to a ρ-equivalent "horizontal" holomorphic map from the universal covering of M to a classifying space of polarized Hodge structures expressed as a homogeneous space G/V with an appropriate complex structure.

Theorem 2.2 ([Gri3]). *Let M be a complex projective manifold. For any polarized complex variation of Hodge structure $(E=\bigoplus_{p+q=w}E^{p,q},D,h)$ of w over M, the flat bundle (E,D) is semi-simple.*

By the Griffiths transversality, the differential D on $A^{*}(M,E)$ decomposes $D=\partial+\theta+\bar{\partial}+\bar{\theta}$ so that:

$$\partial: A^{a,b}(M,E^{c,d})\to A^{a+1,b}(M,E^{c,d}),$$

$$\bar{\partial}: A^{a,b}(M,E^{c,d})\to A^{a,b+1}(M,E^{c,d}),$$

$$\theta: A^{a,b}(M,E^{c,d})\to A^{a+1,b}(M,E^{c-1,d+1})$$

and

$$\bar{\theta} : A^{a,b}(M, E^{c,d}) \to A^{a,b+1}(M, E^{c+1,d-1}).$$

We define

$$A^*(M,E)^{p,q} = \bigoplus_{a+c=p, b+d=q} A^{a,b}(M, E^{c,d}),$$

$D' = \partial + \bar{\theta} : A^*(M,E)^{p,q} \to A^*(M,E)^{p+1,q}$ and $D'' = \bar{\partial} + \theta : A^*(M,E)^{p,q} \to A^*(M,E)^{p,q+1}$. By the flatness $DD = 0$, we have

$$D'D' = D''D'' = D'D'' + D''D' = 0.$$

We have the double complex

$$(A^*(M,E)^{p,q}, D', D'')$$

as the usual Dolbeault complex.

Definition 2.3. *A polarized real variation of Hodge structure of weight w over M is $(E_{\mathbb{R}}, E = \bigoplus_{p+q=w} E^{p,q}, D, Q)$ so that:*

- *(E, D) is a flat complex vector bundle with a real structure $E_{\mathbb{R}} \subset E$,*
- *Q is a parallel $(-1)^w$-symmetric bilinear form on $E_{\mathbb{R}}$.*
- *defining $h(u,v) = Q(u, \bar{v})$ on E, $(E = \bigoplus_{p+q=w} E^{p,q}, D, h)$ is a polarized complex variation of Hodge structure and*
- *$\overline{E^{p,q}} = E^{q,p}$.*

Polarized real variations of Hodge structure extend Hodge structures on the de Rham cohomology of compact Kähler manifolds.

Theorem 2.4 ([Zu]). *Let M be a compact Kähler manifold. For any polarized real variation of Hodge structure of weight w $(E_{\mathbb{R}}, E = \bigoplus_{p+q=w} E^{p,q}, D, Q)$ over M, the filtration $F^r = \bigoplus_{r \leq p} A^*(M,E)^{p,q}$ induces a functorial real Hodge structure of weight $i + w$ on the cohomology $H^i(M, E_{\mathbb{R}})$.*

3. Basic variations of Hodge structures over foliated manifolds

Let $(M, \mathcal{F})$ be a foliated manifold of co-dimension $2q$. $(M, \mathcal{F})$ is *transverse holomorphic* if there is a foliation atlas $\{U_\alpha\}$ with local submersions $f_\alpha : U_\alpha \to \mathbb{C}^q$

and transition functions $\tau_{\alpha\beta} : f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ satisfying the relations $f_\alpha = \tau_{\alpha\beta} f_\beta$ such that each $\tau_{\alpha\beta}$ is biholomorphic. Let $T\mathcal{F}$ be the tangent bundle of $\mathcal{F}$ and $N\mathcal{F} = TM/T\mathcal{F}$ the normal bundle. Then we have the canonical decomposition $N\mathcal{F}_\mathbb{C} = N^{1,0}\mathcal{F} \oplus N^{0,1}\mathcal{F}$ satisfying $\overline{N^{1,0}\mathcal{F}} = N^{0,1}\mathcal{F}$.

On a smooth manifold M, if we have a sub-bundle $E \subset TM_\mathbb{C}$ of complex co-dimension q so that $E + \overline{E} = TM_\mathbb{C}$ and E is involutive, then M admits a transverse holomorphic foliation $\mathcal{F}$ such that $T\mathcal{F}_\mathbb{C} = E \cap \overline{E}$ and $N^{1,0}\mathcal{F} = E/E \cap \overline{E}$ (see [DK]).

A differential form ω on M is called *basic* if the equations

$$i_X\omega = 0 = \mathcal{L}_X\omega \tag{1}$$

hold for any $X \in T\mathcal{F}$. We denote by $A^*_B(M)$ the subspace of basic forms in the de Rham complex $A^*(M)$. Then $A^*_B(M)$ is sub-complex of the de Rham complex $A^*(M)$. Denote by $H^*_B(M)$ the cohomology of the basic de Rham complex $A^*_B(M)$. Suppose $(M, \mathcal{F})$ is transverse holomorphic. Corresponding to the decomposition $N\mathcal{F}_\mathbb{C} = N^{1,0}\mathcal{F} \oplus N^{0,1}\mathcal{F}$, we have the bigrading

$$A^r_B(M)_\mathbb{C} = \bigoplus_{p+q=r} A^{p,q}_B(M)$$

as well as the decomposition of the exterior differential $d_{|A^r_B(M)_\mathbb{C}} = \partial_B + \overline{\partial}_B$ on $A^r_B(M)_\mathbb{C}$, so that

$$\partial_B : A^{p,q}_B(M) \to A^{p+1,q}_B(M) \quad \text{and} \quad \overline{\partial}_B : A^{p,q}_B(M) \to A^{p,q+1}_B(M)\,.$$

A *basic vector bundle* E over a foliated manifold $(M, \mathcal{F})$ is a C^∞ vector bundle over M which has local trivializations with respect to an open covering $M = \bigcup_\alpha U_\alpha$ satisfying the condition that each transition function $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{C})$ is basic on $U_\alpha \cap U_\beta$ i.e. it is constant on the leaves of the foliation $\mathcal{F}$. Suppose $(M, \mathcal{F})$ is transverse holomorphic. We say that a basic vector bundle E is *holomorphic* if we can take each transition function transverse holomorphic. For a basic vector bundle E, a differential form $\omega \in A^*(M, E)$ with values in E is called basic if ω is basic on every U_α, meaning $\omega_{|U_\alpha} \in A^*_{B_\mathcal{F}}(U_\alpha) \otimes \mathbb{C}^r$ for every α. Let

$$A^*_B(M, E) \subset A^*(M, E)$$

denote the subspace of basic forms in the space $A^*(M, E)$ of differential forms with values in E. Corresponding to the decomposition $N\mathcal{F}_\mathbb{C} = N^{1,0}\mathcal{F} \oplus N^{0,1}\mathcal{F}$, we have the bigrading $A^r_B(M, E) = \bigoplus_{p+q=r} A^{p,q}_B(M, E)$. If E is holomorphic, we can extend the operator $\overline{\partial}_B$ to the operator $\overline{\partial}_E : A^{p,q}_B(M, E) \to A^{p,q+1}_B(M, E)$

satisfying $\overline{\partial}_E\overline{\partial}_E = 0$. Conversely, if we have a connection ∇ of E which defined as $\nabla : A^*(M, E) \to A^{*+1}(M, E)$ and decomposing $\nabla = \nabla' + \nabla''$ with $\nabla' : A_B^{p,q}(M, E) \to A_B^{p+1,q}(M, E)$ and $\nabla'' : A_B^{p,q}(M, E) \to A_B^{p,q+1}(M, E)$ we have $\nabla''\nabla'' = 0$, then there exists a unique holomorphic bundle structure on E such that $\nabla'' = \overline{\partial}_E$ like the complex manifold case (see [Ko, Proposition 3.7]).

Example 3.1. *Consider the* 3*-dimensional Heisenberg group*

$$H_3 = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R} \right\}.$$

We have the left-invariant vector fields

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle$$

and the left-invariant differential forms

$$\langle dx, dy, dt - xdy \rangle .$$

We consider the discrete subgroup

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{Z} \right\}$$

and the compact 3*-dimensional manifold* $M = \Gamma\backslash H_3$. *Then the left-invariant vector field* $\frac{\partial}{\partial t}$ *generates a foliation* $\mathcal{F}$ *of co-dimension* 2*-dimensional so that each leaf is an* S^1*-orbit. Consider the parameter* (x, y, t) *as a local coordinates of* $\Gamma\backslash H_3$. *Then the transition functions may be given by* $(x, y, t) \mapsto (a+x, b+y, c+ay+t)$ *for some* $a, b, c \in \mathbb{Z}$. *We have the transverse holomorphic structure on* $(M = \Gamma\backslash H_3, \mathcal{F})$ *by local submersions* $(x, y, t) \mapsto x + \sqrt{-1}y$. *The basic de Rham complex* $A_B^*(M)$ *is*

$$A_B^*(M) = \mathcal{C}^\infty(\mathbb{C})^{\mathbb{Z}+\sqrt{-1}\mathbb{Z}} \otimes \bigwedge \langle dx, dy \rangle.$$

This is identified with the de Rham complex of the complex torus $\mathbb{C}/\mathbb{Z} + \sqrt{-1}\mathbb{Z}$. *Hence,*

$$H_B^*(M) \cong \bigwedge \langle dx, dy \rangle$$

and

$$H_B^{p,q}(M) \cong \bigwedge^p \langle dz \rangle \otimes \bigwedge^q \langle d\bar{z} \rangle.$$

Consider the $\mathcal{C}^\infty$-trivial complex line bundle $E = M \times \mathbb{C}$. For each local coordinate (x, y, t), taking nowhere vanishing section e^{-Ct} of E for a constant $C \in \mathbb{C}$ as a local frame, the transition functions will be written by the basic functions e^{c+ay}. For $C \neq 0$, this gives a non-trivial basic vector bundle structure E_C on E. Consider the connection $\nabla = d + C(dt - xdy)$ on E. Then ∇ is defined as $\nabla : A_B^*(M, E) \to A_B^{*+1}(M, E)$. The curvature of ∇ is $-Cdx \wedge dy$. The basic cohomology class of $\frac{1}{2\pi\sqrt{-1}} Cdx \wedge dy$ is the basic first Chern class of the basic vector bundle E_C. Like the usual Chern class, this is an invariant of a basic vector bundle. Thus $\{E_C\}$ is a family of basic vector bundles such that $E_C \ncong E_{C'}$ for $C \neq C'$. Obviously, $\nabla''\nabla'' = 0$ and hence we have a holomorphic structure on each E_C.

We consider a flat vector bundle (E, D) over M as a basic vector bundle by local flat frames. Then, $A_B^*(M, E)$ is a sub-complex of the de Rham complex $A^*(M, E)$ equipped with the differential associated to the flat connection D. Denote by $H_B^*(M, E)$ the cohomology of $A_B^*(M, E)$.

A *polarized complex basic variation of Hodge structure of weight* w over $(M, \mathcal{F})$ is $(E = \bigoplus_{p+q=w} E^{p,q}, D, h)$ so that

1. (E, D) is a complex flat vector bundle.
2. $\bigoplus_{p+q=w} E^{p,q}$ is a direct sum of basic C^∞-subbundles,
3. The basic Griffiths transversality condition

   $$\begin{aligned} &D : A_B^0(M, E^{p,q}) \\ &\to A_B^{0,1}(M, E^{p+1,q-1}) \oplus A_B^{1,0}(M, E^{p,q}) \oplus A_B^{0,1}(M, E^{p,q}) \oplus A_B^{1,0}(M, E^{p-1,q+1}) \end{aligned}$$

   holds and
4. h is a parallel Hermitian form so that the decomposition $\bigoplus_{p+q=w} E^{p,q}$ is orthogonal and h is positive on $E^{p,q}$ for even p and negative for odd p.

By the basic Griffiths transversality, the differential D on $A_B^*(M, E)$ decomposes $D = \partial + \theta + \bar{\partial} + \bar{\theta}$ so that:

$$\partial : A_B^{a,b}(M, E^{c,d}) \to A_B^{a+1,b}(M, E^{c,d}),$$

$$\bar{\partial} : A_B^{a,b}(M, E^{c,d}) \to A_B^{a,b+1}(M, E^{c,d}),$$

$$\theta : A_B^{a,b}(M, E^{c,d}) \to A^{a+1,b}(M, E^{c-1,d+1})$$

and

$$\bar{\theta} : A_B^{a,b}(M, E^{c,d}) \to A_B^{a,b+1}(M, E^{c+1,d-1}).$$

We define

$$A_B^*(M, E)^{p,q} = \bigoplus_{a+c=p, b+d=q} A_B^{a,b}(M, E^{c,d}),$$

$D' = \partial + \bar{\theta} : A_B^*(M, E)^{p,q} \to A_B^*(M, E)^{p+1,q}$ and $D'' = \bar{\partial} + \theta : A_B^*(M, E)^{p,q} \to A_B^*(M, E)^{p,q+1}$. By the flatness $DD = 0$, we have

$$D'D' = D''D'' = D'D'' + D''D' = 0.$$

We have the double complex

$$(A_B^*(M, E)^{p,q}, D', D'')$$

as the usual Dolbeault complex. By $D''D' = 0$, we have $\bar{\partial}\bar{\partial} = 0$. As [Ko, Chapter 1. Proposition 3.7], each $E^{p,q}$ can be seen as a holomorphic basic vector bundle.

A *polarized real basic variation of Hodge structure of weight w* over $(M, \mathcal{F})$ is $(E_{\mathbb{R}}, E = \bigoplus_{p+q=w} E^{p,q}, D, Q)$ so that:

- (E, D) is a flat complex vector bundle with a real structure $E_{\mathbb{R}} \subset E$,
- Q is a parallel $(-1)^w$-symmetric bilinear form on $E_{\mathbb{R}}$.
- defining $h(u, v) = Q(u, \bar{v})$ on E, $(E = \bigoplus_{p+q=w} E^{p,q}, D, h)$ is a polarized complex basic variation of Hodge structure and
- $\overline{E^{p,q}} = E^{q,p}$.

THEOREM 3.2. *Let $(M, \mathcal{F})$ be a foliated manifold. Suppose $(M, \mathcal{F})$ is transverse holomorphic and admits a transverse Kähler structure ω. Then the filtration $F^r = \bigoplus_{r \le p} A^*(M, E)^{p,q}$ induces a functorial real Hodge structure of weight $i + w$ on the cohomology $H_B^i(M, E_{\mathbb{R}})$.*

Proof. We have the Hodge theory on the basic differential forms (see [EKA] and [KT]). As usual, we have the Kähler identities

$$[\Lambda, D'] = -\sqrt{-1}(D'')^* \quad \text{and} \quad [\Lambda, D''] = (D')^*$$

on the operators D' and D''. The proof is given by the standard argument. □

4. Variations of Hodge structure over Sasakian manifolds

Let M be a $(2n+1)$-dimensional real smooth manifold. A *CR-structure* on M is an n-dimensional complex sub-bundle $T^{1,0}$ of the complexified tangent bundle $TM_{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$ such that $T^{1,0} \cap \overline{T^{1,0}} = \{0\}$ and $T^{1,0}$ is integrable (i.e., the locally defined sections of $T^{1,0}$ are closed under the Lie bracket operation). We shall denote $\overline{T^{1,0}}$ by $T^{0,1}$. For a CR-structure $T^{1,0}$ on M, there is a unique sub-bundle S of rank $2n$ of the real tangent bundle TM together with a vector bundle homomorphism $I : S \longrightarrow S$ satisfying the conditions that

1. $I^2 = -\mathrm{Id}_S$, and
2. $T^{1,0}$ is the $\sqrt{-1}$-eigenbundle of I.

A $(2n+1)$-dimensional manifold M equipped with a triple $(T^{1,0}, S, I)$ as above is called a *CR-manifold*. A *contact CR-manifold* is a CR-manifold M with a contact 1-form η on M such that $\ker \eta = S$. Let ξ denote the Reeb vector field for the contact form η i.e. the unique vector field characterized by $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. On a contact CR-manifold, the above homomorphism I extends to entire TM by setting $I(\xi) = 0$.

Definition 4.1. *A contact CR-manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$ is a* strongly pseudo-convex CR-manifold *if the Hermitian form (called the Levi form) L_η on S_x defined by $L_\eta(X, Y) = d\eta(X, IY)$, $X, Y \in S_x$, is positive definite for every point $x \in M$.*

Given any strongly pseudo-convex CR-manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$, there is a canonical Riemann metric g_η on M, called the *Webster metric*, which is defined to be

$$g_\eta(X, Y) := L_\eta(X, Y) + \eta(X)\eta(Y)\,, \quad X, Y \in T_xM\,, \ x \in M\,.$$

For another strongly pseudo-convex CR-manifold $(M', (T^{1,0\prime}, S', I'), (\eta', \xi'))$, a map $f : M \to M'$ is *CR-holomorphic* if $df \circ I = I' \circ df$. For any $x \in M$, by

$\ker(I_x) = \langle \xi_x \rangle$, we have $df_x(\xi_x) = c(x)\xi'_{f(x)}$ and $f^*\eta' = c(x)\eta$ for some positive function $c(x)$ on M.

Remark 4.2. *For two CR manifolds* $(M, (T^{1,0}, S, I))$ *and* $(M', (T^{1,0\prime}, S', I'))$, *a map* $f : M \to M'$ *is a* CR-map *if* $df(T_x^{1,0}) \subset T_{f(x)}^{1,0\prime}$ *for any* $x \in M$. *On strongly pseudo-convex CR-manifolds, CR-holomorphic maps are CR maps. But the converse may not be true.*

For a complex manifold (N, J), a map $f : M \to N$ is *CR-holomorphic* if $df \circ I = J \circ df$. By $0 = df \circ I(\xi) = J \circ df(\xi)$, we have $df_x(\xi_x) = 0$ for any $x \in M$. A map $f : M \to N$ satisfies $df_x(\xi_x) = 0$ and $df_x(T^{1,0}) \subset T^{1,0}N$ for any $x \in M$ if and only if $f : M \to N$ is CR-holomorphic.

Definition 4.3. *A Sasakian manifold is a strongly pseudo-convex CR-manifold*

$$(M, (T^{1,0}, S, I), (\eta, \xi))$$

such that for any section ζ *of* $T^{1,0}$, $[\xi, \zeta]$ *is also a section of* $T^{1,0}$.

Let $(M, (T^{1,0}, S, I), (\eta, \xi))$ be a compact Sasakian manifold. The flow of Reeb vector field ξ defines a 1-dimensional foliation $\mathcal{F}_\xi$ on M. By the definition of the Sasakian manifold, the CR-structure $T^{1,0}$ defines a transverse holomorphic structure on the foliated manifold $(M, \mathcal{F}_\xi)$. Furthermore, the closed basic 2-form $d\eta$ is a transverse Kähler structure with respect to this transverse holomorphic structure.

Example 4.4. *Consider the Heisenberg Lie group* H_3 *as in Example 3.1. Then, the complex sub-bundle* $T^{1,0}$ *in* $TH_{3\mathbb{C}}$ *generated by*

$$\left\langle \frac{\partial}{\partial x} - \sqrt{-1}\left(\frac{\partial}{\partial y} + x\frac{\partial}{\partial t}\right)\right\rangle$$

is a left-invariant CR structure on H_3. *Take* $\eta = dt - xdy$ *and* $\xi = \frac{\partial}{\partial t}$. *Then* $\eta = dt - xdy$ *is a contact structure and* ξ *is its Reeb vector field. Since we have*

$$\left[\xi, \frac{\partial}{\partial x} - \sqrt{-1}\left(\frac{\partial}{\partial y} + x\frac{\partial}{\partial t}\right)\right] = 0,$$

we obtain a left-invariant Sasakian structure on H_3. *By the left-invariance, this induces a Sasakian structure on the compact quotient* $\Gamma\backslash H_3$.

The main purpose of this paper is to study a basic polarized variation of Hodge structure over the foliated manifold $(M, \mathcal{F}_\xi)$. We first show the semi-simplicity as Theorem 2.2.

Theorem 4.5. *Let* $(M, (T^{1,0}, S, I), (\eta, \xi))$ *be a compact Sasakian manifold. For any basic polarized complex variation of Hodge structure* $(E = \bigoplus_{p+q=w} E^{p,q}, D, h)$ *of weight* w *over* $(M, \mathcal{F}_\xi)$*, the flat bundle* (E, D) *is semi-simple.*

Proof. Define the Hermitian metric H on E so that $H(v,w) = (-1)^p h(v,w)$ for $v, w \in E^{p,q}$. Since $\bigoplus_{p+q=w} E^{p,q}$ is a direct sum of basic C^∞-subbundles, H is basic. But it may not be parallel. Consider $\phi = \theta + \bar{\theta}$ and $\nabla = D - \phi$. Then ∇ is a unitary connection and ϕ is a basic 1-form with values in the self-adjoint part of $\mathrm{End}(E)$ corresponding to H. Applying [BK, Theorem 4.2], we can say that the Hermitian metric H is harmonic (i.e. $\nabla^*\phi = 0$). By [Co, Proposition 3.2], the flat bundle (E, D) is semi-simple.

□

Remark 4.6. *In* [Co]*, Corlette proved the semi-simplicity for polarized complex variations of Hodge structures over compact Kähler manifolds by using harmonic metrics. This is different from Griffiths's proof in* [Gri3]*. The proof of Theorem 4.5 is based on Corlette's idea.*

For two Sasakian manifolds $(M, (T^{1,0}, S, I), (\eta, \xi))$ and $(M', (T^{1,0\prime}, S', I'), (\eta', \xi'))$, consider a CR-holomorphic map $f : M \to M'$. We have $f^*\eta' = c(x)\eta$ for some positive function $c(x)$ on M. Since $d\eta'$ is basic, we have

$$df^*\eta'(\xi_x, X) = f^*d\eta'(\xi_x, X) = d\eta'(df(\xi_x), df(X)) = d\eta'(c(x)\xi'_{f(x)}, df(X)) = 0$$

for any $X \in TM_x$. By $\mathcal{L}_\xi d\eta = 0$, this implies

$$\mathcal{L}_\xi(c)\eta = \mathcal{L}_\xi(f^*\eta') = (i_\xi d + di_\xi)(f^*\eta') = di_\xi f^*\eta' = dc$$

and so $dc = 0$ on the sub-bundle $S \subset TM$. This implies the following claim (cf. [DT, Lemma 4.1]).

Claim 4.7. *$c(x)$ is constant.*

Proof. Since $TM = S \oplus \langle \xi \rangle$, it is sufficient to prove $dc(\xi_x) = 0$ for any $x \in M$. We use the Tanaka-Webster connection:

Proposition 4.8 ([Ta], [We]). *For a strongly pseudo-convex CR-manifold*

$$(M,\,(T^{1,0},\,S,\,I),\,(\eta,\,\xi))$$

there exists a unique affine connection ∇^{TW} on TM such that the following hold:

1. $\nabla^{TW}(C^{\infty}(S)) \subset A^1(M,\,S)$*, where $A^k(M,\,S)$ is the space of differential k-forms on M with values in the vector bundle S.*
2. $\nabla^{TW}\xi = 0$*,* $\nabla^{TW}I = 0$*,* $\nabla^{TW}d\eta = 0$*,* $\nabla^{TW}\eta = 0$ *and* $\nabla^{TW}g_\eta = 0$*.*
3. *The torsion T^{TW} of the affine connection ∇^{TW} satisfies the equation*

$$T^{TW}(X,Y) = -d\eta(X,Y)\xi$$

for all $X,\,Y \in S_x$ and $x \in M$.

Since $d\eta$ is non-degenerate on S, we can take local sections X, Y of S such that $d\eta(X_x, Y_x) \neq 0$. By the third property of the connection ∇^{TW} in Proposition 4.8, we have

$$\nabla^{TW}_X Y - \nabla^{TW}_Y X - [X,Y] = -d\eta(X,Y)\xi.$$

By the first property in Proposition 4.8, $\nabla^{TW}_X Y - \nabla^{TW}_Y X$ is a local section of S. Hence

$$-d\eta(X,Y)dc(\xi) = dc(\nabla^{TW}_X Y - \nabla^{TW}_Y X) - X(Y(c)) + Y(X(c)) = 0.$$

This implies $dc(\xi_x) = 0$. □

Remark 4.9. *Usually, morphisms between CR manifolds are CR-maps. But we would like to adopt CR-holomorphic maps as morphisms between Sasakian manifolds rather than CR maps. A CR-holomorphic map between Sasakian manifolds can be regarded as an analogue of a holomorphic map between Kähler manifolds (see* [IP], [Pe]*). For example, CR-holomorphic maps between Sasakian manifolds are harmonic and an analogue of the Siu theorem (*[Siu]*) follows.*

Since the pull-back $f^*\varphi$ of a basic function φ on $(M', \mathcal{F}_{\xi'})$ is a basic function on $(M, \mathcal{F}_\xi)$, the pullback f^*E of a basic vector bundle E over $(M', \mathcal{F}_{\xi'})$ is a basic vector bundle E over $(M, \mathcal{F}_\xi)$. The pull-back f^* on the de Rham complex is restricted as $f^* : A^*_B(M') \to A^*_B(M)$ and this map is a morphism of double complexes. For a basic polarized complex variation of Hodge structure of w $(E = \bigoplus_{p+q=w} E^{p,q}, D, h)$

over $(M', \mathcal{F}_{\xi'})$, the pull-back $(f^*E = \bigoplus_{p+q=w} f^*E^{p,q}, f^*D, f^*h)$ over $(M, \mathcal{F}_\xi)$. The pull-back $f^* : A^*(M', E) \to A^*(M, f^*E)$ is restricted as $f^* : A^*_B(M', E) \to A^*_B(M, f^*E)$ and this map is a morphism of double complexes.

Like complex case, we see the correspondence between basic polarized complex variations of Hodge structure over Sasakian manifolds and equivariant maps into classifying spaces of polarized Hodge structures. For a compact Sasakian manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$, we take the universal cover $p : \tilde{M} \to M$ associated with a base point $x \in M$. We consider the Sasakian manifold $(\tilde{M}, (\tilde{T^{1,0}}, \tilde{S}, \tilde{I}), (\tilde{\eta}, \tilde{\xi}))$. Let $(E = \bigoplus_{p+q=w} E^{p,q}, D, h)$ a basic polarized complex variation of Hodge structure of weight w over $(M, \mathcal{F}_\xi)$. Let $G = \mathrm{Aut}(E_x, h_x)$ and $V = \Pi_{p+q=w} \mathrm{Aut}(E^{p,q}_x, h_x)$. Take the monodromy $\rho : \pi_1(M, x) \to GL(E_x)$ of the flat bundle (E, D). Since h is parallel, we have $\rho(\pi_1(M, x)) \subset G$. Let $\mathfrak{g}$ be the Lie algebra of G. The decomposition $E = \bigoplus_{p+q=w}$ gives the Hodge structure $\mathfrak{g}_{\mathbb{C}} = \bigoplus_r \mathfrak{g}^{-r,r}$. For the Lie algebra $\mathfrak{v}$ of V, we have $\mathfrak{v} = \mathfrak{g} \cap \mathfrak{g}^{0,0}$. Consider the Homogeneous space G/V. We identify

$$T(G/V) = (G \times \mathfrak{g}/\mathfrak{v})/V = \left(G \times \mathfrak{g} \cap \left(\bigoplus_{r \neq 0} \mathfrak{g}^{-r,r}\right)\right)/V.$$

It is known that G/V is the classifying space for h_x-polarized complex Hodge structures of type $\{d^{p,q}\}$ where $d^{p,q} = \dim E^{p,q}_x$. This means that every h_x-polarized complex Hodge structure of type $\{d^{p,q}\}$ is given by $gE^{p,q}_x$ for a unique $gV \in G/V$. The Hermitian metric H as above corresponds to a ρ-equivariant map $\Phi : \tilde{M} \to GL(E_x)/\mathrm{Aut}(E_x, H_x)$ so that $d\Phi_y = \phi_{p(y)}$ where we identify $E = \pi_1(M, x) \backslash (\tilde{M} \times E_x)$ and $T(GL(E_x)/\mathrm{Aut}(E_x, H_x)) = (GL(E_x) \times S(E_x, H_x))/\mathrm{Aut}(E_x, H_x)$ for the vector space $S(E_x, H_x)$ of the self-adjoint linear operators on E_x. Since H comes from polarized complex Hodge structures on E_x, we can define $\Phi : \tilde{M} \to G/V$.

Defining the complex structure on G/V whose holomorphic tangent bundle is $T^{1,0}(G/V) = \left(G \times \bigoplus_{r>0} \mathfrak{g}^{-r,r}\right)/V$. We have the holomorphic sub-bundle

$$T^{1,0}_h(G/V) = \left(G \times \mathfrak{g}^{-1,1}\right)/V \subset T^{1,0}(G/V).$$

Now, for any $y \in$ we have $d\Phi_y(\tilde{\xi}_x) = 0$ and the basic Griffiths transversality implies

$$d\Phi_y(T^{1,0}_y \tilde{M}) = \theta(T^{1,0}_{p(y)} M) \subset T^{1,0}_h(G/V)_{\Phi(y)}.$$

In particular, the map $\Phi : \tilde{M} \to G/V \subset GL(E_x)/\mathrm{Aut}(E_x, H_x)$ is CR-holomorphic.

Conversely, if we have:

- a polarized complex Hodge structure $(E_x = \bigoplus_{p+q=w} E_x^{p,q}, h_x)$ on a vector space E_x,
- a representation $\rho : \pi_1(M, x) \to G$ and
- a CR-holomorphic ρ-equivariant map $\Phi : \tilde{M} \to G/V$ such that $d\Phi_y(T_y^{1,0}\tilde{M}) \subset T_h^{1,0}(G/V)_{\Phi(y)}$ for any $y \in \tilde{M}$

where we define G, V and $T_h^{1,0}(G/V)$ in the same manner as above, then we obtain a basic polarized complex variation of Hodge structure $(E = \bigoplus_{p+q=w} E^{p,q}, D, h)$ so that $E = \pi_1(M, x)\backslash(\tilde{M} \times_\rho E_x)$ and for each $y \in \tilde{M}$, $E_y^{p,q} = \Psi(y)E_x^{p,q}$ by taking $\Psi(y) \in G$ as $\Phi(y) = \Psi(y)V \in G/V$.

Remark 4.10. *Similarly a basic polarized real variation of Hodge structure $(E_{\mathbb{R}}, E = \bigoplus_{p+q=w} E^{p,q}, D, Q)$ corresponds to*

- *a polarized real Hodge structure $(E_{\mathbb{R},x}, E_x = \bigoplus_{p+q=w} E_x^{p,q}, h_x)$ on a vector space E_x,*
- *a representation $\rho : \pi_1(M, x) \to G$ and*
- *a CR-holomorphic ρ-equivariant map $\Phi : \tilde{M} \to G/V$ such that $d\Phi_y(T_y^{1,0}\tilde{M}) \subset T_h^{1,0}(G/V)_{\Phi(y)}$ for any $y \in \tilde{M}$*

where $G = \mathrm{Aut}(E_{\mathbb{R},x}, Q_x)$, $V = \{g \in G : gE_x^{p,q} = E_x^{p,q}\ \forall (p,q)\}$ and for the Lie algebra $\mathfrak{g}$ of G with the Hodge structure $\mathfrak{g}_{\mathbb{C}} = \bigoplus_r \mathfrak{g}^{-r,r}$ associated with $E_x = \bigoplus_{p+q=w} E_x^{p,q}$, $T_h^{1,0}(G/V) = \left(G \times \mathfrak{g}^{-1,1}\right)/V \subset T^{1,0}(G/V)$.

Remark 4.11. *For $t \in U(1)$, we have $\tau(t) \in V$ so that $\tau(t)v = t^{p-q}v$ for $v \in E_x^{p,q}$. We consider the 1-dimensional sub-torus $S \subset V$ associated with $\tau : U(1) \to V$. Then S is contained in the center C of V. Since C is a torus and a direct factor of V, we can take a closed subgroup $V' \subset V$ such that $V = V' \times S$. Consider the homogeneous space G/V'. Then we have $T(G/V') = \left(G \times \mathfrak{g} \cap \left(\bigoplus_{r\neq 0} \mathfrak{g}^{-r,r}\right) \bigoplus \mathfrak{s}\right)/V'$ where $\mathfrak{s}$ is the Lie algebra of S. We define the CR structure $T^{1,0} = \left(G \times \bigoplus_{r>0} \mathfrak{g}^{-r,r}\right)/V'$. It may be interesting to extend CR-holomorphic maps $\Phi : \tilde{M} \to G/V$ corresponding to basic polarized complex variations of Hodge structure to CR-holomorphic maps $\Phi' : \tilde{M} \to G/V'$.*

We can see G/V' admits a pseudo-Sasakian structure. Since the action of V' on $\mathfrak{s}$ is trivial, we have a real left-invariant vector field ξ on G/V' corresponding to $\mathfrak{s}$ and a real left-invariant 1-form η on G/V' corresponding to $\mathfrak{s}^*$ such *that $\xi(\eta) = 1$. Then $i_\xi d\eta = 0$ and $[\xi, T^{1,0}] \subset T^{1,0}$. We define the symmetric bilinear form B on $\mathfrak{g} \cap \left(\bigoplus_{r\neq 0} \mathfrak{g}^{-r,r}\right) \bigoplus \mathfrak{s}$ such that $B(X,Y) = -Tr(XY)$. Then B polarizes the Hodge structure $\bigoplus_{r\neq 0} \mathfrak{g}^{-r,r} \oplus \mathfrak{s}_{\mathbb{C}}$ where $\mathfrak{s}_{\mathbb{C}}$ is the component of type $(0,0)$ (see* [Sch, Lemma 8.8]*). For $X, Y \in \mathfrak{g}^{-r,r}$, for some $T \in \mathfrak{s}$, we have $B([X,\overline{Y}],T) = -B(X,[T,\overline{Y}]) = -r\sqrt{-1}B(X,\overline{Y})$. This implies that the left-invariant Hermitian form $\sqrt{-1}d\eta(X,\overline{Y}) = -\sqrt{-1}\eta([X,\overline{Y}])$ on $T^{1,0}$ has the same rank and same signature as $B(X,\overline{Y})$. Thus η is a contact form on G/V' which is compatible with the CR-structure $T^{1,0}$ and its Levi form L_η has the signature (s,t) where $s = \sum_{p\, even} \dim E^{p,q}$ and $t = \sum_{p\, odd} \dim E^{p,q}$.*

We give an analogue of Theorem 2.4.

Theorem 4.12. *Let $(M, (T^{1,0}, S, I), (\eta, \xi))$ be a compact Sasakian manifold. For a polarized real basic variation of Hodge structure of weight w over $(M, \mathcal{F})$ is $(E_{\mathbb{R}}, E = \bigoplus_{p+q=w} E^{p,q}, D, Q)$, there exists a functorial real Hodge structure on the cohomology $H^i(M, E_{\mathbb{R}})$ of weight $i+w$ for $1 \leq i \leq n$ or of weight $i+w+1$ for $n+1 \leq i \leq 2n$.*

Proof. As [Ka], we have the quasi-isomorphisms

$$A^*(M, E_{\mathbb{R}}) \leftarrow \ker D^c \oplus \ker D^c \wedge \eta \rightarrow H_B^*(M, E_{\mathbb{R}}) \oplus H_B^*(M, E_{\mathbb{R}}) \otimes \langle \eta \rangle.$$

By Theorem 3.2, there exists a functorial Hodge structure on $H_B^i(M, E_{\mathbb{R}})$ of weight $i+w$. Putting the Hodge structure on $\langle \eta \rangle$ so that η is of type $(1,1)$, we have the Hodge structure on $H_B^{i-1}(M, E_{\mathbb{R}}) \otimes \langle \eta \rangle$ of weight $i+w+1$. As the usual Lefschetz decomposition, the map $H_B^r(M, E_{\mathbb{R}}) \ni a \mapsto a \wedge d\eta \in H_B^{r+2}(M, E_{\mathbb{R}})$ is injective for $r \leq n-1$ and surjective for $r \geq n-1$. Thus the cohomology $H^i(M, E_{\mathbb{R}})$ is canonically isomorphic to

$$H_B^i(M, E_{\mathbb{R}})/H_B^{i-2}(M, E_{\mathbb{R}}) \wedge d\eta$$

for $i \leq n$ or canonically isomorphic to

$$\ker\left(H_B^{i-1}(M, E_{\mathbb{R}}) \ni a \mapsto a \wedge d\eta \in H_B^{i+1}(M, E_{\mathbb{R}})\right) \otimes \langle \eta \rangle$$

for $n+1 \leq i \leq 2n$. Thus, the Hodge structure as in the statement exists.

We show the functoriality. Let $f : M \to M'$ be a CR-holomorphic map. Via the quasi-isomorphisms as above, the pull-back $f^* : A^*(M', E) \to A^*(M, f^*E)$

corresponds to the map

$$f^* : H^*_B(M', E_{\mathbb{R}}) \oplus H^*_B(M', E_{\mathbb{R}}) \otimes \langle \eta' \rangle \to H^*_B(M, f^*E_{\mathbb{R}}) \oplus H^*_B(M, f^*E_{\mathbb{R}}) \otimes \langle \eta \rangle.$$

Since the map $f^* : H^i_B(M', E_{\mathbb{R}}) \to H^i_B(M, f^*E_{\mathbb{R}})$ is a morphism of Hodge structures and $f^*\eta' = c\eta$ for some constant c, the map

$$f^* : H^*_B(M', E_{\mathbb{R}}) \oplus H^*_B(M', E_{\mathbb{R}}) \otimes \langle \eta' \rangle \to H^*_B(M, f^*E_{\mathbb{R}}) \oplus H^*_B(M, f^*E_{\mathbb{R}}) \otimes \langle \eta \rangle.$$

induces a morphism of Hodge structures on the cohomology. Hence the map $f^* : H^i(M', E_{\mathbb{R}}) \to H^i(M, f^*E_{\mathbb{R}})$ is a morphism of Hodge structures.

□

5. Remarks on Sasakian manifolds and Kähler manifolds(orbifolds)

Let X be a complex projective manifold with a Hodge metric i.e. a Kähler metric whose Kähler form ω determines an integral cohomology class $[\omega] \in H^2(M, \mathbb{Z})$. Then, we have a principal circle bundle (Boothby-Wang fibration) $\pi : M \to X$ and a 1-form η on M such that $d\eta = \pi^*\omega$. It is known that we can take η a contact form on M and defining the CR-structure $T^{1,0}$ by the horizontal lifting of $T^{1,0}X$, η and $T^{1,0}$ constitute a Sasakian structure on M (see [Bl]). Pulling back of variations of Hodge structure over X, we obtain basic variations of Hodge structure over the Sasakian manifold M.

An n-dimensional $\mathcal{C}^\infty$(complex) *orbifold* is a paracompact Hausdorff space X with a family $\mathcal{U} = (\{U_\alpha, \tilde{U}_\alpha, \Gamma_\alpha, \phi_\alpha\})_\alpha$ such that :

- For each α, $\tilde{U}_\alpha$ is a connected open subset of $\mathbb{R}^n$ ($\mathbb{C}^n$) containing the origin, Γ_α is a finite subgroup in the orthogonal group $O(n)$ (the unitary group $U(n)$) and ϕ_α is a Γ_α-invariant continuous map from $\tilde{U}_\alpha$ into an open subset U_α in M which induces a homeomorphism between $\tilde{U}_\alpha/\Gamma_\alpha$ and U_α.
- $M = \bigcup_\alpha U_\alpha$.
- For two $\{U_\alpha, \tilde{U}_\alpha, \Gamma_\alpha, \phi_\alpha\}$ and $\{U_\beta, \tilde{U}_\beta, \Gamma_\alpha, \phi_\alpha\}$ with $x \in U_\alpha \cap U_\beta$, there exists $\{U_\gamma, \tilde{U}_\gamma, \Gamma_\gamma, \phi_\gamma\}$ such that $x \in U_\gamma$ and there are $\mathcal{C}^\infty$(holomorphic) embeddings $\lambda_{\alpha\gamma} : \tilde{U}_\gamma \to \tilde{U}_\alpha$ and $\lambda_{\beta\gamma} : \tilde{U}_\gamma \to \tilde{U}_\beta$ satisfying $\phi_\alpha \circ \lambda_{\alpha\gamma} = \phi_\gamma$ and $\phi_\beta \circ \lambda_{\beta\gamma} = \phi_\gamma$.

If each finite group Γ_α is trivial, an orbifold $(X,\mathcal{U})$ is a smooth manifold. A Riemannian (Hermitian) metric on a $\mathcal{C}^\infty$(complex) orbifold $(X,\mathcal{U})$ is a family $(g_\alpha)_\alpha$ consisting of a Riemannian (Hermitian) metrics on each $\tilde{U}_\alpha$ such that each g_α is

Γ_α-invariant and any $\lambda_{\alpha\gamma}$ is an isometry. A Hermitian metric on a complex orbifold $(X,\mathcal{U})$ is Kähler if each g_α is a Kähler metric on $\tilde{U}_\alpha$. For a Lie group G, a principal G-orbibundle over a $\mathcal{C}^\infty$ orbifold $(X,\mathcal{U})$ consists principal G-bundles $p_\alpha : B_\alpha \to U_\alpha$ with homomorphisms $h_\alpha : \Gamma_\alpha \to G$ such that:

- $p_\alpha(bh_\alpha(\gamma)) = \gamma^{-1}p(b)$ for $b \in B_\alpha$ and $\gamma \in \Gamma_\alpha$.
- For any $\lambda_{\alpha\gamma} : \tilde{U}_\gamma \to \tilde{U}_\alpha$, there exists a bundle map $\Lambda_{\alpha\gamma} : p_\alpha^{-1}(\lambda_{\alpha\gamma}(\tilde{U}_\gamma)) \to B_\gamma$ such that for $g_\gamma \in \Gamma_\gamma$ and $g_\alpha \in \Gamma_\alpha$ with $\lambda_{\alpha\gamma} \circ g_\gamma = g_\alpha \circ \lambda_{\alpha\gamma}$, $h_\gamma(g_\gamma) \circ \Lambda_{\alpha\gamma} = \Lambda_{\alpha\gamma} \circ h_\alpha(g_\alpha)$.

Similarly we define a (holomorphic) vector orbibundle over a $\mathcal{C}^\infty$ (complex) orbifold $(X,\mathcal{U})$, replacing $p_\alpha : B_\alpha \to U_\alpha$ with vector bundles and $h_\alpha : \Gamma_\alpha \to G$ with homomorphisms from Γ_α into linear transformations. For a (holomorphic) vector orbibundle $E \to X$, we define a section of E as a family of Γ_α-equivariant sections which are compatible with the transition structure of E. In the usual manner, we define the de Rham (Dolbeault) complex $(A^*(X), d)$ $((A^{*,*}(X), \partial, \bar{\partial}))$ of a $\mathcal{C}^\infty$ (complex) orbifold $(X,\mathcal{U})$. We also define a connection on a vector orbibundle $E \to X$ as a differential operator D on the space $A^*(X, E)$ of differential forms with values in E and we say a connection D is flat if $DD = 0$. Let M be a complex orbifold. We define a polarized complex (real) variation of Hodge structure of weight w over a complex orbifold M by the same manner as in Definition 2.1 (2.3).

A compact Sasakian manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$ is called *quasi-regular* if every leaf of the foliation $\mathcal{F}_\xi$ is closed. For any compact Sasakian manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$, we can take another contact form η' with the Reeb vector field ξ' so that $(M, (T^{1,0}, S, I), (\eta', \xi'))$ is quasi-regular (see [BG, Section 8.2.3]. Under the quasi-regular condition, the leaf space $M/\mathcal{F}_\xi$ admits a canonical Kähler orbifold structure and M is the total space of a principal S^1-orbibundle over $M/\mathcal{F}_\xi$(see [BG]). Precisely, the basic two form $d\eta$ induces a Kähler structure on the orbifold $M/\mathcal{F}_\xi$ which is a representative a integral class of the orbifold cohomology. We note that the orbifold $M/\mathcal{F}_\xi$ is a projective algebraic variety by the result in [1]. Since M is a smooth manifold, each homomorphism $h_\alpha : \Gamma_\alpha \to S^1$ for this principal S^1-orbibundle is injective. Pulling back of variations of Hodge structure over $M/\mathcal{F}_\xi$, we obtain basic variations of Hodge structure over the Sasakian manifold M.

6. Examples

Consider $SL_2(\mathbb{R})$ and its universal covering $\pi : \widetilde{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R})$. Let

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad Z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $SL_2(\mathbb{R})$, we have $\mathfrak{sl}_2(\mathbb{R}) = \langle X, Y, Z \rangle$ and $[X, Z] = -2Y$, $[Y, Z] = 2X$ and $[X, Y] = -2Z$. We regard $\mathfrak{sl}_2(\mathbb{R})$ as the left-invariant vector fields. Then, $W = \frac{1}{2}(X - \sqrt{-1}Y)$ defines a left-invariant CR structure $T^{1,0}$ on $\widetilde{SL}_2(\mathbb{R})$. Consider the dual $\mathfrak{sl}_2(\mathbb{R})^* = \langle x, y, z \rangle$ as the left-invariant differential forms. Then, $\eta = z$ is a left-invariant contact structure on $\widetilde{SL}_2(\mathbb{R})$ and we have the strongly pseudo-convex CR-manifold $(\widetilde{SL}_2(\mathbb{R}), (T^{1,0}, S, I), (\eta, \xi))$ with $\xi = Z$. Since $[Z, W] = 2\sqrt{-1}W$, this is a Sasakian manifold.

Let $\widetilde{\Gamma} \subset \widetilde{SL}_2(\mathbb{R})$ be a cocompact discrete subgroup. Consider the Sasakian manifold $(\widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}), (T^{1,0}, S, I), (\eta, \xi))$. Let $\rho_k : SL_2(\mathbb{R}) \to GL_{k+1}(\mathbb{R})$ be the irreducible representation which is the k-th symmetric product of the standard representation $SL_2(\mathbb{R}) \to SL_2(\mathbb{R})$. For the representation $\tilde{\rho_k} = \rho_k \circ \pi : \widetilde{\Gamma} \to GL_{k+1}(\mathbb{R})$, we define the flat bundle $E^k_{\mathbb{R}} = \widetilde{\Gamma}\backslash(\widetilde{SL}_2(\mathbb{R}) \times_{\tilde{\rho_k}} \mathbb{R}^{k+1})$ with the natural flat connection D. Corresponding $v \in \mathbb{R}^{k+1}$ to the section $\tilde{\rho_k}(g)v$ of $E^k_{\mathbb{R}}$, we have the trivialization $E^k_{\mathbb{R}} = \widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}) \times \mathbb{R}^{k+1}$ such that $D = d + \omega$ with

$$\omega = \rho_k(X) \otimes x + \rho_k(Y) \otimes y + \rho_k(Z) \otimes z.$$

Let $\mathbb{C}^{k+1} = \oplus_{0 \le p \le k} V_{k-2p}$ so that $\rho_k(Z)(v) = (k-2p)\sqrt{-1}v$ for $v \in V_{k-2p}$. Then we have

$$\rho_k(W)V_{k-2p} \subset V_{k-2p+2} \qquad \text{and} \qquad \rho_k(\overline{W})V_{k-2p} \subset V_{k-2p-2}.$$

Define the subbundle $E^{p,k-q} = \widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}) \times V_{k-2p}$ of $E^k = E^k_{\mathbb{R}} \otimes \mathbb{C} = \widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}) \times \mathbb{C}^{k+1}$. Taking the 1-parameter subgroup associated with the vector field $\xi = Z$, this can be considered as an $S^1 = SO(2)$-action (see [RV]). This S^1-action is extended to a S^1-action on $E^k_{\mathbb{R}}$. We can see that for $t \in S^1$ and $([g], v) \in \widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}) \times \mathbb{R}^{k+1}$, $t \cdot (\widetilde{\Gamma}g, v) = (\widetilde{\Gamma}gt, \rho_k(t)^{-1}v)$. By this condition, $E^k = \bigoplus_{p+q=k} E^{p,q}$ is a direct sum of basic $\mathcal{C}^\infty$-subbundles. Let $D = \nabla + \theta + \bar{\theta}$ where

$$\nabla = d + \rho_k(Z) \otimes z, \qquad \theta = \frac{1}{2}\rho_k(W) \otimes w \qquad \text{and} \qquad \bar{\theta} = \frac{1}{2}\rho_k(\overline{W}) \otimes \bar{w}$$

where $w = x + \sqrt{-1}y$. Since we have

$$\nabla : A^0_B(\widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}), E^{p,q}) \to A^1_B(\widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}), E^{p,q}),$$

$$\theta : A^{0,0}_B(\widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}), E^{p,q}) \to A^{1,0}_B(\widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}), E^{p-1,q+1})$$

and

$$\bar{\theta} : A^{0,0}_B(\widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}), E^{p,q}) \to A^{0,1}_B(\widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}), E^{p+1,q-1}),$$

the basic Griffiths transversality condition holds. We define a parallel $(-1)^k$-symmetric bilinear form Q on $E^k_{\mathbb{R}}$ induced by the skew-symmetric form on $\mathbb{R}^2$ associated with the determinant on $GL_2(\mathbb{R})$. Finally, we have the polarized real basic variation of Hodge structure $(E^k_{\mathbb{R}}, E^k = \bigoplus_{p+q=k} E^{p,q}, D, Q)$ of weight k over $(\widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R}), \mathcal{F}_\xi)$.

Remark 6.1. *The compact Sasakian manifold* $(\widetilde{\Gamma}\backslash\widetilde{SL}_2(\mathbb{R})$, $(T^{1,0}$, S, $I)$, $(\eta$, $\xi))$ *is quasi-regular. The leaf space* $M/\mathcal{F}_\xi$ *is a hyperbolic orbifold Riemann surface* $\pi(\widetilde{\Gamma})\backslash H$ *where* H *is the upper-half plane with the hyperbolic metric. The basic vector bundle* $E^{p,q}$ *is the pull-back of the vector orbibundle* L^{p-q} *over* $\pi(\widetilde{\Gamma})\backslash H$ *where* L *is a square root of the canonical bundle of the orbifold Riemann surface* $\pi(\widetilde{\Gamma})\backslash H$*.*

Let G be a connected non-compact simple Lie group and K a maximal compact subgroup of G. We assume that G is of Hermitian type i.e. G/K is a Hermitian symmetric space of non-compact type. Consider the universal covering $\pi : \widetilde{G} \to G$ and the subgroup $\widetilde{K} = \pi^{-1}(K)$. Then $\widetilde{K}$ has a unique maximal compact subgroup $\widetilde{K}_1$ which is a maximal compact subgroup in $\widetilde{G}$. Let $\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}(\widetilde{G})$, $\mathfrak{k} = \mathrm{Lie}(K) = \mathrm{Lie}(\widetilde{K})$ and $\mathfrak{k}_1 = \mathrm{Lie}(\widetilde{K}_1)$. Denote by $\mathfrak{c}$ the the center of $\mathfrak{k}$. We have $\dim \mathfrak{c} = 1$ and $\mathfrak{k} = \mathfrak{c} \oplus \mathfrak{k}_1$.

Let $\mathfrak{p} \subset \mathfrak{g}$ be the orthogonal complement of $\mathfrak{k}$ associated with the Cartan-Killing form. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then, we can take $T \in \mathfrak{c}$ so that $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ for

$$\mathfrak{p}_+ = \{X \in \mathfrak{p}_{\mathbb{C}} | [T, X] = \sqrt{-1}X\} \qquad \text{and} \qquad \mathfrak{p}_- = \{X \in \mathfrak{p}_{\mathbb{C}} | [T, X] = -\sqrt{-1}X\}.$$

The decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ induces a G-invariant complex structure J on G/K and a $\widetilde{G}$-invariant CR structure $T^{1,0}$ on $\widetilde{G}/\widetilde{K}_1$ so that $T^{1,0}_{e\widetilde{K}} = \mathfrak{p}_+$. Consider the

$\widetilde{G}$-invariant 1-forms $\left((\mathfrak{g}/\mathfrak{k}_1)^*\right)^{\mathfrak{k}_1} = \mathfrak{c}^* \subset A^1(\widetilde{G}/\widetilde{K}_1)$ and 2-forms $\left(\bigwedge^2 (\mathfrak{g}/\mathfrak{k}_1)^*\right)^{\mathfrak{k}_1} = \left(\bigwedge^2 \mathfrak{p}^*\right)^{\mathfrak{k}_1} \subset A^2(\widetilde{G}/\widetilde{K}_1)$. Then $\left(\bigwedge^2 \mathfrak{p}^*\right)^{\mathfrak{k}_1}$ is spanned by G-invariant Kähler form ω on G/K and the differential $d : \mathfrak{c}^* \to \left(\bigwedge^2 \mathfrak{p}^*\right)^{\mathfrak{k}_1}$ is bijective. Take $\eta \in \mathfrak{c}^*$ so that $d\eta = \omega$ and $\xi \in \mathfrak{c}$ so that $\eta(\xi) = 1$. Then we have the Sasakian manifold $(\widetilde{G}/\widetilde{K}_1, (T^{1,0}, S, I), (\eta, \xi))$. The homogeneous space $\widetilde{G}/\widetilde{K}_1$ is a principal $\mathbb{R}$-bundle over G/K. We notice that this Sasakian manifold can be seen as the Boothby-Wang fibration of the Hermitian symmetric space G/K (see [BaK]).

Let $\widetilde{\Gamma} \subset \widetilde{G}$ be a cocompact discrete subgroup so that $\widetilde{\Gamma}$ acts freely on $\widetilde{G}/\widetilde{K}_1$. Then we have the compact Sasakian manifold $(\widetilde{\Gamma}\backslash\widetilde{G}/\widetilde{K}_1, (T^{1,0}, S, I), (\eta, \xi))$. We assume that $\pi(\widetilde{\Gamma})$ is discrete in G. We do not assume $\pi(\widetilde{\Gamma})$ acts freely on G/K. Hence the quotient $\pi(\widetilde{\Gamma})\backslash G/K$ is a Kähler orbifold.

We construct polarized complex basic variation of Hodge structure over $(\widetilde{\Gamma}\backslash\widetilde{G}/\widetilde{K}_1, \mathcal{F}_\xi)$ by the idea of Locally homogeneous variations of Hodge structure as in [Zu2]. Let Z be the center of K. Then we have an isomorphism $Z \cong U(1)$. For the adjoint group $Ad(G)$ of G and the subgroup $Z^{Ad} \subset Ad(G)$ corresponding to Z, the homomorphism $Z \to Z^{Ad}$ is a covering map. Denote by μ the degree of this covering map. Let $\rho : G \to GL(V)$ be an irreducible representation of G on a finite-dimensional $\mathbb{C}$-vector space V. Let $V_n = \{v \in V | \rho(z)v = z^n v, z \in Z\}$ where we regard $Z = U(1)$. Then we have non-negative integers m, n such that $m \geq \frac{n}{\mu}$ and $V = \bigoplus_{k=0}^{m} V_{n-k\mu}$. We have

$$\rho_*(\mathfrak{p}_+)V_{n-k\mu} \subset V_{n-k\mu+\mu}, \qquad \rho_*(\mathfrak{p}_-)V_{n-k\mu} \subset V_{n-k\mu-\mu}$$

and

$$\rho_*(\mathfrak{k})V_{n-k\mu} \subset V_{n-k\mu}.$$

Taking a compact real form of $\mathfrak{g}$, we can define a Hermitian inner product $(,)$ on V satisfying

$$(\rho_*(X)v_1, v_2) = -(v_1, \rho_*(X)v_2)$$

for $X \in \mathfrak{k}$ and

$$(\rho_*(X)v_1, v_2) = (v_1, \rho_*(X)v_2)$$

for $X \in \mathfrak{p}$ (see [MM, Lemma 3.1]).

Let $\tilde{\rho} = \rho \circ \pi : \widetilde{G} \to GL(V)$ We consider the flat bundles

$$E = \widetilde{\Gamma}\backslash(\widetilde{G}/\widetilde{K}_1 \times_{\tilde{\rho}} V) \qquad \text{and} \qquad E' = \widetilde{\Gamma}\backslash(\widetilde{G} \times_{\tilde{\rho}} V).$$

Corresponding $v \in V$ to the section $\tilde{\rho}(g)v$ of E, we have the trivialization $E' = \widetilde{\Gamma}\backslash\widetilde{G} \times V$. We can see that for $k \in \widetilde{K}$ and $([g], v) \in \widetilde{\Gamma}\backslash\widetilde{G} \times V$, $k \cdot (\widetilde{\Gamma}g, v) = (\widetilde{\Gamma}gk, \tilde{\rho}(k)^{-1}v)$. We have $E = E'/\widetilde{K}_1$ and define the subbundle

$$E^{k,n-k} = (\widetilde{\Gamma}\backslash\widetilde{G} \times V_{n-k\mu})/\widetilde{K}_1 \subset E = E'/\widetilde{K}_1.$$

Then they are basic and we have $E = \bigoplus_{p+q=n} E^{p,q}$. By the relations $\rho_*(\mathfrak{p}_+)V_{n-k\mu} \subset V_{n-k\mu+\mu}$ and $\rho_*(\mathfrak{p}_-)V_{n-k\mu} \subset V_{n-k\mu-\mu}$, the basic Griffiths transversality condition holds. By the relation $(\rho_*(X)v_1, v_2) = -(v_1, \rho_*(X)v_2)$ for $X \in \mathfrak{k}$, the Hermitian inner product $(,)$ induces Hermitian metric H on E and the direct sum $E = \bigoplus_{p+q=n} E^{p,q}$ is orthogonal with respect to H. Define the Hermitian form h on E so that $E = \bigoplus_{p+q=n} E^{p,q}$ is orthogonal with respect to h and $h = (-1)^p H$ on $E^{p,q}$. Then, by the relations $\rho_*(\mathfrak{p}_+)V_{n-k\mu} \subset V_{n-k\mu+\mu}$, $\rho_*(\mathfrak{p}_-)V_{n-k\mu} \subset V_{n-k\mu-\mu}$ and $(\rho_*(X)v_1, v_2) = (v_1, \rho_*(X)v_2)$ for $X \in \mathfrak{p}$, h is a parallel Hermitian form on the flat bundle E. Hence, we have the polarized complex basic variation of Hodge structure $(E = \bigoplus_{p+q=n} E^{p,q}, D, h)$ of weight n over $(\widetilde{\Gamma}\backslash\widetilde{G}/\widetilde{K}_1, \mathcal{F}_\xi)$.

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