Josai Mathematical Monographs
Vol. 13 (2021) pp. 53–68

Construction of contractible complete quaternionic almost Hermitian manifolds with compact isometry group

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Abstract. The $4n+3$-dimensional quaternionic Heisenberg nilpotent Lie group $\mathcal{M}$ admits a quaternionic contact structure. There exists a three dimensional simply connected non-abelian solvable Lie group $\mathcal{R}$ acting properly on $\mathcal{M}$. We show that the quotient of $\mathcal{M}$ by $\mathcal{R}$ admits a quaternionic almost Hermitian metric g. Moreover, one almost complex structure, say J from the quaternionic structure is shown to be integrable for which $(\mathcal{M}/\mathcal{R}, J)$ is a Bochner flat Kähler manifold.

1. Introduction

A *quaternionic contact structure* is a codimension 3-subbundle D on a $4n+3$-dimensional smooth manifold X such that $\mathsf{D}+[\mathsf{D},\mathsf{D}] = TX$. (See [3].) If there exists an $\mathrm{Im}\,\mathbb{H}$-valued 1-form $\omega = \omega_1 i + \omega_2 j + \omega_3 k$ on X such that $\ker\,\omega = \bigcap_{\alpha=1}^{3} \ker\,\omega_\alpha = \mathsf{D}$ and $\overset{3}{\wedge}\,\omega \overset{n}{\wedge}\, d\omega \neq 0$ on X, then ω is said to be a quaternionic contact form. The endomorphisms $\{J_1, J_2, J_3\}$ defined by

$$J_\gamma = (d\omega_\beta|\mathsf{D})^{-1} \circ (d\omega_\alpha|\mathsf{D}) : \mathsf{D}\to\mathsf{D} \;\; ((\alpha,\beta,\gamma) \sim (1,2,3)) \tag{1}$$

constitutes a hypercomplex structure on D. Then $(X, (\mathsf{D}, \omega, \{J_\alpha\}_{\alpha=1}^{3}))$ is called a *quaternionic contact* manifold (*qc*-manifold for short). See [3], [4], [1] for the definition and the reference therein.

Using the equality $d\omega_\alpha(X,Y) = d\omega_\beta(J_\gamma X, Y)$ on D, we have the reciprocity:

$$d\omega_1(J_1X,Y) = d\omega_2(J_2X,Y) = d\omega_3(J_3X,Y). \tag{2}$$

This shows $d\omega_\alpha(J_\alpha X, J_\alpha Y) = d\omega_\alpha(X,Y)$. If the distribution $\mathsf{E} = \{\xi \mid d\omega_1(\xi, A) = d\omega_2(\xi, A) = d\omega_3(\xi, A) = 0,\ ^{\forall} A \in TX\}$ generates a three dimensional local abelian

2020 Mathematics Subject Classification. Primary 53C55; Secondary 57S25, 51M10.
Key Words and Phrases. Quaternionic Hermitian structure, Quaternionic contact structure, Quaternionic Heisenberg Lie group.

Lie group preserving the qc-structure $(\mathsf{D}, \{\omega_\alpha, J_\alpha\}_{\alpha=1}^3)$ on X, then it is shown in [4], [7] that X is a *qc-Einstein manifold of vanishing qc-scalar curvature*. Furthermore if a local abelian group extends to an $\mathbb{R}^3$-action, then the smooth quotient manifold $X/\mathbb{R}^3$ supports a hyperKähler structure. (Of course this is always true *locally*.)

In this paper, given a $4n+3$-dimensional qc-manifold $(X, \mathsf{D}, \{\omega, J_\alpha\}_{\alpha=1}^3)$, we take only one-form ω_1 to study the distribution $\mathsf{E}_1 = \{\xi \mid d\omega_1(\xi, A) = 0, \ ^\forall A \in TX\}$. Let $\mathrm{Aut}_{qc}(X)$ $(= \mathrm{Aut}_{qc}(X, \mathsf{D}, \{J_\alpha\}_{\alpha=1}^3))$ be the group of qc-transformations of X. We have the subgroup of $\mathrm{Aut}_{qc}(X)$ defined by

$$\mathrm{E}(X,\omega) = \{h \in \mathrm{Diff}(X) \mid h^*\omega = a\cdot\omega\cdot\bar{a},\ h_*J_\alpha = \sum_{\beta=1}^{3} a_{\alpha\beta}J_\beta h_*\} \tag{3}$$

for some smooth maps $a : X \to \mathrm{Sp}(1)$ and maps $(a_{\alpha\beta}) \leq \mathrm{SO}(3)$ obtained by the conjugation of a. If E_1 generates a three dimensional Lie subgroup $\mathcal{R} \leq \mathrm{E}(X,\omega)$ consisting of qc-transformations of X. Then we study the quotients of X by subgroups of $\mathcal{R}$.

THEOREM 1.1. *Suppose E_1 generates a three dimensional simply connected Lie group $\mathcal{R}$ lying in $\mathrm{E}(X,\omega)$. Then the quotient manifold $Y = X/\mathcal{R}$ admits a quaternionic almost Hermitian metric g compatible with an induced quaternionic structure $\{\hat{J}_\alpha\}_{\alpha=1}^3$. If $g_\omega = \sum_{i=1}^3 \omega_i\cdot\omega_i + d\omega_1\circ J_1$ is a Riemannian metric on X, then the projection π gives a Riemannian submersion : $\mathcal{R}\to(X, g_\omega) \xrightarrow{\pi} (Y, g)$.*

Moreover if $\mathcal{R}$ is a nontrivial solvable Lie group of the form $\mathbb{R}^2 \rtimes \mathbb{R}$ in which $\mathbb{R}^2$ preserves $\{J_\alpha\}_{\alpha=1}^3$ and $\omega_1|_{T\mathbb{R}^2} = 0$. Put $X_1 = X/\mathbb{R}^2$. Then (ω_1, J_1) induces a strictly pseudoconvex pseudo-Hermitian structure (ω_1', J_1') on X_1 and a Kähler structure $(\Omega_1, \hat{J}_1)$ on Y for which π gives rise to a Sasaki fibering : $\mathbb{R}\to X_1 \xrightarrow{\pi_1} Y$ such that $\Omega_1 = g\circ\hat{J}_1$, $\pi_1^*\Omega_1 = d\omega_1'$ and $\pi_{1*}\circ J_1' = \hat{J}_1\circ\pi_{1*}|_{\ker\,\omega_1'}$.

Here a *quaternionic almost Hermitian metric* g is a Riemannian metric on Y such that $g(J_\alpha\boldsymbol{u}, J_\alpha\boldsymbol{v}) = g(\boldsymbol{u},\boldsymbol{v})$ with respect to a quaternionic structure $\{\hat{J}_\alpha\}_{\alpha=1}^3$. We apply these results to the quaternionic Heisenberg Lie group $\mathcal{M}$ which has the standard qc-structure $(\mathsf{D}_0, \omega_0, \{J_\alpha\}_{\alpha=1}^3)$ where ω_0 is the standard qc-form on $\mathcal{M}$ with $\mathsf{D}_0 = \ker\,\omega_0$.

THEOREM 1.2. *There is a qc-structure $(\mathsf{D}_0, \{\eta_\alpha, J_\alpha\}_{\alpha=1}^3)$ on $\mathcal{M}$ where $\eta = \eta_1 i + \eta_2 j + \eta_3 k$ is a qc-form qc-conformal to ω_0. This induces a quaternionic almost Hermitian structure $(g, \{\hat{J}_\alpha\}_{\alpha=1}^3)$ on the quaternion space $\mathbb{H}^n$. Moreover the quotient $\mathcal{M}/\mathbb{R}^2$ is isomorphic to the Heisenberg nilpotent Lie group $\mathcal{N}_1$ such that*

(i) *There is a strictly pseudoconvex pseudo-Hermitian structure* (η'_1, J'_1) *on* $\mathcal{N}_1$ *where* $\mathbb{R}\to(\mathcal{N}_1,(\eta'_1,J'_1)) \xrightarrow{\pi_1} (\mathbb{H}^n, \{g, \hat{J}_1\})$ *is a Sasaki fibering over the complete Bochner flat Kähler manifold* $(\mathbb{H}^n, \{g, \hat{J}_1\})$.
(ii) *The holomorphic isometry group* $\mathrm{Isom}_h(\mathbb{H}^n,(g,\hat{J}_1))$ *is isomorphic to* $\mathrm{U}(2n)$.
(iii) *The quaternioic almost Hermitian isometry group* $\mathrm{Isom}_{qh}(\mathbb{H}^n,(g,\{\hat{J}_\alpha\}_{\alpha=1}^3))$ *is isomorphic to* $\mathrm{Sp}(n)\cdot S^1$.

In particular $(\mathbb{H}^n,(g,\hat{J}_1))$ *is not holomorphically flat or* $(\mathbb{H}^n,(g,\{\hat{J}_\alpha\}_{\alpha=1}^3))$ *is not flat.*

2. Proper action of $\mathcal{R}$

Put $d\omega_1\circ J_1(\boldsymbol{u},\boldsymbol{v}) = d\omega_1(J_1\boldsymbol{u},\boldsymbol{v})$ ($^\forall\boldsymbol{u},\boldsymbol{v}\in\mathsf{D}$). It follows that $\mathrm{E}(X,\omega)$ leaves $d\omega_1\circ J_1|\mathsf{D}$ invariant. Using the Carnot-Carathéodory metric for ω (see [8]), we note that

LEMMA 2.1. *Any closed subgroup of* $\mathrm{E}(X,\omega)$ *acts properly.*

Fix a form ω_1 from among ω_α's ($\alpha=1,2,3$). Let

$$\mathsf{E}_1 = \{\xi \mid d\omega_1(\xi, A) = 0,\ ^\forall A\in TX\}. \tag{4}$$

Suppose the distribution E_1 generates a 3-dimensional Lie group $\mathcal{R}$ of *qc*-transformations of X. If $\mathcal{R}\le\mathrm{E}(X,\omega)$, then as in the proof of [7, Proposition 2.3], it follows $\mathcal{R}$ is a closed subgroup in $\mathrm{E}(X,\omega)$ and so $\mathcal{R}$ acts properly on X. Furthermore if $\mathcal{R}$ is simply connected, then it acts freely on X. Put $Y = X/\mathcal{R}$ as a $4n$-dimensional smooth manifold.

PROPOSITION 2.2. *Suppose* $\mathcal{R}$ *is a simply connected Lie subgroup of* $\mathrm{E}(X,\omega)$.
(i) *The quotient manifold* $Y = X/\mathcal{R}$ *admits a quaternionic structure* $\{\hat{J}_\alpha\}_{\alpha=1}^3$.
(ii) Y *admits a quaternionic almost Hermitian metric* g *compatible with* $\{\hat{J}_\alpha\}_{\alpha=1}^3$.

PROOF. (i) Let $\mathcal{R}\to X \xrightarrow{\pi} Y$ be the principal bundle. As $\mathcal{R}$ is simply connected, there is a section $s: Y\to X$. Given $y\in Y$ and $\hat{\boldsymbol{u}}_y\in T_yY$, choose $\boldsymbol{u}_{s(y)}\in D_{s(y)}$ such that $\pi_*\boldsymbol{u}_{s(y)} = \hat{\boldsymbol{u}}_y$. Since $\pi_*: D_{s(y)}\to T_yY$ is an isomorphism, $\boldsymbol{u}_{s(y)}$ is uniquely determined. Define a quaternionic structure on Y to be

$$\hat{J}_\alpha(\hat{\boldsymbol{u}}_y) = \pi_*(J_\alpha\boldsymbol{u}_{s(y)})\ \ (\alpha=1,2,3). \tag{5}$$

Thus $\{\hat{J}_\alpha\}_{\alpha=1,2,3}$ gives a quaternionic structure on Y.
Note that this does not imply $\pi: (X,\{J_\alpha\}_{\alpha=1}^3)\to(Y,\{\hat{J}_\alpha\}_{\alpha=1}^3)$ is equivariant.

(ii) Let $\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}} \in T_yY$. Choose $\boldsymbol{u}, \boldsymbol{v} \in D_{\tilde{y}}$ such that $\pi_*\boldsymbol{u} = \hat{\boldsymbol{u}}$, $\pi_*\boldsymbol{v} = \hat{\boldsymbol{v}}$ with $\pi(\tilde{y}) = y$. Define a positive definite 2-form on Y to be

$$g(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}) = d\omega_\alpha(J_\alpha \boldsymbol{u}, \boldsymbol{v}) = d\omega_\beta(J_\beta \boldsymbol{u}, \boldsymbol{v}) = d\omega_\gamma(J_\gamma \boldsymbol{u}, \boldsymbol{v}). \tag{6}$$

Noting $\mathcal{R} \leq \mathrm{E}(X, \omega)$ by the hypothesis, it is easy to check that (6) is well-defined as well as $g(\hat{J}_\alpha \hat{\boldsymbol{u}}_y, \hat{J}_\alpha \hat{\boldsymbol{v}}_y) = g(\hat{\boldsymbol{u}}_y, \hat{\boldsymbol{v}}_y)$. Thus g is a compatible metric on Y with respect to $\{\hat{J}_\alpha\}_{\alpha=1}^3$. □

Put $\pi_*\boldsymbol{u} = \hat{\boldsymbol{u}}, \pi_*\boldsymbol{v} = \hat{\boldsymbol{v}} \in TY$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathsf{D}$. Define Ω_α to be

$$\Omega_\alpha(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}) = g(\hat{\boldsymbol{u}}, \hat{J}_\alpha \hat{\boldsymbol{v}}) = d\omega_\alpha(\boldsymbol{u}, \boldsymbol{v}) \quad (\alpha = 1, 2, 3). \tag{7}$$

COROLLARY 2.3. *Suppose $\mathcal{R} \leq \mathrm{E}(X, \omega)$. Then the following holds:*

$$\begin{aligned} &\Omega_1(\hat{J}_1 \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}) = \Omega_2(\hat{J}_2 \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}) = \Omega_3(\hat{J}_3 \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}) \quad (\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}} \in TY). \\ &\qquad \pi^*\Omega_1 = d\omega_1 \ \text{on } X. \end{aligned} \tag{8}$$

In particular, Ω_1 is a closed 2-form on Y.

Remark 2.4. *Let $\mathcal{R}$ be a 3-dimensional Lie group generated by E_1 consisting of qc-transformations of X, that is $\mathcal{R} \leq \mathrm{Aut}_{qc}(X, \mathsf{D})$. Suppose each $\alpha \in \mathcal{R}$ satisfies $\alpha^*\omega = a \cdot \omega \cdot \bar{a}$ such that $a \in \mathrm{Sp}(1)$. Then it follows $\mathcal{R} \leq \mathrm{E}(X, \omega)$, but not necessarily $\mathcal{R} \leq \mathrm{Isom}(X, g_\omega)$. In fact we determine $\{\xi_\alpha, \alpha = 1, 2, 3\}$ uniquely such as $\omega_\alpha(\xi_\beta) = \delta_{\alpha\beta}$, $d\omega_\alpha(\xi_\alpha, \boldsymbol{v}) = 0$ ($^\forall\, \boldsymbol{v} \in \mathsf{D}$). Then J_α on D naturally extends to the whole X by $J_\alpha(\xi_1) = J_\alpha(\xi_2) = J_\alpha(\xi_3) = 0$. If $g_\omega = \sum_{i=1}^3 \omega_i \cdot \omega_i + d\omega_1 \circ J_1$ is the canonical Riemannian metric on X, then $g_\omega(\xi_\alpha, \boldsymbol{v}) = 0$ by the definition but it is not true whether or not α_* preserves $\{\xi_\alpha, \alpha = 1, 2, 3\}$. In general note neither $\mathcal{R} \leq \mathrm{E}(X, \omega)$ nor $\mathrm{E}(X) \leq \mathrm{Isom}(X, g_\omega)$.*

3. A pseudo-Hermitian structure on $X/\mathcal{R}_1$ and Kähler structure Ω_1 on the complex manifold $(Y, \hat{J}_1)$

Let $\mathsf{E}_1 = \{\xi_1, \xi_2, \xi_3\}$ generate a three dimensional simply connected Lie group $\mathcal{R}$. We assume further

(9)
- (i) $\mathcal{R}$ is solvable such that $[\mathcal{R}, \mathcal{R}] = \mathcal{R}_1$ which induces $\{\xi_2, \xi_3\}$.
- (ii) $\mathcal{R} \leq \mathrm{E}(X, \omega)$.
- (iii) For each $t \in \mathcal{R}$, $t_* J_1 = J_1 t_*$ and $t_*\{J_2, J_3\} = \{J_2, J_3\}$ on D.

Put $X_1 = X/\mathcal{R}_1$. Then $\mathbb{R} = \mathcal{R}/\mathcal{R}_1$ acts properly on X_1. The bundle $\mathcal{R} \to X \xrightarrow{\pi} Y$ induces a principal bundle:

$$\begin{array}{ccccc} \mathbb{R} & \longrightarrow & X_1 & \xrightarrow{\pi_1} & Y \\ || & & || & & \\ \mathcal{R}/\mathcal{R}_1 & \longrightarrow & X/\mathcal{R}_1. & & \end{array} \tag{10}$$

Let D be the *qc*-structure on X. Let $p_1 : X \to X_1$ be the projection such that $\pi_1 \circ p_1 = \pi$. Note that $p_{1*} : \mathsf{D} \to p_{1*}(\mathsf{D})$ is an isomorphism at each point of X_1. Put $\hat{\mathsf{D}} = p_{1*}(\mathsf{D})$ on X_1. If $\mathsf{E}_1 = \{\xi_1, \xi_2, \xi_3\}$, then recall $\omega_1(\xi_1) = 1$, $\omega_1(\xi_2) = \omega_2(\xi_3) = 0$ for which $\{\xi_2, \xi_3\}$ generates $\mathcal{R}_1$. Then $p_1 : X \to X_1$ induces a one-form $\hat{\omega}_1$ on X_1 such that $p_1^* \hat{\omega}_1 = \omega_1$.

LEMMA 3.1. *$\hat{\omega}_1$ is a contact structure on X_1 such that* $\ker \hat{\omega}_1 = \hat{\mathsf{D}}$.

PROOF. As $\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge (d\omega_1)^{2n} \neq 0$ on X and $TX = \mathsf{E}_1 \oplus \mathsf{D}$ with $T\mathcal{R} = \mathsf{E}_1 = \{\xi_1, \xi_2, \xi_3\}$, it follows $p_1^*(\hat{\omega}_1 \wedge (d\hat{\omega}_1)^{2n}) = \omega_1 \wedge (d\omega_1)^{2n}|_{\{\xi_1, \mathsf{D}\}} \neq 0$. Thus, $\hat{\omega}_1 \wedge (d\hat{\omega}_1)^{2n} \neq 0$ on X_1. Since $\ker \omega_1 = \{\xi_2, \xi_3\} \oplus \mathsf{D}$ by the assumption (9), it implies $\ker \hat{\omega}_1 = \hat{\mathsf{D}}$. □

Note that $p_{1*} : \mathsf{D} \to \hat{\mathsf{D}} \subset TX_1$ is an isomorphism at each point of X_1. Since J_1 is an almost complex structure on TX_1 and $t_* J_1 = J_1 t_*$ by the hypothesis, J_1 induces an almost complex structure $\hat{\mathsf{J}}_1$ on $\hat{\mathsf{D}}$ with the commutative diagram:

$$\begin{array}{ccc} \mathsf{D} & \xrightarrow{J_1} & \mathsf{D} \\ {\scriptstyle p_{1*}}\downarrow & & {\scriptstyle p_{1*}}\downarrow \\ \hat{\mathsf{D}} & \xrightarrow{\hat{\mathsf{J}}_1} & \hat{\mathsf{D}}. \end{array}$$

If $\mathsf{D} \otimes \mathbb{C} = \mathsf{D}^{1,0} \oplus \mathsf{D}^{0,1}$ is the eigenspace decomposition for J_1, then p_{1*} maps $\mathsf{D}^{1,0}$ isomorphically onto $\hat{\mathsf{D}}^{1,0}$ where $\hat{\mathsf{D}} \otimes \mathbb{C} = \hat{\mathsf{D}}^{1,0} \oplus \hat{\mathsf{D}}^{0,1}$ is the eigenspace decomposition for $\hat{\mathsf{J}}_1$. We have the following. (See [1].)

LEMMA 3.2. *For any $X, Y \in \mathsf{D}^{1,0}$, there is an element $\boldsymbol{u} \in \mathsf{D} \otimes \mathbb{C}$ such that*

$$[X, Y] = a(\xi_2 - i\xi_3) + \boldsymbol{u} \quad ({}^{\exists}\, a \in \mathbb{R}).$$

PROOF. As $d\omega_2(X, Y) = g(X, J_2 Y)$, $d\omega_2 : \mathsf{D}^{1,0} \times \mathsf{D}^{1,0} \to \mathbb{C}$ is non-degenerate. Put

$d\omega_2(X,Y) = -\dfrac{a}{2}$ for some $a \in \mathbb{R}$. Then $\omega_2([X,Y]) = a$ so that $\omega_2([X,Y]-a\xi_2) = 0$. Put

$$\boldsymbol{v} = [X,Y] - a\xi_2 \in \ker\, \omega_2 \otimes \mathbb{C}. \tag{11}$$

$$\begin{aligned} d\omega_3(X,Y) &= d\omega_3(J_3(-J_3X),Y) = d\omega_2(J_2(-J_3X),Y) \\ &= -d\omega_2(J_1X,Y) = -id\omega_2(X,Y) = \frac{ia}{2}, \end{aligned}$$

it follows $\omega_3([X,Y]) = -ia$. By (11), $\omega_3(\boldsymbol{v}) = -ia$, or $\omega_3(\boldsymbol{v} + ia\xi_3) = 0$. Putting $\boldsymbol{u} = \boldsymbol{v} + ia\xi_3$, it follows

$$\boldsymbol{u} = \boldsymbol{v} + ia\xi_3 \in \ker\, \omega_3 \otimes \mathbb{C}. \tag{12}$$

Thus,

$$[X,Y] = \boldsymbol{v} + a\xi_2 = \boldsymbol{u} - ia\xi_3 + a\xi_2 = \boldsymbol{u} + a(\xi_2 - i\xi_3). \tag{13}$$

As

$$-\omega_1([X,Y]) = 2d\omega_1(X,Y) = 2d\omega_1(J_1X,J_1Y) = 2d\omega_1(iX,iY) = -2d\omega_1(X,Y),$$

note $\omega_1([X,Y]) = 0$. Since $\omega_1([X,Y]) = \omega_1(\boldsymbol{u} + a(\xi_2 - i\xi_3)) = \omega_1(\boldsymbol{u})$ by (13), $\omega_1(\boldsymbol{u}) = 0$. Similarly $\omega_2(\boldsymbol{u}) = \omega_2(\boldsymbol{v} + ia\xi_3) = \omega_2(\boldsymbol{v}) = 0$ by (11). By (12), $\omega_3(\boldsymbol{u}) = 0$. As a consequence, it follows $\boldsymbol{u} \in \mathsf{D} \otimes \mathbb{C}$. □

Since ξ_β generates an element of $\mathcal{R}$, $[\xi_\beta, \mathsf{D}] \subset \mathsf{D}$. Noting $\omega_\alpha(\xi_\beta) = \delta_{\alpha\beta}$, for any $\boldsymbol{v} \in \mathsf{D}$ it follows

$$2d\omega_\alpha(\xi_\beta, \boldsymbol{v}) = \xi_\beta\omega_\alpha(\boldsymbol{v}) - \boldsymbol{v}\omega_\alpha(\xi_\beta) - \omega_\alpha([\xi_\beta, \boldsymbol{v}]) = 0. \tag{14}$$

PROPOSITION 3.3. *For the element $\boldsymbol{u}$ of Lemma 3.2, $J_1\boldsymbol{u} = i\boldsymbol{u}$.*

PROOF. First recall from [9, Proposition 3.10 (p.35)] that for any k-form θ ($k \geq 1$),

$$\mathcal{L}_X(\iota_Y\theta) - \iota_Y\mathcal{L}_X\theta = \iota_{[X,Y]}\theta. \tag{15}$$

For $X \in \mathsf{D}^{1,0}$, $\boldsymbol{v} \in \mathsf{D}$ we obtain that

$$\begin{aligned} \iota_X d\omega_2(\boldsymbol{v}) &= d\omega_2(X, \boldsymbol{v}) = -d\omega_2(J_2(J_2X), \boldsymbol{v}) \\ &= -d\omega_3(J_3(J_2X), \boldsymbol{v}) = d\omega_3(J_1X, \boldsymbol{v}) \\ &= id\omega_3(X, \boldsymbol{v}) = i\iota_X d\omega_3(\boldsymbol{v}). \end{aligned} \tag{16}$$

It follows by (14) that $\iota_X d\omega_\alpha(\xi_\beta) = d\omega_\alpha(X, \xi_\beta) = 0$ for $X \in \mathsf{D} \otimes \mathbb{C}$ and ξ_β $(\beta = 1, 2, 3)$. Since $TX = \mathsf{E}_1 \oplus \mathsf{D}$, (16) implies

$$\iota_X d\omega_2 = i\iota_X d\omega_3 \quad \text{on } TX. \tag{17}$$

Noting $\mathcal{L}_X d\omega_2 = (d\iota_X + \iota_X d)d\omega_2 = d\iota_X d\omega_2$, substitute (17) so that

$$\mathcal{L}_X d\omega_2 = d(i\iota_X d\omega_3) = id\iota_X d\omega_3 = i(\mathcal{L}_X - \iota_X d)d\omega_3 = i\mathcal{L}_X d\omega_3. \tag{18}$$

Applying $Y \in \mathsf{D}^{1,0}$, we have

$$\iota_Y \mathcal{L}_X d\omega_2 = i\iota_Y \mathcal{L}_X d\omega_3. \tag{19}$$

For $Y \in \mathsf{D}^{1,0}$, it follows similarly

$$\iota_Y d\omega_2 = i\iota_Y d\omega_3 \quad \text{on } TX. \tag{20}$$

Apply $\theta = d\omega_3$ to (15).

$$\begin{aligned} \mathcal{L}_X(\iota_Y d\omega_2) &= \mathcal{L}_X(i\iota_Y d\omega_3) = i\mathcal{L}_X(\iota_Y d\omega_3) \ ((20)) \\ &= i(\iota_Y \mathcal{L}_X d\omega_3 + \iota_{[X,Y]} d\omega_3) = \iota_Y \mathcal{L}_X d\omega_2 + i\iota_{[X,Y]} d\omega_3 \quad ((19)) \end{aligned} \tag{21}$$

Comparing to (15) with $\theta = d\omega_2$,

$$i\iota_{[X,Y]} d\omega_3 = \iota_{[X,Y]} d\omega_2 \text{ on } TX. \tag{22}$$

By Lemma 3.2 with $[X, Y] = a(\xi_2 - i\xi_3) + \boldsymbol{u}$, for any $\boldsymbol{w} \in \mathsf{D}$,

$$\begin{aligned} id\omega_3([X, Y], \boldsymbol{w}) &= d\omega_2([X, Y], \boldsymbol{w}) \\ id\omega_3(a(\xi_2 - i\xi_3) + \boldsymbol{u}, \boldsymbol{w}) &= d\omega_2(a(\xi_2 - i\xi_3) + \boldsymbol{u}, \boldsymbol{w}) \\ id\omega_3(\boldsymbol{u}, \boldsymbol{w}) &= d\omega_2(\boldsymbol{u}, \boldsymbol{w}). \end{aligned} \tag{23}$$

On the other hand, putting $\boldsymbol{w}=-J_2\boldsymbol{v}$,

$$
\begin{aligned}
id\omega_3(\boldsymbol{u},\boldsymbol{w}) &= id\omega_3(\boldsymbol{u},-J_2\boldsymbol{v}) = id\omega_3(J_2(\boldsymbol{v}),\boldsymbol{u})\\
&= id\omega_3(J_3J_1(\boldsymbol{v}),\boldsymbol{u}) = id\omega_1(J_1(J_1\boldsymbol{v}),\boldsymbol{u})\\
&= -id\omega_1(\boldsymbol{v},\boldsymbol{u}) = -d\omega_1(\boldsymbol{v},i\boldsymbol{u})\\
d\omega_2(\boldsymbol{u},\boldsymbol{w}) &= d\omega_2(\boldsymbol{u},-J_2(\boldsymbol{v})) = d\omega_2(J_2(\boldsymbol{v}),\boldsymbol{u})\\
&= d\omega_1(J_1(\boldsymbol{v}),\boldsymbol{u}) = -d\omega_1(\boldsymbol{v},J_1\boldsymbol{u}),
\end{aligned}
\tag{24}
$$

thus $d\omega_1(\boldsymbol{v},i\boldsymbol{u}) = d\omega_1(\boldsymbol{v},J_1\boldsymbol{u})$. Since $d\omega_1(\boldsymbol{v},i\boldsymbol{u}-J_1\boldsymbol{u})=0$ for any $\boldsymbol{v}\in\mathsf{D}$, the non-degeneracy of $d\omega_1$ implies $i\boldsymbol{u}-J_1\boldsymbol{u}=0$. □

Theorem 3.4. *Suppose* (9) *is satisfied, that is* $\mathcal{R}\le \mathrm{E}(X,\omega)$, $t_*J_1=J_1t_*$, $t_*\{J_2,J_3\}=\{J_2,J_3\}$ *on* D *for each* $t\in\mathcal{R}$. *Then*
(i) $(\hat\omega_1,\hat{\mathsf{J}}_1)$ *is a strictly pseudoconvex pseudo-Hermitian structure on* X_1 *for which* $\mathbb{R}\to X_1\xrightarrow{\pi_1}Y$ *gives rise to a pseudo-Hermitian fibering such that* $\pi_1^*\Omega_1=d\hat\omega_1$.
(ii) $\hat J_1$ *is a complex structure on* Y. *In particular,* $(Y,(\Omega_1,\hat J_1))$ *is a Kähler manifold.*

Proof. (i) It is sufficient to show that $\hat{\mathsf{J}}_1$ is integrable on $\hat{\mathsf{D}}$. Let $\hat{\mathsf{D}}\otimes\mathbb{C}=\hat{\mathsf{D}}^{1,0}\oplus\hat{\mathsf{D}}^{0,1}$ be the eigenspace decomposition for $\hat{\mathsf{J}}_1$. As the isomorphism $p_{1*}:\mathsf{D}\to\hat{\mathsf{D}}$ satisfies $p_{1*}\circ J_1=\hat{\mathsf{J}}_1\circ p_{1*}$, note that $p_{1*}\mathsf{D}^{1,0}=\hat{\mathsf{D}}^{1,0}$. Since $[X,Y]=a(\xi_2-i\xi_3)+\boldsymbol{u}$ ($^\exists\, a\in\mathbb{R}$), it follows $p_{1*}([X,Y])=[p_{1*}X,p_{1*}Y]=p_{1*}\boldsymbol{u}$. Put $p_{1*}X=\hat X$, $p_{1*}Y=\hat Y\in\hat{\mathsf{D}}^{1,0}$ and $\hat{\boldsymbol{u}}=p_{1*}\boldsymbol{u}\in\hat{\mathsf{D}}$. We have $[\hat X,\hat Y]=\hat{\boldsymbol{u}}$. As $J_1\boldsymbol{u}=i\boldsymbol{u}$ by Proposition 3.3, we obtain $\hat{\mathsf{J}}_1([\hat X,\hat Y])=i[\hat X,\hat Y]$. Hence $[\hat X,\hat Y]\in\mathsf{D}^{1,0}$. $\hat{\mathsf{J}}_1$ is integrable.

Let $\mathbb{R}\to X_1\xrightarrow{\pi_1}Y$ be the principal bundle. Here $\mathbb{R}$ induces the vector field $\hat\xi_1$ $(=p_{1*}\xi_1)$ such that $\hat\omega_1(\hat\xi_1)=1$. As $\pi^*\Omega_1=d\omega_1$ (cf. (8)), it is easy to see that

$$
\pi_1^*\Omega_1=d\hat\omega_1. \tag{25}
$$

(ii) Since $\pi_{1*}:\hat{\mathsf{D}}\to TY$ is an isomorphism at each point, recall from (5) that there is an almost complex structure $\hat J_1$ on Y which satisfies $\pi_{1*}\hat{\mathsf{J}}_1=\hat J_1\pi_{1*}$. Then we check that $\hat J_1$ is a complex structure on Y. Let $\pi_{1*}(\hat{\mathsf{D}}^{0,1})=TY^{1,0}$ where $TY\otimes\mathbb{C}=TY^{1,0}\oplus TY^{0,1}$ with respect to $\hat J_1$. If we note $[\hat{\mathsf{D}}^{1,0},\hat{\mathsf{D}}^{1,0}]\subset\hat{\mathsf{D}}^{1,0}$ by Theorem 3.4, then $\pi_{1*}(\hat{\mathsf{J}}_1[\hat{\boldsymbol{u}},\hat{\boldsymbol{v}}])=\pi_{1*}(i[\hat{\boldsymbol{u}},\hat{\boldsymbol{v}}])=i[\pi_{1*}\hat{\boldsymbol{u}},\pi_{1*}\hat{\boldsymbol{v}}]$ for $\hat{\boldsymbol{u}},\hat{\boldsymbol{v}}\in\hat{\mathsf{D}}^{1,0}$. Since $\pi_{1*}(\hat{\mathsf{J}}_1[\hat{\boldsymbol{u}},\hat{\boldsymbol{v}}])=\hat J_1\pi_{1*}[\hat{\boldsymbol{u}},\hat{\boldsymbol{v}}]=\hat J_1[\pi_{1*}\hat{\boldsymbol{u}},\pi_{1*}\hat{\boldsymbol{v}}]$, it follows $\hat J_1[\pi_{1*}\hat{\boldsymbol{u}},\pi_{1*}\hat{\boldsymbol{v}}]=i[\pi_{1*}\hat{\boldsymbol{u}},\pi_{1*}\hat{\boldsymbol{v}}]$. Noting $\pi_{1*}\hat{\boldsymbol{u}},\pi_{1*}\hat{\boldsymbol{v}}\in TY^{1,0}$, $\hat J_1$ is integrable on Y. As $\Omega_1(\hat J_1\hat{\boldsymbol{u}},\hat J_1\hat{\boldsymbol{v}})=\Omega_1(\hat{\boldsymbol{u}},\hat{\boldsymbol{v}})$ from (25), $(\Omega_1,\hat J_1)$ is a Kähler structure on Y with $g=\Omega_1\circ\hat J_1$. □

This proves Theorem 1.1 of Introduction.

4. Quaternionic contact structure on $\mathcal{M}$

Let $\mathbb{H}^n$ be the $4n$-dimensional vector space over the quaternion field $\mathbb{H}$ and $\mathrm{Im}\,\mathbb{H} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ identifies the three dimensional vector space $\mathbb{R}^3$. The quaternionic Heisenberg Lie group $\mathcal{M}$ is the product $\mathbb{R}^3 \times \mathbb{H}^n$ with group law: $(t, z) \cdot (s, w) = (t + s - \mathrm{Im}\langle z, w\rangle, z + w)$ where $\langle z, w\rangle = {}^t\bar{z}w$ is the Hermitian inner product on $\mathbb{H}^n$. ($(t, s) \in \mathbb{R}^3 = \mathrm{Im}\,\mathbb{H}$, and $z, w \in \mathbb{H}^n$.) $\mathcal{M}$ is a nilpotent Lie group whose center is $[\mathcal{M}, \mathcal{M}] = \mathbb{R}^3$ consisting of elements $(t, 0)$. See [1] for the details. Let ω_0 be an $\mathrm{Im}\,\mathbb{H}$-valued one-form on $\mathcal{M}$ defined by

$$\omega_0 = dt_1 i + dt_2 j + dt_3 k + \mathrm{Im}\langle z, dz\rangle. \tag{26}$$

Put $\omega_0 = \omega_1 i + \omega_2 j + \omega_3 k$. The codimension 3-subbundle $\mathsf{D}_0 = \bigcap_{i=1}^{3} \ker\omega_i = \ker\omega_0$ on $\mathcal{M}$ together with the hypercomplex structure $\{J_1, J_2, J_3\}$ is called the *standard qc-structure* on $\mathcal{M}$. Alternatively, for this hypercomplex structure, if $\pi : \mathcal{M} \to \mathbb{H}^n$ is the canonical projection (homomorphism), then $\pi_* : \mathsf{D} \to T\mathbb{H}^n$ is an isomorphism at each point for which each J_α on D is defined by the commutative rule:

$$\pi_* \circ J_\alpha = \hat{J}_\alpha \circ \pi_* \tag{27}$$

where $\{\hat{J}_\alpha, \alpha = 1, 2, 3\}$ is the *standard quaternionic structure* $\{i, j, k\}$ on $\mathbb{H}^n$, that is $\hat{J}_1 z = z\bar{i}$, $\hat{J}_2 z = z\bar{j}$, $\hat{J}_3 z = z\bar{k}$ respectively. By calculation, D satisfies $[\mathsf{D}, \mathsf{D}] \equiv \langle d/dt_1, d/dt_2, d/dt_3\rangle \bmod \ker\,\omega_0$.

Let $\mathrm{Sim}(\mathcal{M}) = \mathcal{M} \rtimes (\mathrm{Sp}(n)\cdot\mathrm{Sp}(1) \times \mathbb{R}^+)$ be the full group of *qc*-transformations of $\mathcal{M}$ (cf. [1]). If $h = \big((t, u), A \cdot \alpha\lambda\big)$ ($\lambda \in \mathbb{R}^+, \alpha \in \mathrm{Sp}(1)$) is an element of $\mathrm{Sim}(\mathcal{M})$, then h acts on $\mathcal{M}$ as

$$hp = h(s, z) = (t + \lambda^2\alpha s\bar{\alpha} - \mathrm{Im}\langle u, \lambda A z\bar{\alpha}\rangle, u + \lambda A z\bar{\alpha}) \quad ({}^\forall\, p = (s, z) \in \mathcal{M}). \tag{28}$$

Then it follows

$$h^*\omega_0 = \lambda^2 \cdot \alpha\omega_0\bar{\alpha}. \tag{29}$$

Thus every element of $\mathrm{E}(\mathcal{M})$ leaves $\mathsf{D}_0 = \ker\,\omega_0$ invariant in general. On the other hand, the euclidean group of $\mathcal{M}$ is denoted by $\mathrm{E}(\mathcal{M}) = \mathrm{E}(\mathcal{M}, \omega_0) = \mathcal{M} \rtimes \big(\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)\big)$ which is a normal subgroup of $\mathrm{Sim}(\mathcal{M})$. By (29), each element $h = \big((t, v), (A \cdot \alpha)\big) \in \mathrm{E}(\mathcal{M})$ satisfies

$$h^*\omega_0 = \alpha\omega_0\bar{\alpha}. \tag{30}$$

Remark 4.1. *In particular, if $h \in \mathcal{M} \rtimes \mathrm{Sp}(n)$, then $h^*\omega_0 = \omega_0$. For the Riemannian metric $g_{\omega_0} = \sum_{i=1}^3 \omega_i \cdot \omega_i + d\omega_1 \circ J_1$ on $\mathcal{M}$, the identity component* $\mathrm{Isom}(\mathcal{M}, g_{\omega_0})^0$ *coincides with* $\mathrm{E}(\mathcal{M})$. *Note that this metric is neither a* 3*-Sasaki metric nor defines a quaternionic CR structure on $\mathcal{M}$.*

5. Conformal change of ω_0

Fix a number $a \geq 0$. For each $i = 1, 2, 3$, define the elements $\rho(t_1) = \big(((t_1, 0, 0), 0),\ I \cdot e^{-iat_1})\big), \rho(t_2) = \big(((0, t_2, 0), 0),\ I)\big),\ \rho(t_3) = \big(((0, 0, t_3), 0),\ I)\big)$ in $(\mathbb{R}^3, 0) \rtimes S^1 \leq \mathrm{E}(\mathcal{M})$. More precisely, the action of $\rho(t_i)$ on $\mathcal{M}$ has the form:

$$
\begin{aligned}
&\rho(t_1)\big((s_1 i, s_2 j, s_3 k), z\big) = \big((t_1 + s_1)i, e^{-iat_1}(s_2 j, s_3 k)e^{iat_1}), ze^{iat_1}\big),\\
&\rho(t_2)\big((s_1 i, s_2 j, s_3 k), z\big) = \big((s_1 i, (s_2 + t_2)j, s_3 k), z\big),\\
&\rho(t_3)\big((s_1 i, s_2 j, s_3 k), z\big) = \big((s_1 i, s_2 j, (s_3 + t_3)k), z\big)
\end{aligned}
\tag{31}
$$

for $\big((s_1 i, s_2 j, s_3 k), z\big) \in \mathcal{M} = \mathbb{R}^3 \times \mathbb{H}^n$. As $\rho(t_1)$ normalizes the vector space $\mathbb{R}^2 = \langle \rho(t_2), \rho(t_3) \rangle$, $\langle \rho(t_1), \rho(t_2), \rho(t_3) \rangle$ forms a 3-dimensional solvable Lie group $\mathcal{R}_1$ such that

$$
\mathcal{R}_1 = \mathbb{R}^2 \rtimes \{\rho(t_1)\} \leq \mathbb{R}^3 \rtimes (\{1\} \times S^1). \tag{32}
$$

Take $x_1 + ix_2 + jx_3 + kx_4$ as the standard real coordinate of $z_1 \in \mathbb{H}$ and identify $\mathbb{H}^n$ with $\mathbb{R}^{4n}$. Let $\langle \dfrac{d}{dt_1}, \dfrac{d}{dt_2}, \dfrac{d}{dt_3} \rangle$ be the standard basis of $\mathbb{R}^3 = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ also. From (31) it follows

$$
\frac{d(e^{-iat_1}(s_2 j, s_3 k)e^{iat_1})}{dt_1}\Big|_{t_1=0} = 2as_3\frac{d}{dt_2} - 2as_2\frac{d}{dt_3}.
$$

Then $\mathcal{R}_1$ induces three vector fields $\langle \xi_1, \dfrac{d}{dt_2}, \dfrac{d}{dt_3} \rangle$ at $(s, z) \in \mathcal{M}$ such that

$$
\begin{aligned}
\xi_1 = &\frac{d}{dt_1} + 2a(s_3\frac{d}{dt_2} - s_2\frac{d}{dt_3})\\
&+ \sum_{k=1}^{n} a(x_{4k-3}\frac{d}{dx_{4k-2}} - x_{4k-2}\frac{d}{dx_{4k-3}} + x_{4k}\frac{d}{dx_{4k-1}} - x_{4k-1}\frac{d}{dx_{4k}}).
\end{aligned}
\tag{33}
$$

Since $\omega_0 = \omega_1 i + \omega_2 j + \omega_3 k$, using (26) it follows

$$
\omega_1(\xi_1) = 1 + a(|z_1|^2 + \cdots + |z_n|^2). \tag{34}
$$

As we assumed $a \geq 0$, note $\omega_1(\xi_1) \geq 1$. We introduce new one-forms on $\mathcal{M}$:

$$\eta_1 = \frac{1}{\omega_1(\xi_1)}\omega_1,\ \eta_2 = \frac{1}{\omega_1(\xi_1)}\omega_2,\ \eta_3 = \frac{1}{\omega_1(\xi_1)}\omega_3. \tag{35}$$

PROPOSITION 5.1. *Then the following hold.*

$$\begin{aligned} &\eta_1(\xi_1) = 1,\ \eta_1(\frac{d}{dt_2}) = \eta_1(\frac{d}{dt_3}) = 0,\\ &d\eta_1(\xi_1, X) = d\eta_1(\frac{d}{dt_2}, X) = d\eta_1(\frac{d}{dt_3}, X) = 0\ (^\forall\, X \in T\mathcal{M}),\\ &\mathrm{E}_1 = \{\xi \mid d\eta_1(\xi, X) = 0\} = \{\xi_1,\ \frac{d}{dt_2},\ \frac{d}{dt_3}\}. \end{aligned} \tag{36}$$

In addition, $\eta_2(\frac{d}{dt_2}) = \eta_3(\frac{d}{dt_3}) = \frac{1}{\omega_1(\xi_1)} > 0.$

PROOF. It follows from (30) that $\rho(t_i)_*\mathsf{D} = \mathsf{D}$ for each $\rho(t_i) \in \mathcal{R}$ $(i = 1, 2, 3)$. Since $\eta_1(\xi_1) = 1$ from (35), calculate $2d\eta_1(\xi_1, \boldsymbol{u}) = -\eta_1([\xi_1, \boldsymbol{u}]) = 0$ $(^\forall\, \boldsymbol{u} \in \mathsf{D})$.

For $\alpha = 2, 3$, $\omega_1(\frac{d}{dt_\alpha}) = 0$. As $\langle\rho(t_1)\rangle$ normalizes $\mathbb{R}^2 = \langle\rho(t_2), \rho(t_3)\rangle$, it follows $[\xi_1, \frac{d}{dt_\alpha}] \in \langle\frac{d}{dt_2}, \frac{d}{dt_3}\rangle$ so that $d\eta_1(\xi_1, \frac{d}{dt_\alpha}) = 0$. This shows that $d\eta_1(\xi_1, X) = 0$ $(^\forall\, X \in T\mathcal{M})$. Similarly at $p = ((t_1, t_2, t_3), z_1, \ldots, z_n) \in \mathcal{M}$,

$$2d\eta_1(\frac{d}{dt_\alpha}, \frac{d}{dt_1}) = \frac{d}{dt_\alpha}(\frac{1}{1 + a\sum_{i=1}^n |z_i|^2}) - \eta_1([\frac{d}{dt_\alpha}, \frac{d}{dt_1}]) = 0,$$

also $2d\eta_1(\frac{d}{dt_\alpha}, \boldsymbol{u}) = -\eta_1([\frac{d}{dt_\alpha}, \boldsymbol{u}]) = 0$ $(^\forall\, \boldsymbol{u} \in \mathsf{D})$. It follows $d\eta_1(\frac{d}{dt_\alpha}, X) = 0$ $(^\forall\, X \in T\mathcal{M})$. We obtain $\mathrm{E}_1 = \{\xi_1,\ \frac{d}{dt_2},\ \frac{d}{dt_3}\}$. □

Remark that $d\eta_2(\frac{d}{dt_2}, X)$, $d\eta_2(\frac{d}{dt_3}, X)$ are not zero in general. Since $d\eta_\alpha$ is conformal to $d\omega_\alpha$ on D_0,

LEMMA 5.2. *The hypercomplex structure obtained from* $\{\eta_\alpha\}$ *is the same as* $\{J_\alpha\}_{\alpha=1}^3$ *of* $\{\omega_\alpha\}_{\alpha=1,2,3}$.

By (2), we have $d\eta_\alpha(J_\alpha X, J_\alpha Y) = d\eta_\alpha(X, Y)$ $(\alpha = 1, 2, 3)$. Using (27), (31), note that

$$\rho(t_1)_* J_1 = J_1 \rho(t_1)_*,$$

$$(37)\qquad \rho(t_1)_* \begin{bmatrix} J_2 \\ J_3 \end{bmatrix} = \begin{bmatrix} \cos 2at_1 & -\sin 2at_1 \\ \sin 2at_1 & \cos 2at_1 \end{bmatrix} \begin{bmatrix} J_2 \\ J_3 \end{bmatrix} \rho(t_1)_*.$$

$$\rho(t_2)_* J_\alpha = J_\alpha \rho(t_2)_*,\ \rho(t_3)_* J_\alpha = J_\alpha \rho(t_3)_* \ (\alpha = 1, 2, 3).$$

As above each element of $\mathcal{R}_1 = \mathbb{R}^2 \rtimes \rho(\mathbb{R})$ satisfies (9).

Letting $\eta = \eta_1 i + \eta_2 j + \eta_3 k$ at $p = (s, z) = ((s_1, s_2, s_3), z) \in \mathcal{M}$, it follows $\eta = f \cdot \omega_0$ such that $f(p) = \dfrac{1}{\omega_1((\xi_1)_p)} = \dfrac{1}{1 + a|z|^2}$ where $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$ (cf. (34)). As η is conformal to ω_0, $\mathrm{E}(\mathcal{M}, \eta) \le \mathrm{Sim}(\mathcal{M})$.

PROPOSITION 5.3. *If* $h \in \mathrm{E}(\mathcal{M}, \eta)$, *then* $h \in \mathbb{R}^3 \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1))$. *As a consequence,* $\mathrm{E}(\mathcal{M}, \eta) = \mathbb{R}^3 \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1))$. *In addtion,* $\mathcal{R}_1 \le \mathrm{E}(\mathcal{M}, \eta)$.

PROOF. For $h = \big((t, u), A \cdot \alpha\lambda\big)$ $(\lambda \in \mathbb{R}^+, \alpha \in \mathrm{Sp}(1))$, (28) implies

$$(38)\qquad h^* f(p) = f(hp) = \frac{1}{\omega_1((\xi_1)_{hp})} = \frac{1}{1 + a|u + \lambda A z \bar{\alpha}|^2}.$$

Since $h \in \mathrm{E}(\mathcal{M}, \eta)$, as in (30), $h^*\eta = \beta\eta\bar{\beta}$ for some $\beta \in \mathrm{Sp}(1)$. Noting $\eta = f \cdot \omega_0$, the action of $\mathrm{Sim}(\mathcal{M})$ shows that $h^*\eta = h^* f \cdot h^*\omega_0 = h^* f \cdot \lambda^2 \alpha\omega_0\bar{\alpha}$. Then the equality $\beta\eta\bar{\beta} = h^* f \cdot \lambda^2 \alpha\omega_0\bar{\alpha}$ implies

$$|\eta| = |\beta\eta\bar{\beta}| = |h^* f \cdot \lambda^2 \alpha\omega_0\bar{\alpha}| = h^* f \cdot \lambda^2 |\omega_0|.$$

(Here $|t_1 i + t_2 j + t_3 k| = \sqrt{t_1^2 + t_2^2 + t_3^2}$ is the norm.) Evaluated at p, it follows $|\eta| = f(p)|\omega_0|$, that is $f(p)|\omega_0| = h^* f(p) \cdot \lambda^2 |\omega_0|$. Thus $f(p) = f(hp)\lambda^2$ which implies

$$(39)\qquad \frac{1}{1 + a|z|^2} = \frac{\lambda^2}{1 + a|u + \lambda A z \bar{\alpha}|^2}$$

by (38). Take $z_0 = \dfrac{1}{1 - \lambda} A^{-1} u \alpha \in \mathbb{H}^n$, that is $A z_0 \bar{\alpha} = u + \lambda A z_0 \bar{\alpha}$. Substitute this into (39):

$$\frac{1}{1 + a|z_0|^2} = \frac{\lambda^2}{1 + a|A z_0 \bar{\alpha}|^2} = \frac{\lambda^2}{1 + a|z_0|^2},$$

thus $\lambda = 1$ $(\lambda > 0)$. Then $|z|^2 = |u + A z \bar{\alpha}|^2$ from (39). Taking $z = 0$ shows $u = 0$.

So $h \in \mathbb{R}^3 \rtimes (\mathrm{Sp}(n)\cdot \mathrm{Sp}(1))$. Conversely if $h = ((t,0), A\cdot\alpha) \in \mathbb{R}^3 \rtimes (\mathrm{Sp}(n)\cdot \mathrm{Sp}(1))$, then $h^*\omega_0 = \alpha\omega_0\bar{\alpha}$. Then $h^*f(p) = f(hp) = \dfrac{1}{1+|Az\bar{\alpha}|^2} = f(p)$. It follows $h^*\eta = h^*f \cdot h^*\omega_0 = f\cdot \alpha\omega_0\bar{\alpha} = \alpha\eta\bar{\alpha}$. Obviously $h_*J_\alpha = \displaystyle\sum_{\beta=1}^{3} a_{\alpha\beta}J_\beta$. Hence $h \in \mathrm{E}(\mathcal{M},\eta)$ by the definition. □

We apply Theorem 3.4 to obtain

Theorem 5.4. (I) *There exists a quaternionic almost Hermitian structure* $(g, \{\hat{J}_\alpha\}_{\alpha=1}^3)$ *on the quaternion vector space* $\mathbb{H}^n$ *such that* $\mathcal{R}{\to}(\mathcal{M}, g_\omega) \xrightarrow{\pi} (\mathbb{H}^n, g)$ *is a Riemannian submersion.*
(II) *There is a strictly pseudoconvex pseudo-Hermitian structure* $(\hat{\eta}_1, J_1')$ *on the nilpotent Lie group* $\mathcal{N}_1 = \mathcal{M}/\mathbb{R}^2$ *such that*

$$\rho(\mathbb{R}) \longrightarrow (\mathcal{N}_1, (\hat{\eta}_1, J_1')) \xrightarrow{\pi_1} (\mathbb{H}^n, (g, \hat{J}_1)) \tag{40}$$

gives rise to a pseudo-Hermitian (Sasaki) fibering over the complete Kähler manifold $(\mathbb{H}^n, g)$ *such that* $\pi_1^*\Omega_1 = d\hat{\eta}_1$.

Proof. Let $\rho(t_1) = \big(((t_1,0,0),0),\, I\cdot e^{-iat_1})\big)$ where $\mathcal{R} = \mathbb{R}^2 \rtimes \rho(\mathbb{R}) \leq \mathbb{R}^3 \rtimes (1\cdot S^1)$ from (32). Then $\mathcal{R}/\mathbb{R}^2 = \rho(\mathbb{R})$ such that $\rho(t_1) = \big((t_1,0),\, e^{-iat_1})\big) \in \mathbb{R}\times S^1$. By Theorem 3.4, note that $Y = \mathcal{N}_1/\rho(\mathbb{R}) = \mathcal{M}/\mathcal{R} = \mathbb{H}^n$. (I) follows from Proposition 2.2. $(\mathbb{H}^n, g, \hat{J}_1)$ is a complete Kähler manifold by [6, Proposition 3.5]. □

This proves (i) of Theorem 1.2 of Introduction.

6. Isometry group of $\mathbb{H}^n$

Let $\rho : \mathbb{R}\to\mathbb{R}^3 \rtimes S^1$ be the representation as in (31) where $\mathbb{R}^3 \rtimes S^1 \leq \mathrm{E}(\mathcal{M},\eta) = \mathbb{R}^3 \rtimes (\mathrm{Sp}(n)\cdot\mathrm{Sp}(1))$ (cf. Proposition 5.3). If $C_{\mathrm{E}(\mathcal{M},\eta)}(\rho(\mathbb{R}))$ is the centralizer of $\rho(\mathbb{R})$, then it follows $C_{\mathrm{E}(\mathcal{M},\eta)}(\rho(\mathbb{R})) = \mathbb{R}\rtimes(\mathrm{Sp}(n)\cdot S^1)$ where $\mathbb{R} = (\mathbb{R},0,0)$ and $S^1 = \{e^{i\theta}\}$. Then the equivariant Riemannian submersion induces the equivariant pseudo-Hermitian (Sasaki) fibration of (II) of Theorem 5.4:

$$\begin{array}{ccccc}
\mathcal{R} & \longrightarrow & (\mathbb{R}^3 \rtimes \mathrm{Sp}(n)\cdot S^1, \mathcal{M}) & \xrightarrow{\pi} & (\mathrm{Sp}(n)\cdot S^1, \mathbb{H}^n) \\
\big\downarrow{/\mathbb{R}^2} & & \big\downarrow{/\mathbb{R}^2} & & \| \\
\rho(\mathbb{R}) & \longrightarrow & (\mathbb{R}\rtimes(\mathrm{Sp}(n)\cdot S^1), \mathcal{N}_1) & \xrightarrow{\pi_1} & (\mathrm{Sp}(n)\cdot S^1, \mathbb{H}^n).
\end{array} \tag{41}$$

Denote $\mathrm{Isom}_{qh}(\mathbb{H}^n,(g,\{\hat{J}_\alpha\}_{\alpha=1}^3))$ the quaternionic almost Hermitian isometry group and $\mathrm{Isom}_h(\mathbb{H}^n,(g,\hat{J}_1))$ the holomorphic isometry group respectively. Note from (41)

$$\mathrm{Sp}(n)\cdot S^1 \leq \mathrm{Isom}_{qh}(\mathbb{H}^n,(g,\{\hat{J}_\alpha\}_{\alpha=1}^3)). \tag{42}$$

Let $(\mathcal{N},\hat{\omega}_0,J_0)$ be the standard pseudo-Hermitian structure on the $2n+1$-dimensional Heisenberg nilpotent Lie group. Moreover, $\mathcal{R}_1 \leq \mathrm{E}(\mathcal{M},\eta)$ by (31).group $\mathcal{N}$ such that $\mathrm{Psh}(\mathcal{N},(\hat{\omega}_0,J_0)) = \mathcal{N}\rtimes \mathrm{U}(2n)$ is the group of pseudo-Hermitian transformations of $\mathcal{N}$. Consider the representation $\rho_1:\mathbb{R}\rightarrow\mathcal{N}\rtimes\mathrm{U}(2n)$ defined by

$$\rho_1(t_1)=\big((t_1,0),\mathrm{diag}\,(e^{iat_1})\big)\in(\mathbb{R},0)\times\mathrm{U}(2n). \tag{43}$$

As in [7], there is the commutative diagram:

$$\begin{array}{ccc}
\rho(\mathbb{R}) & \xrightarrow{\ \mu\ } & \rho_1(\mathbb{R})\\
\downarrow & & \downarrow\\
\mathcal{N}_1 & \xrightarrow{\ \varphi\ } & \mathcal{N}\\
{\scriptstyle\hat{\pi}}\downarrow & & {\scriptstyle p}\downarrow\\
\mathbb{H}^n & \xrightarrow{\ \hat{\varphi}\ } & \mathbb{C}^{2n},
\end{array} \tag{44}$$

satisfying that

(1) $\varphi:\mathcal{N}_1=\mathcal{M}/\mathbb{R}^2=\mathbb{R}\times\mathbb{H}^n\rightarrow\mathcal{N}=\mathbb{R}\times\mathbb{C}^{2n}$ is a Lie group isomorphism between nilpotent Lie groups defined by $\varphi(s,(u+vj))=(s,(u,\bar{v}))$ where $z=u+vj\in\mathbb{H}^n$ $(u,v\in\mathbb{C}^n)$. Note that $\varphi(\rho(t_1)(s,u+vj))=\rho_1(t_1)\varphi(s,u+vj)$.

(2) $\hat{\varphi}:\mathbb{H}^n\rightarrow\mathbb{C}^{2n}$ is the induced diffeomorphism : $\hat{\varphi}(u+wj)=(u,\bar{v})$. If $\rho_1(\mathbb{R})$ induces the vector field ξ_1 on $\mathcal{N}$, then it follows $\hat{\omega}_0(\xi_1)=1+a(|u|^2+|v|^2)$. Putting $\hat{\omega}_1=\dfrac{1}{\hat{\omega}_0(\xi_1)}\hat{\omega}_0$, ξ_1 turns to the Reeb field of $\hat{\omega}_1$.

(3) It follows $\varphi^*\hat{\omega}_1=\hat{\eta}_1$ and $\varphi_*\circ J_1'=J_0'\circ\varphi_*$ where J_0' is the anti-complex structure of J_0 on $\mathcal{N}$. As a consequence, φ becomes a pseudo-Hermtian diffeomorphism of $(\mathcal{N}_1,(\hat{\eta}_1,J_1'))$ onto a strictly pseudoconvex pseudo-Hermitian structure $(\mathcal{N},(\hat{\omega}_1,J_0'))$.

Since $\rho_1(\mathbb{R})$ induces the Reeb field of $\hat{\omega}_1$ on $\mathcal{N}$, the uniqueness of Reeb field implies that $\mathrm{Psh}(\mathcal{N},(\hat{\omega}_1,J_0'))$ centralizes $\rho_1(\mathbb{R})$ and so $C_{\mathrm{Psh}(\mathcal{N})}(\rho_1(\mathbb{R}))=\mathrm{Psh}(\mathcal{N},(\hat{\omega}_1,J_0'))$.

As the centralizer of $\rho_1(\mathbb{R})$ in $\mathrm{Psh}(\mathcal{N})$ is $\mathbb{R}\times \mathrm{U}(2n)$ from (43),

$$\mathrm{Psh}(\mathcal{N},(\hat{\omega}_1, J_0')) = \mathbb{R}\times \mathrm{U}(2n). \tag{45}$$

Recall from [2, Proposition 3.4] that there is an exact sequence:

$$1\longrightarrow \rho_1(\mathbb{R}) \longrightarrow \mathrm{Psh}(\mathcal{N},(\hat{\omega}_1, J_0')) \xrightarrow{\phi} \mathrm{Isom}_h(\mathbb{C}^{2n},(g_0,\hat{J}_0'))\longrightarrow 1, \tag{46}$$

where $(g_0,\hat{J}_0')$ is a *complete Kähler* metric on the complex vector space $\mathbb{C}^{2n} = \mathcal{N}/\rho_1(\mathbb{R})$. (Compare [6, Proposition 3.5].) Hence we have

Proposition 6.1. *The holomorphic isometry group* $\mathrm{Isom}_h(\mathbb{C}^{2n},(g_0,\hat{J}_0'))$ *is isomorphic to* $\mathrm{U}(2n)$.

If $\hat{\varphi} : \mathbb{H}^n \to \mathbb{C}^{2n}$ is the diffeomorphism of **(2)**, then it follows from **(3)** that $\hat{\varphi}^* g_0 = g$ with $\hat{\varphi}_*\circ \hat{J}_1 = \hat{J}_0'\circ\hat{\varphi}_*$. Let $\mu : \mathrm{Diff}(\mathbb{H}^n)\to \mathrm{Diff}(\mathbb{C}^{2n})$ be an isomorphism defined by $\mu(h) = \hat{\varphi}\circ h\circ\hat{\varphi}^{-1}$. Then $(\mu,\hat{\varphi}) : (\mathrm{Isom}_h(\mathbb{H}^n,(g,\hat{J}_1)),\mathbb{H}^n)\longrightarrow(\mathrm{Isom}_h(\mathbb{C}^{2n},(g_0,\hat{J}_0')),\mathbb{C}^{2n})$ is an equivariant holomorphic isometry such that $\mu\big(\mathrm{Isom}_h(\mathbb{H}^n,(g,\hat{J}_1))\big) = \mathrm{U}(2n) = S^1\cdot \mathrm{SU}(2n)$. By forgetting the almost complex structures $\{J_2, J_3\}$, there is a natural inclusion:

$$\mathrm{Sp}(n)\cdot S^1 \leq \mathrm{Isom}_{qh}(\mathbb{H}^n,(g,\{\hat{J}_\alpha\}_{\alpha=1}^3)) \leq \mathrm{Isom}_h(\mathbb{H}^n,(g,\hat{J}_1)) = S^1\cdot \mathrm{SU}(2n).$$

Theorem 6.2. $\mathrm{Isom}_{qh}(\mathbb{H}^n,(g,\{\hat{J}_\alpha\}_{\alpha=1}^3)) = \mathrm{Sp}(n)\cdot S^1$.

Combining Proposition 6.1 with this theorem proves (ii), (iii) of Theorem 1.2.

Acknowledgements. This work was supported by the President Research grants 2019 at Josai University.

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