Josai Mathematical Monographs
Vol. 13 (2021) pp. 69–91

Area-minimizing cones over minimal embeddings of R-spaces

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Abstract. We prove area-minimizing properties of some cones over minimal embeddings of R-spaces by constructing area-nonincreasing retractions.

1. Introduction

Let C_B be the cone over a submanifold B of the unit sphere S^{n-1} in $\mathbb{R}^n$. The cone C_B is minimal in $\mathbb{R}^n$ if and only if B is minimal in S^{n-1}. We call a cone C_B area-minimizing if its truncated cone C_B^1 has the least area among all integral currents with the same boundary B. Solutions of Plateau's problem can have singularities as integral currents. At an isolated conical singularity, the tangent cone is area-minimizing. Hence, in order to understand such singularities, we should study area-minimizing properties of minimal cones.

Lawlor [9] gave a sufficient condition, so-called the *curvature criterion*, for a cone to be area-minimizing, using an area-nonincreasing retraction. With this criterion, he obtained a complete classification of area-minimizing cones over products of spheres and the first examples of area-minimizing cones over nonorientable manifolds. Kerckhove [8] proved that some cones over isolated orbits of the adjoint representations of SU(n) and SO(n) are area-minimizing. A symmetric R-space can be minimally embedded in the sphere in a canonical way. Hirohashi, Kanno and Tasaki [3] constructed area-nonincreasing retractions onto the cones over symmetric R-spaces associated with symmetric pairs of type B_l. Furthermore, Kanno [6] proved that cones over some symmetric R-spaces are area-minimizing. Recently, independently of our study, Z.Z. Tang and Y.S. Zhang [11] showed area minimizing properties of the cones over the focal submanifolds of isoparametric hypersurfaces in the unit sphere except for some low dimensional cases.

In this paper, we study area-minimizing properties of cones over minimal embeddings of R-spaces, not only symmetric R-spaces. In Theorem 3.4, we give a construction of retractions generalizing the method given in [3]. Applying this theorem we give some examples of area-minimizing cones over minimal embeddings

2020 Mathematics Subject Classification. Primary 49Q05; Secondary 53C42.
Key Words and Phrases. area-minimizing surface, R-space, cone.

of R-spaces. In Section 5, we discuss area-minimizing properties of cones over products of R-spaces.

2. Preliminaries

2.1. Area-minimizing cones.

Let B be a submanifold of the unit sphere S^{n-1} in $\mathbb{R}^n$. We define the cone C_B and the truncated cone C_B^1 over B by

$$C_B = \{tx \in \mathbb{R}^n \mid 0 \leq t, x \in B\},$$
$$C_B^1 = \{tx \in \mathbb{R}^n \mid 0 \leq t \leq 1, x \in B\}.$$

Both C_B and C_B^1 have an isolated singularity at the origin $0 \in \mathbb{R}^n$.

Definition 2.1. A cone C_B is called area-minimizing if C_B^1 has the least area among all integral currents with boundary B.

Let V and W be two vector spaces with inner products, and let $F : V \to W$ be a linear map. Suppose $\dim V = n \geq \dim W = m$. We define the Jacobian JF of F by

$$JF = \sup\{\|F(v_1) \wedge \cdots \wedge F(v_m)\|\},$$

where $\{v_1, \ldots, v_m\}$ runs over all orthonormal systems of V. If F is not surjective, then $JF = 0$. If F is surjective, then

$$JF = \|F(v_1) \wedge \cdots \wedge F(v_m)\|$$

for any orthonormal basis of $(\ker F)^{\perp}$.

Definition 2.2. A retraction $\Phi : \mathbb{R}^n \to C_B$ is called differentiable if $\Phi : \mathbb{R}^n \setminus \Phi^{-1}(0) \to C_B \setminus \{0\}$ is C^1. A differentiable retraction Φ is called area-nonincreasing if $J(d\Phi)_x \leq 1$ holds for all $x \in \mathbb{R}^n \setminus \Phi^{-1}(0)$.

Proposition 2.3. *Let B be a compact submanifold of the unit sphere S^{n-1} in $\mathbb{R}^n$. Suppose that there exists an area-nonincreasing retraction Φ from $\mathbb{R}^n$ to C_B. Then C_B is area-minimizing.*

Proof. Let S be an integral current which has the same boundary B as C_B^1. Since

$\Phi(S) \supset C_B^1$, we have

$$\begin{aligned}\mathrm{Vol}(C_B^1) &\leq \mathrm{Vol}(\Phi(S)) = \mathrm{Vol}(\Phi(S) \setminus \{0\}) \leq \int_{S\setminus\Phi^{-1}(0)} \|d\Phi(e_1 \wedge \cdots \wedge e_k)\| d\mu_S \\ &\leq \int_{S\setminus\Phi^{-1}(0)} J(d\Phi)_x d\mu_S \leq \int_{S\setminus\Phi^{-1}(0)} 1 d\mu_S \leq \int_S 1 d\mu_S = \mathrm{Vol}(S),\end{aligned}$$

where $\{e_1, \ldots, e_k\}$ is an orthonormal frame of S. □

If C_B is area-minimizing, then C_B is minimal in $\mathbb{R}^n$. Therefore, to find area-minimizing cones, it suffices to consider cones over minimal submanifolds of S^{n-1}. For this purpose, we use s-representations, which are the linear isotropy representations of Riemannian symmetric spaces.

2.2. Riemannian symmetric pairs and restricted root systems.

Let G be a connected Lie group and θ be an involutive automorphism of G. We denote by $F(\theta, G)$ the fixed point set of θ, and we denote by $F(\theta, G)_0$ the identity component of $F(\theta, G)$. For a closed subgroup K of G, the pair (G, K) is said to be a Riemannian symmetric pair if $F(\theta, G)_0 \subset K \subset F(\theta, G)$ and $\mathrm{Ad}(K)$ is compact. Let (G, K) be a Riemannian symmetric pair, and $\mathfrak{g}$ and $\mathfrak{k}$ be Lie algebras of G and K, respectively. We immediately see that

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid d\theta(X) = X\}.$$

We put

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid d\theta(X) = -X\}.$$

We denote by $\langle \cdot, \cdot \rangle$ an inner product on $\mathfrak{g}$ which is invariant under the actions of $\mathrm{Ad}(K)$ and $d\theta$. Then $\langle \cdot, \cdot \rangle$ induces a left-invariant metric on G and a G-invariant metric on $M = G/K$ to be a Riemannian symmetric space respectively, which we use the same symbol $\langle \cdot, \cdot \rangle$. Since $d\theta$ is involutive, we have an orthogonal direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

This decomposition is called the canonical decomposition of $(\mathfrak{g}, \mathfrak{k})$. For the origin $o \in G/K$, we can identify the tangent space $T_o(G/K)$ with $\mathfrak{m}$ by the differential of the natural projection $\pi : G \to G/K$.

In this paper, we consider only Riemannian symmetric spaces of compact type.

We suppose that G is compact and semisimple. Take and fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$ and a maximal abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ including $\mathfrak{a}$. For $\lambda \in \mathfrak{t}$, we put

$$\tilde{\mathfrak{g}}_\lambda = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = \sqrt{-1}\langle \lambda, H\rangle X \ (H \in \mathfrak{t})\}$$

and define the root system $\tilde{R}$ of $\mathfrak{g}$ by

$$\tilde{R} = \{\lambda \in \mathfrak{t} \setminus \{0\} \mid \tilde{\mathfrak{g}}_\lambda \neq \{0\}\}.$$

For $\lambda \in \mathfrak{a}$, we put

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = \sqrt{-1}\langle \lambda, H\rangle X \ (H \in \mathfrak{a})\}$$

and define the restricted root system R of $(\mathfrak{g}, \mathfrak{k})$ by

$$R = \{\lambda \in \mathfrak{a} \setminus \{0\} \mid \mathfrak{g}_\lambda \neq \{0\}\}.$$

Denote the orthogonal projection from $\mathfrak{t}$ to $\mathfrak{a}$ by $H \mapsto \overline{H}$. We extend a basis of $\mathfrak{a}$ to that of $\mathfrak{t}$ and define a lexicographic orderings $>$ on $\mathfrak{a}$ and $\mathfrak{t}$ with respect to these basis. Then for $H \in \mathfrak{t}$, $\overline{H} > 0$ implies $H > 0$. We denote by $\tilde{F}$ the fundamental system of $\tilde{R}$ with respect to $>$, by F the fundamental system of R with respect to $>$. We define

$$\tilde{R}_0 = \{\lambda \in \tilde{R} \mid \overline{\lambda} = 0\}, \quad \tilde{F}_0 = \{\alpha \in \tilde{F} \mid \overline{\alpha} = 0\}.$$

Then we have

$$R = \{\overline{\lambda} \mid \lambda \in \tilde{R} \setminus \tilde{R}_0\}, \quad F = \{\overline{\alpha} \mid \alpha \in \tilde{F} \setminus \tilde{F}_0\}.$$

We denote the set of positive roots by

$$\tilde{R}_+ = \{\lambda \in \tilde{R} \mid \lambda > 0\}, \quad R_+ = \{\lambda \in R \mid \lambda > 0\}.$$

We put

$$\mathfrak{k}_0 = \{X \in \mathfrak{k} \mid [H, X] = 0 \ (H \in \mathfrak{a})\}$$

and for each $\lambda \in R_+$

$$\mathfrak{k}_\lambda = \mathfrak{k} \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}), \quad \mathfrak{m}_\lambda = \mathfrak{m} \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}).$$

We then have the following lemma.

LEMMA 2.4 ([10]). *(1) We have orthogonal direct sum decompositions:*

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in R_+} \mathfrak{k}_\lambda \ , \quad \mathfrak{m} = \mathfrak{a} + \sum_{\lambda \in R_+} \mathfrak{m}_\lambda \ .$$

(2) For each $\mu \in \tilde{R}_+ \setminus \tilde{R}_0$ there exist $S_\mu \in \mathfrak{k}$ and $T_\mu \in \mathfrak{m}$ such that

$$\{S_\mu \mid \ \mu \in \tilde{R}_+, \ \overline{\mu} = \lambda\}, \quad \{T_\mu \mid \mu \in \tilde{R}_+, \ \overline{\mu} = \lambda\}$$

are, respectively, orthonormal bases of $\mathfrak{k}_\lambda$ and $\mathfrak{m}_\lambda$ and that for any $H \in \mathfrak{a}$

$$[H, S_\mu] = \langle \mu, H \rangle T_\mu, \quad [H, T_\mu] = -\langle \mu, H \rangle S_\mu.$$

For each $\lambda \in R_+$ we put $m(\lambda) = \dim \mathfrak{m}_\lambda = \dim \mathfrak{k}_\lambda$. $m(\lambda)$ is called the multiplicity of λ. We define a subset D of $\mathfrak{a}$ by

$$D = \bigcup_{\lambda \in R_+} \{H \in \mathfrak{a} \mid \langle \lambda, H \rangle = 0\}.$$

Each connected component of $\mathfrak{a} \setminus D$ is called a Weyl chamber. We define the fundamental Weyl chamber $\mathcal{C}$ by

$$\mathcal{C} = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle > 0 \ (\alpha \in F)\}.$$

The closure of $\mathcal{C}$ is given by

$$\overline{\mathcal{C}} = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle \geq 0 \ (\alpha \in F)\}.$$

For each subset $\Delta \subset F$, we define a subset $\mathcal{C}^\Delta \subset \overline{\mathcal{C}}$ by

$$\mathcal{C}^\Delta = \{H \in \overline{\mathcal{C}} \mid \langle \alpha, H \rangle > 0 \ (\alpha \in \Delta), \ \langle \beta, H \rangle = 0 \ (\beta \in F \setminus \Delta)\}.$$

Then we have the following lemma.

LEMMA 2.5 ([3]). *(1) For $\Delta_1 \subset F$*

$$\overline{\mathcal{C}^{\Delta_1}} = \bigcup_{\Delta \subset \Delta_1} \mathcal{C}^\Delta$$

is a disjoint union. In particular $\overline{\mathcal{C}} = \bigcup_{\Delta \subset F} \mathcal{C}^\Delta$ is a disjoint union.

(2) $\Delta_1 \subset \Delta_2$ *if and only if* $\mathcal{C}^{\Delta_1} \subset \overline{\mathcal{C}^{\Delta_2}}$, *for* $\Delta_1, \Delta_2 \subset F$.

For each $\alpha \in F$ we define $H_\alpha \in \mathfrak{a}$ by

$$\langle H_\alpha, \beta \rangle = \delta_{\alpha\beta} \ (\beta \in F),$$

where $\delta_{\alpha\beta}$ is Kronecker's delta. Then for $\Delta \subset F$ we have

$$\mathcal{C}^{\Delta} = \left\{ \sum_{\alpha \in \Delta} x_\alpha H_\alpha \;\middle|\; x_\alpha > 0 \right\}.$$

3. Construction of retractions

The notation of the preceding section will be preserved. The linear isotropy representation of a Riemannian symmetric space G/K is called an s-representation. The s-representation of G/K on $T_o(G/K)$ and the adjoint representation $\mathrm{Ad}(K)$ on $\mathfrak{m}$ are equivalent. Since an s-representation is an orthogonal representation, for a unit vector $H \in \mathfrak{m}$, the orbit $\mathrm{Ad}(K)H$ is a submanifold of the unit sphere $S \subset \mathfrak{m}$. Orbits of s-representations are called R-spaces. The orbit space of an s-representation is homeomorphic to $\overline{\mathcal{C}}$, more precisely for any $X \in \mathfrak{m}$, there exists $k \in K$ and unique $H \in \overline{\mathcal{C}}$ such that $X = \mathrm{Ad}(k)H$. The decomposition of $\overline{\mathcal{C}}$ in Lemma 2.5 is the decomposition of the orbit type. From the following theorem, we can see that for each orbit type, there exists a unique minimal orbit.

Theorem 3.1 ([5]). *For any nonempty subset* $\Delta \subset F$, *there exists a unique* $H \in S \cap \mathcal{C}^{\Delta}$ *such that the linear isotropy orbit* $\mathrm{Ad}(K)H$ *is a minimal orbit of* S.

Corollary 3.2. *An isolated orbit (i.e.* $\Delta = \{\alpha\}$*) is a minimal submanifold of* S.

Kitagawa and Ohnita ([7]) calculated the mean curvature vector m_H of $\mathrm{Ad}(K)H$ in $\mathfrak{m}$ at H:

$$m_H = -\sum_{\lambda \in \tilde{R}_+ \setminus \tilde{R}_+^{\Delta}} \frac{\overline{\lambda}}{\langle \lambda, H \rangle}.$$

This expression is used in the proof of Theorem 3.1. We consider cones over minimal embeddings of R-spaces that obtained in this way, and construct retractions.

Lemma 3.3 ([3]). *Suppose* ϕ *is a mapping of* $\overline{\mathcal{C}}$ *into itself such that*

$\phi(\mathcal{C}^{\Delta}) \subset \overline{\mathcal{C}^{\Delta}}$ *for each* $\Delta \subset F$*. Then* ϕ *extends to a mapping* Φ *of* $\mathfrak{m}$ *as*

$$\Phi(X) = \mathrm{Ad}(k)\phi(H)$$

for each $X = \mathrm{Ad}(k)H$ $(k \in K, H \in \overline{\mathcal{C}})$*.*

The following theorem is a generalization of Proposition 2.6 in [3].

THEOREM 3.4. *For* $A \in \overline{\mathcal{C}}$*, we put* $\Delta_0 = \{\alpha \in F \mid \langle \alpha, A\rangle > 0\}$*. Let* $f : \overline{\mathcal{C}} \to \mathbb{R}_{\geq 0}$ *be a continuous function. Define a continuous mapping* $\phi : \overline{\mathcal{C}} \to \{tA \mid t \geq 0\}$ *by* $\phi(x) = f(x)A$*. If* f *satisfies*

(1) $f(tA) = t$ $(t \geq 0)$*,*

(2) $f|_{\mathcal{C}^{\Delta}} = 0$ $(\Delta \subset F$ *with* $\Delta_0 \not\subset \Delta)$*,*

then ϕ *extends to a retraction* $\Phi : \mathfrak{m} \to C_{\mathrm{Ad}(K)A}$*.*

PROOF. First, we show that ϕ satisfies the assumption of Lemma 3.3. For $\Delta \subset F$ if $\Delta_0 \subset \Delta$, then $\mathcal{C}^{\Delta_0} \subset \overline{\mathcal{C}^{\Delta}}$. Hence

$$\phi(\mathcal{C}^{\Delta}) = \{tA \mid t \geq 0\} \subset \mathcal{C}^{\Delta_0} \subset \overline{\mathcal{C}^{\Delta}}$$

holds. If $\Delta_0 \not\subset \Delta$, then $\phi(\mathcal{C}^{\Delta}) = \{0\}$ since $f|_{\mathcal{C}^{\Delta}} = 0$. Therefore, ϕ satisfies the assumption of Lemma 3.3. We also get

$$\begin{aligned}\Phi(\mathfrak{m}) &= \{\mathrm{Ad}(k)f(H)A \mid k \in K, H \in \overline{\mathcal{C}}\}\\ &= \{t\mathrm{Ad}(k)A \mid k \in K,\ t \geq 0\} = C_{\mathrm{Ad}(K)A}.\end{aligned}$$

Thus Φ is a surjection from $\mathfrak{m}$ onto $C_{\mathrm{Ad}(K)A}$. Next we show that Φ is continuous. Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{m}$ with limit $P_\infty \in \mathfrak{m}$. Points P_n and P_∞ can be expressed as $P_n = \mathrm{Ad}(k_n)H_n$, $P_\infty = \mathrm{Ad}(k_\infty)H_\infty$ where $k_n, k_\infty \in K$ and $H_n, H_\infty \in \overline{\mathcal{C}}$. Since the projection $\mathfrak{m} \to \overline{\mathcal{C}}; X = \mathrm{Ad}(k)H \mapsto H$ is continuous, we have $\lim_{n\to\infty} H_n = H_\infty$. We put $\Delta_\infty = \{\alpha \in F \mid \langle \alpha, H_\infty\rangle > 0\}$, $Z_K^{H_\infty} = \{k \in K \mid \mathrm{Ad}(k)H_\infty = H_\infty\}$ and $Z_K^{\Delta_\infty} = \{k \in K \mid \mathrm{Ad}(k)|_{\overline{\mathcal{C}^{\Delta_\infty}}} = \mathrm{id}\}$. Since $Z_K^{H_\infty} = Z_K^{\Delta_\infty}$ ([3]), for any accumulation point $\tilde{k} \in K$ of $\{k_n\}_{n\in\mathbb{N}}$, $\mathrm{Ad}(\tilde{k})|_{\overline{\mathcal{C}^{\Delta_\infty}}} = \mathrm{Ad}(k_\infty)|_{\overline{\mathcal{C}^{\Delta_\infty}}}$. Thus, we have $\lim_{n\to\infty} \mathrm{Ad}(k_n)|_{\overline{\mathcal{C}^{\Delta_\infty}}} = \mathrm{Ad}(k_\infty)|_{\overline{\mathcal{C}^{\Delta_\infty}}}$. Therefore

$$\lim_{n\to\infty} \Phi(P_n) = \lim_{n\to\infty} \mathrm{Ad}(k_n)f(H_n)A = \mathrm{Ad}(k_\infty)f(H_\infty)A = \Phi(P_\infty)$$

Hence Φ is a retraction from $\mathfrak{m}$ onto $C_{\mathrm{Ad}(K)A}$. □

Proposition 3.5. *Let $\Phi : \mathfrak{m} \to C_{\mathrm{Ad}(K)A}$ be a retraction which constructed by Theorem 3.4. If $\Phi|_{\mathfrak{a}\setminus\Phi^{-1}(\{0\})}$ is C^1, then so is $\Phi|_{\mathfrak{m}\setminus\Phi^{-1}(\{0\})}$. In this case Φ is area-nonincreasing if and only if $J(d\Phi)_x \leq 1$ holds for each $x \in \mathcal{C} \setminus \Phi^{-1}(\{0\})$.*

Proof. If Φ is C^1 at $H \in \overline{\mathcal{C}}$, then Φ is C^1 at $\mathrm{Ad}(k)H$ for all $k \in K$. Thus we assume $H \in \overline{\mathcal{C}} \setminus \Phi^{-1}(\{0\})$. For $H \in \overline{\mathcal{C}} \setminus \Phi^{-1}(\{0\})$, we put $\Delta = \{\alpha \in F \mid \langle \alpha, H\rangle > 0\}$. Since $f(H) > 0$, we get $\Delta_0 \subset \Delta$ and $\mathcal{C}^{\Delta_0} \subset \overline{\mathcal{C}^{\Delta}}$. By Lemma 2.4, we have

$$\mathfrak{m} = \mathfrak{a} + \sum_{\lambda \in \tilde{R}_+ \setminus \tilde{R}_0} \mathbb{R} \cdot T_\lambda.$$

Since $\Phi|_{\mathfrak{a}\setminus\Phi^{-1}(\{0\})}$ is C^1, we consider only T_λ direction for each $\lambda \in \tilde{R}_+ \setminus \tilde{R}_0$. If $\langle \lambda, H\rangle = 0$, then $[T_\lambda, H] = \langle \lambda, H\rangle S_\lambda = 0$ from Lemma 2.4. Thus there exists $k \in Z_K^H = \{k \in K \mid \mathrm{Ad}(k)H = H\}$ such that $\mathrm{Ad}(k)T_\lambda \in \mathfrak{a}$. Therefore

$$\Phi(H + tT_\lambda) = \mathrm{Ad}(k)^{-1}\Phi(\mathrm{Ad}(k)(H + tT_\lambda)).$$

Since $\mathrm{Ad}(k)(H + tT_\lambda) \in \mathfrak{a}$ and $\Phi|_{\mathfrak{a}\setminus\Phi^{-1}(\{0\})}$ is C^1, we have the directional derivative of Φ along T_λ. If $\langle \lambda, H\rangle \neq 0$, then from Lemma 2.4 we have that $c(t) = \mathrm{Ad}\left(\exp(-tS_\lambda/\langle \lambda, H\rangle)\right) H$ is curve in $\mathfrak{m}$ with $c(0) = H$ and $c'(0) = T_\lambda$. Thus

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \Phi(c(t)) &= \left.\frac{d}{dt}\right|_{t=0} \Phi\left(\mathrm{Ad}\left(\exp\frac{-tS_\lambda}{\langle \lambda, H\rangle}\right) H\right) \\ &= \frac{[-S_\lambda, \phi(H)]}{\langle \lambda, H\rangle} = \frac{\langle \lambda, A\rangle}{\langle \lambda, H\rangle} f(H) T_\lambda.\end{aligned}$$

Therefore Φ is a differentiable retraction from $\mathfrak{m}$ into $C_{\mathrm{Ad}(K)A}$. Since $\Phi|_{\mathfrak{m}\setminus\Phi^{-1}(\{0\})}$ is C^1, the mapping $\overline{\mathcal{C}} \setminus \Phi^{-1}(\{0\}) \to \mathbb{R}; x \mapsto J(d\Phi_x)$ is continuous. Hence, if $J(d\Phi_x) \leq 1$ $(x \in \mathcal{C} \setminus \Phi^{-1}(\{0\}))$, then $J(d\Phi_x) \leq 1$ $(x \in \overline{\mathcal{C}} \setminus \Phi^{-1}(\{0\}))$. □

We will compute $J(d\Phi_x)$ of Φ in Theorem 3.4 for $x \in \mathcal{C} \setminus \Phi^{-1}(\{0\})$.

Proposition 3.6. *We denote $R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A\rangle = 0\}$.*

$$J(d\Phi_x) = \|(\mathrm{grad} f)_x\| \prod_{\lambda \in R_+ \setminus R_+^{\Delta_0}} \left(\frac{\langle \lambda, A\rangle}{\langle \lambda, x\rangle} f(x)\right)^{m(\lambda)} \quad (x \in \mathcal{C} \setminus \Phi^{-1}(\{0\})).$$

Proof. From the proof of Proposition 3.5, we have

$$d\Phi_x(H) = df_x(H)A \ (H \in \mathfrak{a}), \quad d\Phi_x(T_\lambda) = \frac{\langle \lambda, A\rangle}{\langle \lambda, x\rangle} f(x) T_\lambda \ (\lambda \in \tilde{R}_+ \setminus \tilde{R}_0)$$

for $x \in \mathcal{C} \setminus \Phi^{-1}(\{0\})$. Thus we get

$$d\Phi_x(\mathfrak{a}) \subset \mathbb{R}A \subset \mathfrak{a}, \quad d\Phi_x\left(\sum_{\mu \in R_+} \mathfrak{m}_\mu\right) \subset \sum_{\mu \in R_+} \mathfrak{m}_\mu.$$

Since $\mathfrak{a}$ and $\sum_{\mu\in R_+} \mathfrak{m}_\mu$ are orthogonal, we have

$$J(d\Phi_x) = J(d\Phi_x|_{\mathfrak{a}}) \times J(d\Phi_x|_{\sum_{\mu \in R_+} \mathfrak{m}_\mu}).$$

We put $J_1(x) = J(d\Phi_x|_{\mathfrak{a}})$, $J_2(x) = J(d\Phi_x|_{\sum_{\mu\in R_+} \mathfrak{m}_\mu})$ and compute each of these.

$$\begin{aligned} J_1(x) &= \sup\{\|d\Phi_x(v)\| \mid v \in \mathfrak{a}, \|v\| = 1\} \\ &= \sup\{\langle (\mathrm{grad} f)_x, v\rangle \mid v \in \mathfrak{a}, \|v\| = 1\} = \|(\mathrm{grad} f)_x\|. \end{aligned}$$

Since $\ker\left(d\Phi_x|_{\sum_{\mu\in R_+}\mathfrak{m}_\mu}\right) = \sum_{\mu \in R_+^{\Delta_0}} \mathfrak{m}_\mu$, $\{T_\lambda \mid \lambda \in \tilde{R}_+, \langle\lambda, A\rangle > 0\}$ is an orthonormal basis of $\ker\left(d\Phi_x|_{\sum_{\mu\in R_+}\mathfrak{m}_\mu}\right)^\perp = \sum_{\mu \in R_+ \setminus R_+^{\Delta_0}} \mathfrak{m}_\mu$. Hence

$$\begin{aligned} J_2(x) &= \left\| \bigwedge_{\lambda \in \tilde{R}_+, \langle\lambda, A\rangle > 0} d\Phi_x(T_\lambda) \right\| = \left\| \bigwedge_{\lambda \in \tilde{R}_+, \langle\lambda, A\rangle > 0} \frac{\langle\lambda, A\rangle}{\langle\lambda, x\rangle} f(x) T_\lambda \right\| \\ &= \prod_{\lambda \in \tilde{R}_+, \langle\lambda, A\rangle > 0} \frac{\langle\lambda, A\rangle}{\langle\lambda, x\rangle} f(x) = \prod_{\lambda \in R_+ \setminus R_+^{\Delta_0}} \left(\frac{\langle\lambda, A\rangle}{\langle\lambda, x\rangle} f(x)\right)^{m(\lambda)}. \end{aligned}$$

Therefore we get

$$J(d\Phi)_x = J_1(x) J_2(x) = \|(\mathrm{grad} f)_x\| \prod_{\lambda \in R_+ \setminus R_+^{\Delta_0}} \left(\frac{\langle\lambda, A\rangle}{\langle\lambda, x\rangle} f(x)\right)^{m(\lambda)}.$$

□

4. Example of area-minimizing cones over R-spaces

Using Theorem 3.4, Proposition 3.5 and Proposition 3.6, we investigate area-minimizing properties of cones over R-spaces. First we consider cones over isolated orbits of s-representations of irreducible symmetric pairs of compact type of rank two. Principal orbits of these representations are homogeneous hypersurfaces in the sphere. The area-minimizing properties of the cones over homogeneous minimal hypersurfaces were investigated in [4] and [9].

We shall follow the notations of root systems in [2]. Partly we used Maxima* for algebraic computations.

4.1. Type $\mathbf{A_2}$.

$$\mathfrak{a} = \{\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 \mid \xi_1 + \xi_3 + \xi_3 = 0\},$$

$$F = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3\}.$$

Then we have $R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. For $\lambda \in R_+$, we put $m = m(\lambda)$. We have

$$H_{\alpha_1} = \frac{1}{3}(2e_1 - e_2 - e_3), \quad H_{\alpha_2} = \frac{1}{3}(e_1 + e_2 - 2e_3).$$

We put

$$A_1 = \frac{H_{\alpha_1}}{\|H_{\alpha_1}\|} = \frac{1}{\sqrt{6}}(2e_1 - e_2 - e_3), \quad A_2 = \frac{H_{\alpha_2}}{\|H_{\alpha_2}\|} = \frac{1}{\sqrt{6}}(e_1 + e_2 - 2e_3).$$

Since $\mathrm{Ad}(K)A_1$ and $\mathrm{Ad}(K)A_2$ are isometric, we consider only the cone over $\mathrm{Ad}(K)A_1$.

4.1.1. Cones over $\mathbf{Ad}(\boldsymbol{K})\boldsymbol{A_1}$.

We put $\Delta_0 = \{\alpha_1\}$ then $R_+^{\Delta_0} = \{\alpha_2\}$. For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{\frac{2}{3}}\left(\langle \alpha_1, x\rangle^2 \left\langle \alpha_1 + \frac{3}{2}\alpha_2, x\right\rangle\right)^{\frac{1}{3}} = \sqrt{\frac{2}{3}}\left(x_1^2\left(x_1 + \frac{3}{2}x_2\right)\right)^{\frac{1}{3}}.$$

Since

(1) $f(tA_1) = \sqrt{\frac{2}{3}}\left(\left(\sqrt{\frac{3}{2}}t\right)^3\right)^{\frac{1}{3}} = t,$

*http://maxima.sourceforge.net/

(2) for each $\Delta \subset F$, if $\Delta_0 \not\subset \Delta$, then $f|_{\mathcal{C}^{\Delta}} = 0$,

we can apply Theorem 3.4 to this case. It is clear that $\Phi|_{\mathfrak{a}\setminus\Phi^{-1}(\{0\})}$ is C^1. Thus Φ is a differentiable retraction by Proposition 3.5. Since

$$\frac{\partial f}{\partial x_1}(x) = \sqrt{\frac{2}{3}}\left(x_1^2\left(x_1 + \frac{3}{2}x_2\right)\right)^{-\frac{2}{3}}(x_1^2 + x_1x_2),$$
$$\frac{\partial f}{\partial x_2}(x) = \sqrt{\frac{2}{3}}\left(x_1^2\left(x_1 + \frac{3}{2}x_2\right)\right)^{-\frac{2}{3}}\frac{x_1^2}{2},$$

we get

$$J_1(x) = \|(\mathrm{grad} f)_x\| = \sqrt{\frac{2}{3}}\left(x_1^2\left(x_1 + \frac{3}{2}x_2\right)\right)^{-\frac{2}{3}}\sqrt{\frac{3}{2}x_1^4 + 3x_1^3x_2 + 2x_1^2x_2^2}.$$

On the other hand,

$$J_2(x) = \left(\frac{\langle\alpha_1, A_1\rangle}{\langle\alpha_1, x\rangle}f(x)\right)^m\left(\frac{\langle\alpha_1+\alpha_2, A_1\rangle}{\langle\alpha_1+\alpha_2, x\rangle}f(x)\right)^m = \left(\frac{\left(x_1\left(x_1+\frac{3}{2}x_2\right)^2\right)^{\frac{1}{3}}}{x_1+x_2}\right)^m.$$

Then

$$(x_1+x_2)^3 - x_1\left(x_1 + \frac{3}{2}x_2\right)^2 = \frac{3}{4}x_1x_2^2 + x_2^3 \geq 0,$$

thus

$$\left(\frac{\left(x_1\left(x_1+\frac{3}{2}x_2\right)^2\right)^{\frac{1}{3}}}{x_1+x_2}\right) \leq 1.$$

We put

$$D = J_1(x)\times\left(\frac{\left(x_1\left(x_1+\frac{3}{2}x_2\right)^2\right)^{\frac{1}{3}}}{x_1+x_2}\right)^2 = \left(\frac{(3x_1^2+6x_1x_2+4x_2^2)^3x_1^2(2x_1+3x_2)^4}{3^32^4(x_1+x_2)^{12}}\right)^{\frac{1}{6}}.$$

Since

$$J(d\Phi)_x = D \times \left(\frac{\left(x_1 \left(x_1 + \frac{3}{2} x_2 \right)^2 \right)^{\frac{1}{3}}}{x_1 + x_2} \right)^{m-2},$$

if $D \leq 1$, then $J(d\Phi_x) \leq 1$ for $m \geq 2$. Since

$$\begin{aligned} & 3^3 2^4 (x_1 + x_2)^{12} - (3x_1^2 + 6x_1x_2 + 4x_2^2)^3 x_1^2 (2x_1 + 3x_2)^4 \\ = & 216 x_1^{10} x_2^2 + 2376 x_1^9 x_2^3 + 11925 x_1^8 x_2^4 + 35838 x_1^7 x_2^5 + 71120 x_1^6 x_2^6 \\ & + 96888 x_1^5 x_2^7 + 91152 x_1^4 x_2^8 + 57888 x_1^3 x_2^9 + 23328 x_1^2 x_2^{10} + 5184 x_1 x_2^{11} + 432 x_2^{12} \\ \geq & 0, \end{aligned}$$

we have $D \leq 1$. Therefore, cones over $\mathrm{Ad}(K)A_1$ are area-minimizing for $m \geq 2$.

4.2. Types $\mathbf{B_2}$, $\mathbf{BC_2}$ and $\mathbf{C_2}$.

Types C_2 and B_2 are isomorphic, thus it suffices to compute the type B_2 case. Moreover setting the multiplicity of long roots to zero, the set of restricted roots of type BC_2 reduces to that of type B_2. We have

$$F = \{\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2\},$$

$$R_+ = \{\alpha_1,\ \alpha_2,\ \alpha_1 + \alpha_2,\ \alpha_1 + 2\alpha_2,\ 2\alpha_1 + 2\alpha_2,\ 2\alpha_2\},$$

$$H_{\alpha_1} = e_1, \quad H_{\alpha_2} = e_1 + e_2,$$

and put

$$m(\alpha_1) = m_1,\ m(\alpha_2) = m_2,\ m(2\alpha_2) = m_3.$$

4.2.1. Cones over $\mathrm{Ad}(K)A_1$.

We put $\Delta_0 = \{\alpha_1\}$, then we have

$$A_1 = \frac{H_{\alpha_1}}{\|H_{\alpha_1}\|} = e_1,$$

and

$$R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A_1 \rangle = 0\} = \{\alpha_2, 2\alpha_2\}.$$

For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{\langle \alpha_1, x\rangle\langle \alpha_1 + 2\alpha_2, x\rangle} = \sqrt{x_1(x_1 + 2x_2)}.$$

Then we can show that f satisfies the condition of Theorem 3.4 and Φ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m_2 + m_3 \geq 2$.

Therefore, cones over $\mathrm{Ad}(K)A_1$ are area-minimizing for $m_2 + m_3 \geq 2$.

4.2.2. Cones over $\mathbf{Ad}(\boldsymbol{K})\boldsymbol{A_2}$.

We put $\Delta_0 = \{\alpha_2\}$, then we have

$$A_2 = \frac{H_{\alpha_2}}{\|H_{\alpha_2}\|} = \frac{e_1 + e_2}{\sqrt{2}}$$

and

$$R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A_2\rangle = 0\} = \{\alpha_1\}.$$

For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{2}\left(\langle \alpha_2, x\rangle^2 \left\langle \frac{3}{2}\alpha_1 + \alpha_2, x\right\rangle\right)^{\frac{1}{3}} = \sqrt{2}\left(x_2^2\left(\frac{3}{2}x_1 + x_2\right)\right)^{\frac{1}{3}}.$$

Then we can show that f satisfies the condition of Theorem 3.4 and Φ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m_2 + m_3 \geq 2$.

Therefore, cones over $\mathrm{Ad}(K)A_2$ are area-minimizing for $m_2 + m_3 \geq 2$.

4.3. Type $\mathbf{G_2}$.

We have

$$F = \{\alpha_1,\ \alpha_2\},$$

$$R_+ = \{\alpha_1,\ \alpha_2,\ \alpha_1 + \alpha_2,\ 2\alpha_1 + \alpha_2,\ 3\alpha_1 + \alpha_2,\ 3\alpha_1 + 2\alpha_2\},$$

$$\langle \alpha_1, \alpha_1\rangle = 1,\ \langle \alpha_1, \alpha_2\rangle = -\frac{3}{2},\ \langle \alpha_2, \alpha_2\rangle = 3,$$

$$H_{\alpha_1} = 4\alpha_1 + 2\alpha_2, \quad H_{\alpha_2} = \frac{2}{3}(3\alpha_1 + 2\alpha_2),$$

and put

$$m = m(\alpha_1) = m(\alpha_2).$$

4.3.1. Cones over $\mathrm{Ad}(K)A_1$.

We put $\Delta_0 = \{\alpha_1\}$ then we have

$$A_1 = \frac{H_{\alpha_1}}{\|H_{\alpha_1}\|}$$

and

$$R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A_1 \rangle = 0\} = \{\alpha_2\}.$$

For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{4\langle \alpha_1, x\rangle \langle \alpha_1 + \alpha_2, x\rangle} = \sqrt{4x_1(x_1 + x_2)}.$$

Then we can show that f satisfies the condition of Theorem 3.4 and Φ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m \geq 2$.

Therefore cones over $\mathrm{Ad}(K)A_1$ are area-minimizing for $m \geq 2$.

4.3.2. Cones over $\mathrm{Ad}(K)A_2$.

We put $\Delta_0 = \{\alpha_2\}$ then we have

$$A_2 = \frac{H_{\alpha_2}}{\|H_{\alpha_2}\|},$$

and

$$R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A_1 \rangle = 0\} = \{\alpha_1\}.$$

For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{\frac{4}{3}\langle \alpha_2, x\rangle \langle 3\alpha_1 + \alpha_2, x\rangle} = \sqrt{\frac{4}{3}x_2(3x_1 + x_2)}.$$

Then we can show that f satisfies the condition of Theorem 3.4 and Φ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m \geq 2$.

Therefore, cones over $\mathrm{Ad}(K)A_2$ are area-minimizing for $m \geq 2$.

By the above computation, we get the following table of cones over isolated orbits of the s-representations of irreducible symmetric spaces of rank two.

type	symmetric pair	multiplicities	A_i	orbit	dim. of orbit and sphere	area-min.
A_2	$(\mathrm{SU}(3), \mathrm{SO}(3))$	$(1,1)$	A_1	$\mathbb{R}P^2$ (symm. R-space)	$(2,4)$	
	$(\mathrm{SU}(3)\times\mathrm{SU}(3), \mathrm{SU}(3))$	$(2,2)$	A_1	$\mathbb{C}P^2$ (symm. R-space)	$(4,7)$	◯ [8]
	$(\mathrm{SU}(6), \mathrm{Sp}(3))$	$(4,4)$	A_1	$\mathbb{H}P^2$ (symm. R-space)	$(8,13)$	◯ [6]
	(E_6, F_4)	$(8,8)$	A_1	$\mathbb{O}P^2$ (symm. R-space)	$(16,25)$	◯
B_2	$(\mathrm{SO}(5)\times\mathrm{SO}(5), \mathrm{SO}(5))$	$(2,2)$	A_1	$\widetilde{G_2}(\mathbb{R}^5)$(symm. R-space)	$(6,9)$	◯[3]
			A_2	$\mathrm{SO}(5)/\mathrm{U}(2)$	$(6,9)$	◯[8]
	$(\mathrm{SO}(5), \mathrm{SO}(2)\times\mathrm{SO}(3))$	$(1,1)$	A_1	(symm. R-space)	$(3,5)$	
			A_2		$(3,5)$	
	$(\mathrm{SO}(4+n), \mathrm{SO}(2)\times\mathrm{SO}(2+n))$	$(1,n)$	A_1	(symm. R-space)	$(n+2, 2n+3)$	◯ $(n\geq 2)$ [3]
			A_2		$(2n+1, 2n+3)$	◯ $(n\geq 2)$
C_2	$(\mathrm{Sp}(2), \mathrm{U}(2))$	$(1,1)$	A_1		$(3,5)$	
			A_2	$\frac{\mathrm{U}(2)}{\mathrm{O}(2)}$(symm. R-space)	$(3,5)$	
	$(\mathrm{Sp}(2)\times\mathrm{Sp}(2), \mathrm{Sp}(2))$	$(2,2)$	A_1		$(6,9)$	◯
			A_2	$\frac{\mathrm{Sp}(2)}{\mathrm{U}(2)}$(symm. R-space)	$(6,9)$	◯
	$(\mathrm{Sp}(4), \mathrm{Sp}(2)\times\mathrm{Sp}(2))$	$(4,3)$	A_1		$(11,15)$	◯
			A_2	$\mathrm{Sp}(2)$(symm. R-space)	$(10,15)$	◯
	$(\mathrm{SU}(4), \mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2)))$	$(2,1)$	A_1		$(5,7)$	◯
			A_2	$\mathrm{U}(2)$ (symm. R-space)	$(4,7)$	◯ [9]
	$(\mathrm{SO}(8), \mathrm{U}(4))$	$(4,1)$	A_1	$\frac{\mathrm{U}(4)}{(\mathrm{Sp}(1)\times\mathrm{U}(2))}$	$(9,11)$	◯
			A_2	$\frac{\mathrm{U}(4)}{\mathrm{Sp}(2)}$ (symm. R-space)	$(6,11)$	◯

type	symmetric pair	multiplicities	A_i	orbit	dim. of orbit and sphere	area-min.
BC_2	$(\mathrm{SU}(4+n), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2+n)))$	$(2, (2n, 1))$	A_1		$(2n+3, 4n+7)$	◯ $(n \geq 1)$
			A_2		$(4n+4, 4n+7)$	◯ $(n \geq 1)$
	$(\mathrm{SO}(10), \mathrm{U}(5))$	$(4, (4, 1))$	A_1	$\mathrm{U}(5)/(\mathrm{Sp}(1) \times \mathrm{U}(3))$	$(13, 19)$	◯
			A_2	$\mathrm{U}(5)/(\mathrm{Sp}(2) \times \mathrm{U}(1))$	$(14, 19)$	◯
	$(\mathrm{Sp}(4+n), \mathrm{Sp}(2) \times \mathrm{Sp}(2+n))$	$(4, (4n, 3))$	A_1		$(4n+11, 8n+15)$	◯ $(n \geq 1)$
			A_2		$(8n+10, 8n+15)$	◯ $(n \geq 1)$
	$(E_6, \mathrm{T}^1 \cdot \mathrm{Spin}(10))$	$(6, (8, 1))$	A_1		$(21, 31)$	◯
			A_2		$(24, 31)$	◯
G_2	$(G_2, \mathrm{SO}(4))$	$(1, 1)$	A_1		$(5, 7)$	
			A_2		$(5, 7)$	
	$(G_2 \times G_2, G_2)$	$(2, 2)$	A_1		$(10, 13)$	◯
			A_2		$(10, 13)$	◯

4.4. Type A_3.

Theorem 3.4 can be applied to cones over minimal orbits, not only isolated orbits. We demonstrate the area-minimizing property for the cone over a minimal orbit, which is not an isolated orbit, of the s-representation of symmetric spaces of type A_3.

$$\mathfrak{a} = \left\{ \sum_{i=1}^{4} \xi_i e_i \mid \sum_{i=1}^{4} \xi_i = 0 \right\},$$

$$F = \{\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3 - e_4\}.$$

Then $R_+ = \{\alpha_1, \alpha_2, \alpha_3,\ \alpha_1 + \alpha_2,\ \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3\}$ and for $\lambda \in R_+$, we put $m(\lambda) = m$. We have

$$H_{\alpha_1} = \frac{1}{4}(3e_1 - e_2 - e_3 - e_4),\ H_{\alpha_2} = \frac{1}{4}(2e_1 + 2e_2 - 2e_3 - 2e_4),\ H_{\alpha_3} = \frac{1}{4}(e_1 + e_2 + e_3 - 3e_4).$$

We put $\Delta_0 = \{\alpha_1, \alpha_3\}$, and we have

$$A = \frac{H_{\alpha_1} + H_{\alpha_3}}{\sqrt{2}} = \frac{e_1 - e_4}{\sqrt{2}}.$$

Then the orbit $\mathrm{Ad}(K)A$ is a minimal submanifold of the sphere $S \subset \mathfrak{m}$. We get

$$R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A \rangle = 0\} = \{\alpha_2\}.$$

For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} + x_3 H_{\alpha_3} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{2}\left(\langle \alpha_1, x\rangle \langle \alpha_3, x\rangle \langle \alpha_1 + \alpha_2, x\rangle \langle \alpha_2 + \alpha_3, x\rangle\right)^{\frac{1}{4}} = \sqrt{2}(x_1 x_3 (x_1 + x_2)(x_2 + x_3))^{\frac{1}{4}}.$$

Then we can show that f satisfies the condition of Theorem 3.4 and Φ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m \geq 4$.

Therefore, cones over $\mathrm{Ad}(K)A$ are area-minimizing for $m \geq 4$. The only symmetric pair which satisfies $m \geq 3$ is $(\mathrm{SU}(6), \mathrm{Sp}(3))$.

5. Reducible cases

In this section, we consider cones over products of two R-spaces. Let (G_i, K_i) $(i = 1, 2)$ be Riemannian symmetric pairs, and put $(G, K) = (G_1 \times$

$G_2, K_1 \times K_2)$. We define the notation for (G_i, K_i) as follows. Let

$$\mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{m}_i \ (i = 1, 2)$$

be the canonical decompositions of Lie algebras $\mathfrak{g}_i$ of G_i. Take and fix a maximal abelian subspace $\mathfrak{a}_i$ in $\mathfrak{m}_i$. We denote by R_i the restricted root system of $(\mathfrak{g}_i, \mathfrak{k}_i)$ with respect to $\mathfrak{a}_i$. We put the fundamental systems F_i of R_i by $F_i = \{\alpha_{i1}, \ \ldots, \alpha_{il_i}\}$. R_{i+} is the set of positive roots in R_i. We set

$$\mathcal{C}_i = \{H \in \mathfrak{a}_i \mid \langle \alpha, H \rangle > 0 \ (\alpha \in F_i)\},$$

$$\mathcal{C}_i^{\Delta} = \{H \in \mathfrak{a}_i \mid \langle \alpha, H \rangle > 0 \ (\alpha \in \Delta), \ \langle \beta, H \rangle = 0 \ (\beta \in F_i \setminus \Delta)\},$$

where $\Delta \subset F_i$. The direct sum of the s-representations of (G_i, K_i) is the s-representation of $(G, K) = (G_1 \times G_2, K_1 \times K_2)$. Then, we have

$$\overline{\mathcal{C}} = \overline{\mathcal{C}_1} \times \overline{\mathcal{C}_2}.$$

For $\Delta \subset F$, Δ is expressed as $\Delta = \Delta_1 \cup \Delta_2$ where $\Delta_i \subset F_i$ $(i = 1, 2)$. By Theorem 3.1 for each Δ_i, there exists $A_i \in \overline{\mathcal{C}}_i$ such that $\mathrm{Ad}(K_i)A_i$ is a minimal orbit of the s-representation of (G_i, K_i). We put $k_i = \dim \mathrm{Ad}(K_i)A_i$ and $k = k_1 + k_2$, then

$$A = \sqrt{\frac{k_1}{k}} A_1 + \sqrt{\frac{k_2}{k}} A_2 \in \overline{\mathcal{C}}$$

is a base point of a minimal orbit of the s-representation of (G, K).

THEOREM 5.1. *Let $\Delta_0 = \Delta_1 \cup \Delta_2$ $(\Delta_i \subset F_i)$. We suppose that for the cone over $\mathrm{Ad}(K_i)A_i$, there exists an area-nonincreasing retraction constructed by a function f_i on $\overline{\mathcal{C}_i}$ in Theorem 3.4, and that the retraction satisfies*

$$\prod_{\lambda \in R_{i+} \setminus R_{i+}^{\Delta_i}} \left(\frac{\langle \lambda, A_i \rangle}{\langle \lambda, x \rangle} f_i(x) \right)^{m(\lambda)} \leq 1 \ (x \in \mathcal{C}_i). \tag{5.1}$$

If $\dim \mathrm{Ad}(K_i)A_i \geq 3$, then there exists an area-nonincreasing retraction $\Phi : \mathfrak{m} \to C_{\mathrm{Ad}(K)A}$ constructed by some function f on $\overline{\mathcal{C}}$ in Theorem 3.4 such that the retrac-

tion satisfies

$$\prod_{\lambda \in R_+ \setminus R_+^{\Delta_0}} \left(\frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)} \leq 1 \ (x \in \mathcal{C}).$$

Proof. Let $k_i = \dim \mathrm{Ad}(K_i)A_i$, $k = k_1 + k_2$ and put $a_i = \sqrt{k_i/k}$. $A = a_1 A_1 + a_2 A_2$ holds. For $x = (x_1, x_2) \in \overline{\mathcal{C}}_1 \times \overline{\mathcal{C}}_2 = \overline{\mathcal{C}}$ we define

$$f(x) = \begin{cases} \dfrac{f_1(x_1) f_2(x_2)}{a_2^3 f_1(x_1) + a_1^3 f_2(x_2)} & (f_1(x_1) \neq 0 \text{ or } f_2(x_2) \neq 0) \\ 0 & (f_1(x_1) = f_2(x_2) = 0) \end{cases}$$

We will show that f satisfies the conditions of Theorem 3.4. We can check easily $f(tA) = t$ for $t \geq 0$. For $\Delta \subset F$ with $\Delta_0 \not\subset \Delta$, using $\Delta_i' \subset F_i$ we can write $\Delta = \Delta_1' \cup \Delta_2'$. Then $\Delta_i \not\subset \Delta_i'$ implies $i = 1$ or $i = 2$. Thus $f_1 = 0$ or $f_2 = 0$ holds on $\mathcal{C}^\Delta$. Therefore $f|_{\mathcal{C}^\Delta} = 0$. Since $\Phi|_{\mathfrak{a} \setminus (\{0\})}$ is C^1, Φ is a differentiable retraction by Proposition 3.5. We calculate $J(d\Phi_x)$ for $x \in \mathcal{C} \setminus f^{-1}(\{0\})$. We put

$$J_1(x) = \|(\mathrm{grad} f)_x\|, \quad J_2(x) = \prod_{\lambda \in R_+ \setminus R_+^{\Delta}} \left(\frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)}$$

for $x = (x_1, x_2) = (x_1^1, \ldots, x_1^{l_1}, x_2^1, \ldots, x_2^{l_2}) \in \mathcal{C} \setminus f^{-1}(\{0\}) = \mathcal{C}_1 \times \mathcal{C}_2 \setminus f^{-1}(\{0\})$. Since

$$\frac{\partial f}{\partial x_1^j} = \frac{\frac{\partial f_1}{\partial x_1^j} a_1^3 f_2(x_2)^2}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^2} \quad (j \in \{1, \ldots, l_1\}),$$
$$\frac{\partial f}{\partial x_2^j} = \frac{\frac{\partial f_2}{\partial x_2^j} a_2^3 f_1(x_1)^2}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^2} \quad (j \in \{1, \ldots, l_2\}),$$

we get

$$(\mathrm{grad} f)_x = \frac{a_1^3 f_2(x_2)^2 (\mathrm{grad} f_1)_{x_1} + a_2^3 f_1(x_1)^2 (\mathrm{grad} f_2)_{x_2}}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^2}$$

and

$$J_1(x) = \|(\mathrm{grad} f)_x\| = \frac{\sqrt{a_1^6 f_2(x_2)^4 \|(\mathrm{grad} f_1)_{x_1}\|^2 + a_2^6 f_1(x_1)^4 \|(\mathrm{grad} f_2)_{x_2}\|^2}}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^2}.$$

Since $R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A \rangle = 0\} = R_{1+}^{\Delta_1} \cup R_{2+}^{\Delta_2}$, we get

$$\begin{aligned} J_2(x) &= \prod_{\lambda \in R_+ \setminus R_+^{\Delta_0}} \left(\frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)} \\ &= \prod_{\lambda \in R_{1+} \setminus R_{1+}^{\Delta_1}} \left(\frac{\langle \lambda, a_1 A_1 \rangle}{\langle \lambda, x_1 \rangle} f(x) \right)^{m(\lambda)} \prod_{\mu \in R_{2+} \setminus R_{2+}^{\Delta_2}} \left(\frac{\langle \mu, a_2 A_2 \rangle}{\langle \mu, x_2 \rangle} f(x) \right)^{m(\mu)} \\ &= \prod_{\lambda \in R_{1+} \setminus R_{1+}^{\Delta_1}} \left(\frac{\langle \lambda, A_1 \rangle}{\langle \lambda, x_1 \rangle} f_1(x_1) \frac{a_1 f(x)}{f_1(x_1)} \right)^{m(\lambda)} \prod_{\mu \in R_{2+} \setminus R_{2+}^{\Delta_2}} \left(\frac{\langle \mu, A_2 \rangle}{\langle \mu, x_2 \rangle} f_2(x_2) \frac{a_2 f(x)}{f_2(x_2)} \right)^{m(\mu)}. \end{aligned}$$

Put

$$J_{2i}(x_i) = \prod_{\lambda \in R_{i+} \setminus R_{i+}^{\Delta_i}} \left(\frac{\langle \lambda, A_i \rangle}{\langle \lambda, x_i \rangle} f_i(x_i) \right)^{m(\lambda)}, \ J_{1i}(x_i) = \|(\mathrm{grad} f_i)_{x_i}\| \ (i = 1, 2).$$

Note that $J_{2i}(x_i) \leq 1$ holds by the assumption (5.1), and $J_{1i}(x_i) J_{2i}(x_i) \leq 1$ holds since the function f_i constructs an area-nonincreasing retraction by Theorem 3.4. Since

$$\sum_{\lambda \in R_{i+} \setminus R_{i+}^{\Delta_i}} m(\lambda) = \dim \mathrm{Ad}(K_i) A_i = k_i,$$

we can write

$$J_2(x) = J_{21}(x_1) J_{22}(x_2) \left(\frac{a_1 f(x)}{f_1(x_1)} \right)^{k_1} \left(\frac{a_2 f(x)}{f_2(x_2)} \right)^{k_2}.$$

Since $J_{2i}(x_i) \leq 1$,

$$J_2(x) \leq \left(\frac{a_1 f(x)}{f_1(x_1)} \right)^{k_1} \left(\frac{a_2 f(x)}{f_2(x_2)} \right)^{k_2}.$$

We put

$$X_1 = \frac{f_2(x_2)}{a_2}, \ X_2 = \frac{f_1(x_1)}{a_1}.$$

Then we have

$$\left(\frac{a_1 f(x)}{f_1(x_1)}\right)^{k_1}\left(\frac{a_2 f(x)}{f_2(x_2)}\right)^{k_2} = \frac{X_1^{k_1}X_2^{k_2}}{(a_1^2X_1+a_2^2X_2)^k}.$$

For $X_1, X_2 > 0$, we define

$$\tilde{D}(X_1,X_2) = \frac{X_1^{k_1}X_2^{k_2}}{(a_1^2X_1+a_2^2X_2)^k}.$$

If $\tilde{D} \leq 1$, then $J_2(x) \leq 1$. Thus we prove $\tilde{D} \leq 1$. Since $\tilde{D}(X_1,X_2) = \tilde{D}(tX_1,tX_2)$ $(t>0)$, in order to prove $\tilde{D} \leq 1$, we show $\tilde{D}|_P \leq 1$ where

$$P = \{(X_1,X_2) \in \mathbb{R}^2 \mid X_1, X_2 > 0,\ a_1^2X_1 + a_2^2X_2 = 1\}.$$

We have $\tilde{D}|_P = X_1^{k_1}X_2^{k_2}$ and $X_2 = \dfrac{1-a_1^2X_1}{a_2^2}$. Since

$$\frac{d\tilde{D}|_P}{dX_1} = k_1X_1^{k_1-1}X_2^{k_2} + X_1^{k_1}(-k_2\frac{a_1^2}{a_2^2})X_2^{k_2-1} = k_1X_1^{k_1-1}X_2^{k_2-1}(X_2-X_1),$$

a critical point of $\tilde{D}|_P$ is only $X_1 = 1$ in P. Further, we get

$$\tilde{D}|_P \to 0 \ \text{ as } X_1 \to 0 \text{ or } \frac{1}{a_1^2}.$$

Hence $\max\{\tilde{D}(X_1,X_2) \mid (X_1,X_2) \in P\} = \tilde{D}(1,1) = 1$. Therefore

$$J_2(x) \leq 1.$$

Then we have

$$\begin{aligned}
J(d\Phi)_x &= J_1(x)J_2(x) = \|(\mathrm{grad} f)_x\| J_2(x)\\
&= \frac{\sqrt{a_1^6 f_2(x_2)^4 J_{11}(x_1)^2 + a_2^6 f_1(x_1)^4 J_{12}(x_2)^2}}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^2} J_{21}(x_1)J_{22}(x_2)\frac{(a_1f_2(x_2))^{k_1}(a_2f_1(x_1))^{k_2}}{(a_2^3f_1(x_1)+a_1^3f_2(x_2))^k}\\
&= \frac{\sqrt{a_1^6 f_2(x_2)^4 J_{11}(x_1)^2 J_{21}(x_1)^2 J_{22}(x_2)^2 + a_2^6 f_1(x_1)^4 J_{12}(x_2)^2 J_{21}(x_1)^2 J_{22}(x_2)^2}}{(a_2^3f_1(x_1)+a_1^3f_2(x_2))^{k+2}}\\
&\quad \times (a_1f_2(x_2))^{k_1}(a_2f_1(x_1))^{k_2}\\
&\leq \frac{\sqrt{a_1^6 f_2(x_2)^4 + a_2^6 f_1(x_1)^4}(a_1f_2(x_2))^{k_1}(a_2f_1(x_1))^{k_2}}{(a_2^3f_1(x_1)+a_1^3f_2(x_2))^{k+2}}
\end{aligned}$$

$$= \frac{\sqrt{a_1^2 X_1^4 + a_2^2 X_2^4}\, X_1^{k_1} X_2^{k_2}}{(a_1^2 X_1 + a_2^2 X_2)^{k+2}}.$$

We define

$$D(X_1, X_2) = J(d\Phi_x)^2 = \frac{(a_1^2 X_1^4 + a_2^2 X_2^4) X_1^{2k_1} X_2^{2k_2}}{(a_1^2 X_1 + a_2^2 X_2)^{2k+4}}.$$

We have $D(tX_1, tX_2) = D(X_1, X_2)$ $(t > 0)$. Similar to the above argument, we consider the maximum value of $D|_P$. Since

$$D|_P = (a_1^2 X_1^4 + a_2^2 X_2^4) X_1^{2k_1} X_2^{2k_2},$$

we get

$$\begin{aligned}
\frac{dD|_P}{dX_1} &= 4\left(a_1^2 X_1^3 - \frac{a_1^2}{a_2^2} a_2^2 X_2^3\right) X_1^{2k_1} X_2^{2k_2} \\
&\quad + (a_1^2 X_1^4 + a_2^2 X_2^4)\left(2k_1 X_1^{2k_1-1} X_2^{2k_2} - 2k_2 \frac{a_1^2}{a_2^2} X_1^{2k_1} X_2^{2k_2-1}\right) \\
&= -2a_1^2 X_1^{2k_1-1} X_2^{2k_2-1}(X_1 - X_2) \\
&\quad \times \left\{\left((k_1 - 3)X_1^4 + (k_2 - 3)X_2^4\right) + 3(X_1 - X_2)^4 + 10(X_1 - X_2)^2\right\}.
\end{aligned}$$

Hence, if $k_1 \geq 3$, $k_2 \geq 3$, then a critical point of $D|_P$ is only $X_1 = 1$ in P. Furthermore, we get

$$D|_P \to 0 \text{ as } X_1 \to 0 \text{ or } \frac{1}{a_1^2}.$$

Thus $\max\{D(X_1, X_2) \mid (X_1, X_2) \in P\} = D(1,1) = 1$. Hence $D \leq 1$. This implies $J(d\Phi_x) \leq 1$. Therefore if $k_1 \geq 3, k_2 \geq 3$, Φ is area nonincreasing. □

Remark 5.2. In 1969, Bombieri, DeGiorgi and Giusti [1] showed that the cone over $S^k \times S^k \subset S^{2k+1} (k \geq 3)$ is area-minimizing. On the other hand, Lawlor [9] proved that the cone over $S^{k_1} \times S^{k_2} \subset S^{k_1+k_2+1}$ are not area-minimizing when $k_1 + k_2 \leq 5$ or $k_1 = 1, k_2 = 5$. Hence, we need the condition $k_1 \geq 3, k_2 \geq 3$ in Theorem 5.1.

Remark 5.3. Area-nonincreasing retractions which we constructed in Section 4 satisfy the assumption of Theorem 5.1. Moreover, an area-nonincreasing retraction that is constructed using Theorem 5.1 satisfies the assumption of Theorem 5.1 again. Therefore, we can apply Theorem 5.1 inductively. This implies that the cone over the k-product of R-spaces ($k \geq 2$) with "$\bigcirc$" in the table in Section 4 is area-minimizing.

Acknowledgements. The second author was partly supported by the Grant-in-Aid for Science Research (C) No. 17K05223, JSPS.

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