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Complex hyperbolic triangle groups of type $(n, n, \infty; k)$

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Abstract. A complex hyperbolic triangle group is a group generated by three complex reflections fixing complex geodesics in complex hyperbolic space. We give a list of non-discrete groups of type $(n, n, \infty; k)$. In particular we show that if $n \geq 14$, then complex hyperbolic triangle groups of type $(n, n, \infty; k)$ are not discrete. This is a survey of our results.

1. Introduction

A complex hyperbolic triangle is a triple (C_1, C_2, C_3) of complex geodesics in complex hyperbolic 2-space $\mathrm{H}^2_{\mathbf{C}}$. We assume that C_k and C_{k-1} either meet at the angle π/p_k for some integer $p_k \geq 3$ or else C_k and C_{k-1} are asymptotic, in which case they make an angle 0 and in this case we write $p_k = \infty$, where the indices are taken mod 3. Let Γ be the group of holomorphic isometries of $\mathrm{H}^2_{\mathbf{C}}$ generated by complex reflections i_1, i_2, i_3 fixing complex geodesics C_1, C_2, C_3, respectively. We call Γ a *complex hyperbolic triangle group*. We can index a complex hyperbolic triangle group by a triple (p_1, p_2, p_3). A group Γ with (p_1, p_2, p_3) is said to be a *complex hyperbolic triangle group of type* (p_1, p_2, p_3), which is denoted by $\Gamma(p_1, p_2, p_3)$. In the real hyperbolic space $\mathrm{H}^2_{\mathbf{R}}$, (p_1, p_2, p_3) determines a unique triangle group (see [1]). On the other hand, the situation in $\mathrm{H}^2_{\mathbf{C}}$ is much different. Actually, for each such triple there is a one real parameter family of complex hyperbolic triangle groups. It is interesting to ask which values of this parameter correspond to discrete groups.

Complex hyperbolic triangle groups were investigated by Picard in the 1880s. Strongly influenced by Picard, Giraud developed the theory of complex hyperbolic space in his work from 1915 to 1921. In 1980, Mostow studied these groups in [17], where he constructed the first non-arithmetic lattices in PU(2,1). The deformation theory of complex hyperbolic triangle groups was begun in [5], where complex hyperbolic triangle groups of type (∞, ∞, ∞) were discussed by using the Cartan's

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angular invariant as the parameter. Since then there have been many developments. Complex hyperbolic triangle groups are the simplest groups, but even in this case we know only a small number of discrete groups. Generally speaking, it is not easy to show a group to be discrete. It seems that this will be done in a case by case fashion. Therefore, we try to restrict the range to search for discrete groups by finding non-discrete groups.

In this paper we restrict our attention to complex hyperbolic triangle groups of type (n, n, ∞) and give a list of non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$. In particular, we show that complex hyperbolic triangle groups of type $(n, n, \infty; k)$ are not discrete for $n \geq 14$.

We now describe the main results of this paper.

Theorem 1.1. *Let* $\Gamma = < i_1, i_2, i_3 >$ *be a complex hyperbolic triangle group of type* $(n, n, \infty; k)$ *with* $k \geq [n/2] + 1$. *The following groups are non-discrete.*

(1) $\Gamma(5, 5, \infty; 3)$.

(2) $\Gamma(6, 6, \infty; 5)$.

(3) $\Gamma(7, 7, \infty; 4), \Gamma(7, 7, \infty; 5), \Gamma(7, 7, \infty; 6)$.

(4) $\Gamma(8, 8, \infty; 5), \Gamma(8, 8, \infty; 7)$.

(5) $\Gamma(9, 9, \infty; k)$ *for* $5 \leq k \leq 8$,

(6) $\Gamma(10, 10, \infty; k)$ *for* $6 \leq k \leq 9$.

(7) $\Gamma(11, 11, \infty; k)$ *for* $6 \leq k \leq 11$.

(8) $\Gamma(12, 12, \infty; k)$ *for* $7 \leq k \leq 16$.

(9) $\Gamma(13, 13, \infty; k)$ *for* $7 \leq k \leq 38$.

(10) $\Gamma(14, 14, \infty; k)$ *for* $k \geq 8$.

(11) $\Gamma(n, n, \infty; k)$ *for any* n (≥ 15).

Corollary 1.2. *If* $\Gamma(n, n, \infty)$ *has type B, then* $\Gamma(n, n, \infty; k)$ *is not discrete.*

2. Preliminaries

We recall some basic notions of complex hyperbolic geometry. Let $\mathbf{C}^{2,1}$ be the complex vector space of dimension 3, equipped with the Hermitian form

$$< Z, W >= Z_1\overline{W}_1 + Z_2\overline{W}_2 - Z_3\overline{W}_3,$$

where $Z = (Z_1, Z_2, Z_3), W = (W_1, W_2, W_3) \in \mathbf{C}^{2,1}$. We call a vector $Z \in \mathbf{C}^{2,1}$ *negative* (respectively *null, positive*) if $< Z, Z >< 0$ (respectively $< Z, Z >= 0$, $< Z, Z >> 0$). Let $\pi : \mathbf{C}^{2,1} - \{0\} \to \mathrm{P}^2_{\mathbf{C}}$ (complex projective space) be the projection map defined by $\pi((Z_1, Z_2, Z_3)) = (Z_1/Z_3, Z_2/Z_3)$. The *complex hyperbolic 2-space* $\mathrm{H}^2_{\mathbf{C}}$ is defined as complex projectivization of the set of negative vectors in $\mathbf{C}^{2,1}$. Let $\mathrm{PU}(2,1)$ be the projectivization of $\mathrm{SU}(2,1)$, that is the group of matrices with determinant 1 which are unitary with respect to the Hermitian form. Non-trivial elements in $\mathrm{PU}(2,1)$ fall into three conjugacy classes, depending on the location and the number of fixed points. An element g is *elliptic* if it has a fixed point in $\mathrm{H}^2_{\mathbf{C}}$, *parabolic* if it has a unique fixed point on the boundary $\partial\mathrm{H}^2_{\mathbf{C}}$, *loxodromic* if it fixes a unique pair of points on $\partial\mathrm{H}^2_{\mathbf{C}}$. Furthermore, we say that an elliptic element g is *regular elliptic* if and only if its eigenvalues are distinct. A parabolic element g is *unipotent* if all eigenvalues of g are 1. Using the discriminant function

$$f(\tau) = |\tau|^4 - 8Re(\tau^3) + 18|\tau|^2 - 27,$$

we can classify elements of $\mathrm{PU}(2,1)$ by traces of the corresponding matrices in $\mathrm{SU}(2,1)$. In [4, Theorem 6.2.4] Goldman states that an element g in $\mathrm{SU}(2,1)$ is regular elliptic if and only if $f(\tau(g)) < 0$, where $\tau(g)$ is the trace of g.

The intrinsic metric on $\mathrm{H}^2_{\mathbf{C}}$ is the Bergman metric. For any pair of points z, w in $\mathrm{H}^2_{\mathbf{C}}$, the *complex hyperbolic distance* $d(z, w)$ is given by:

$$\cosh^2\left(\frac{d(z,w)}{2}\right) = \frac{< Z, W >< W, Z >}{< Z, Z >< W, W >},$$

where Z and W are arbitrary lifts of z and w. We see that the group of holomorphic isometries of $\mathrm{H}^2_{\mathbf{C}}$ is exactly $\mathrm{PU}(2,1)$.

The boundary $\partial\mathrm{H}^2_{\mathbf{C}}$ is homeomorphic to S^3 by the standard stereographic projection and one of representation we choose for this is $(\mathbf{C} \times \mathbf{R}) \cup \{\infty\}$, with points either ∞ or $(\zeta, v)_H$ with $\zeta \in \mathbf{C}$ and $v \in \mathbf{R}$, where $(0, -1, 1) \in \mathbf{C}^{2,1}$ corresponds to ∞. We call $(\zeta, v)_H$ the $H-coordinates$. Let H denote this representation, that is,

$(\mathbf{C}\times\mathbf{R})\cup\{\infty\}$. We define the *Cygan metric* δ by

$$\delta((\zeta,v)_H,(\tau,u)_H) = ||\zeta-\tau|^2 + iv - iu + 2i\mathrm{Im}(\zeta\overline{\tau})|^{\frac{1}{2}}$$

for $(\zeta,v)_H,(\tau,u)_H$ in $H-\{\infty\}$. This metric is thought as the counterpart of the Euclidean metric.

In $\mathrm{H}^2_{\mathbf{C}}$ there are two kinds of totally geodesic subspaces, *totally real totally geodesic subspaces* and *totally geodesic complex subspaces*. The former is isometric to $\mathrm{H}^2_{\mathbf{C}}\cap\mathbf{R}^2$. The latter is isometric to $\mathrm{H}^2_{\mathbf{C}}\cap\mathbf{C}$, which is called a *complex geodesic.* A complex geodesic C is uniquely determined by a positive vector $V\in\mathbf{C}^{2,1}$, that is, $C=\pi(\{U\in\mathbf{C}^{2,1}|<U,V>=0\})$. We call V a *polar vector* to C. Two distinct complex geodesics in $\mathrm{H}^2_{\mathbf{C}}$ intersect in either the empty set or a point. Let C_1 and C_2 be distinct complex geodesics corresponding to polar vectors $V_1,V_2\in\mathbf{C}^{2,1}$, respectively. At a point of intersection, C_1 and C_2 intersect at the *complex angle* ϕ, which is defined as

$$\cos\phi = \frac{|<V_1,V_2>|}{\sqrt{<V_1,V_1><V_2,V_2>}}.$$

For a triple $x=(x_1,x_2,x_3)$ of distinct points in $\partial\mathrm{H}^2_{\mathbf{C}}$ the *Cartan angular invariant* is defined by

$$A(x) = arg\left(-<X_1,X_2><X_2,X_3><X_3,X_1>\right),$$

where $X_i\in\mathbf{C}^{2,1}$ of a lift of x_i with $\pi(X_i)=x_i$. The Cartan angular invariant is independent of chosen lifts. We define the *angular invariant* $T(c)$ of a triple $c=(C_1,C_2,C_3)$ of distinct complex geodesics by

$$T(c) = arg\left(<V_3,V_2><V_1,V_3><V_2,V_1>\right),$$

where V_i is the normalized polar vector to C_i. Both angular invariants are invariant under PU(2,1) and there is a close relation between them (see [23]). A complex hyperbolic triangle in $\mathrm{H}^2_{\mathbf{C}}$ is determined uniquely up to isometry by the three angles and the angular invariant $T(c)$.

Given a complex geodesic C with polar vector V, there is a unique involution i, that fixes every point in C. We call i the *complex reflection* in C. Explicitly i is given by

$$i(Z) = -Z + \frac{2<Z,V>}{<V,V>}V.$$

Details for background material on complex hyperbolic space will be found in [1], [2] and [4]. For material on complex hyperbolic triangle groups see [17] and [27].

3. Complex hyperbolic triangle groups of type (n, n, ∞)

In [25] Schwartz considered ideal triangle groups, that is complex hyperbolic triangle groups of type (∞, ∞, ∞) and proved that if the product $i_1 i_2 i_3$ of generators is regular elliptic, then it is not of finite order, hence the corresponding complex hyperbolic triangle group is not discrete. In [20] Parker explored groups of type (n, n, n) such that $i_1 i_2 i_3$ is regular elliptic. In this case there are some discrete groups. And he classified them. In the same manner as in the proof of Schwartz in [25], Wyss-Gallifent formulated Schwartz's statement for groups of type (n, n, ∞) in [29]. In [24] Pratoussevitch made a refinement on the proof of Wyss-Gallifent. Here we show the result due to Wyss-Gallifent and Pratoussevitch.

LEMMA 3.1. *Let $\Gamma =< i_1, i_2, i_3 >$ be a complex hyperbolic triangle group of type (n, n, ∞). If the product $i_1 i_2 i_3$ of the three generators is regular elliptic, then Γ is non-discrete.*

By conjugation, we may assume that the forms of i_j as follows:

$$i_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$i_2 = \begin{bmatrix} 1 & -2s & -2s \\ -2s & 2s^2-1 & 2s^2 \\ 2s & -2s^2 & -2s^2-1 \end{bmatrix} \text{ and}$$

$$i_3 = \begin{bmatrix} 1 & -2se^{i\theta} & -2se^{i\theta} \\ -2se^{-i\theta} & 2s^2-1 & 2s^2 \\ 2se^{-i\theta} & -2s^2 & -2s^2-1 \end{bmatrix},$$

where $s = \cos(\pi/n)$. In this case the angular invariant $T(c) = \theta$. We take $\cos\theta$ as the parameter of complex hyperbolic triangle groups of type (n, n, ∞). A simple computation yields $\mathrm{trace}(i_1 i_2 i_3) = 8s^2(e^{i\theta} - 1) - 1$. Using the discriminant function $f(\tau)$ and Lemma 3.1, we work out some conditions on $\cos\theta$ for Γ of type (n, n, ∞)

to be non-discrete. We see that if $n < 9$, then the product $i_1 i_2 i_3$ is not regular elliptic and that if $n \geq 9$, then there are two positive numbers α_n and β_n such that $i_1 i_2 i_3$ is regular elliptic for $\cos\theta \in (\alpha_n, \beta_n)$. Note that α_n and β_n are increasing functions of n. By Lemma 3.1, we obtain

THEOREM 3.2. *If $n \geq 9$, then $\Gamma(n,n,\infty)$ is not discrete for $\cos\theta \in (\alpha_n, \beta_n)$.*

Next we use a complex hyperbolic version of Jørgensen's inequality to find out some sufficient conditions on $\cos\theta$ for Γ to be non-discrete. Let g be an element of $\mathrm{PU}(2,1)$. We define the translation length $t_g(p)$ of g at $p \in H$ by $t_g(p) = \delta(g(p), p)$. To state Lemma 3.3, we need the notion of isometric spheres. Let h be an element of $\mathrm{PU}(2,1)$ not fixing ∞. The *isometric sphere* of h is the sphere in the Cygan metric with center $h^{-1}(\infty)$ and radius R_h, where $R_h^2 = \delta(h(z), h(\infty))\delta(z, h^{-1}(\infty))$ for any $z \in H - \{\infty, h^{-1}(\infty)\}$ (see [4], [7] and [8]).

Here we recall a complex hyperbolic version of Shimizu's lemma due to Parker.

LEMMA 3.3. *Let G be a discrete subgroup of $\mathrm{PU}(2,1)$ that contains the unipotent parabolic element g with the form*

$$g = \begin{bmatrix} 1 & \tau & \tau \\ -\overline{\tau} & 1-(|\tau|^2 - it)/2 & -(|\tau|^2 - it)/2 \\ \overline{\tau} & (|\tau|^2 - it)/2 & 1+(|\tau|^2 - it)/2 \end{bmatrix}.$$

The element g fixes ∞ and maps the point with H-coordinates $(\zeta, v)_H$ to the point with H-coordinates $(\zeta + \tau, v + t + 2\mathrm{Im}(\tau\overline{\zeta}))_H$. Let h be any element of G not fixing ∞ and with isometric sphere of radius R_h. Then

$$R_h^2 \leq t_g(h^{-1}(\infty)) t_g(h(\infty)) + 4|\tau|^2.$$

Taking $g = i_{23}$ and $h = i_{1231}$, we apply the contraposition of Lemma 3.3 to $\Gamma(n,n,\infty)$. It follows that there is a positive number γ_n such that $R_h^2 > t_g(h^{-1}(\infty)) t_g(h(\infty)) + 4|\tau|^2$ for $\cos\theta$ with $\gamma_n < \cos\theta < 1$. We have

THEOREM 3.4. *$\Gamma(n,n,\infty)$ is not discrete for $\cos\theta \in (\gamma_n, 1)$.*

Numerical values of α_n, β_n and γ_n can be found in [14]. We note that $\alpha_n < \beta_n < \gamma_n$ for $9 \leq n \leq 28$ and $\alpha_n < \gamma_n < \beta_n$ for $n \geq 29$.

4. Complex hyperbolic triangle groups of type $(n, n, \infty; k)$

Let $\Gamma =< i_1, i_2, i_3 >$ be a complex hyperbolic triangle group of type (n, n, ∞). If the trace of the element $i_1 i_2 i_1 i_3$ is equal to $1 + 2\cos\frac{2\pi}{k}$, where k is a positive integer ≥ 3, then Γ is said to be of *type* $(n, n, \infty; k)$. This group is denoted by $\Gamma(n, n, \infty; k)$. We note that $k \geq [n/2] + 1$. There is a positive number δ_n such that the element $i_1 i_2 i_1 i_3$ in $\Gamma(n, n, \infty)$ is regular elliptic for $\cos\theta > \delta_n$. In [27] Schwartz classified complex hyperbolic triangle groups into two types. It is said that $\Gamma(n, n, \infty)$ has *type B* if there is a positive number k_0 such that $i_1 i_2 i_3$ becomes regular elliptic for $k > k_0$. It is seen that $\delta_n < \alpha_n < \beta_n$ for $9 \leq n \leq 13$ and $\alpha_n < \delta_n < \beta_n$ for $n \geq 14$. Therefore, if $n \geq 14$, then $\Gamma(n, n, \infty)$ has type B.

In this section we discuss complex hyperbolic triangle groups of type $(n, n, \infty; k)$. We have

$$\begin{aligned} trace(i_1 i_2 i_1 i_3) &= 3 - 16s^2\cos\theta + 16s^4 \\ &= 1 + 2\cos\frac{2\pi}{k}. \end{aligned}$$

Considering the intervals (α_n, β_n) and $(\gamma_n, 1)$ in Theorems 3.2 and 3.4, we can deduce which values k correspond to non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$. It is seen that the following groups are not discrete:

$\Gamma(10, 10, \infty; 9); \Gamma(11, 11, \infty; 10), \Gamma(11, 11, \infty; 11);$

$\Gamma(12, 12, \infty; k)$ for $11 \leq k \leq 16$;

$\Gamma(13, 13, \infty; 7)$ and $\Gamma(13, 13, \infty; k)$ for $12 \leq k \leq 38$;

$\Gamma(14, 14, \infty; k)$ for $k \geq 12$;

$\Gamma(15, 15, \infty; 8)$ and $\Gamma(15, 15, \infty; k)$ for $k \geq 13$;

$\Gamma(16, 16, \infty; 9)$ and $\Gamma(16, 16, \infty; k)$ for $k \geq 14$;

$\Gamma(17, 17, \infty; 9)$ and $\Gamma(17, 17, \infty; k)$ for $k \geq 15$;

$\Gamma(18, 18, \infty; 10)$ and $\Gamma(18, 18, \infty; k)$ for $k \geq 16$;

$\Gamma(19,19,\infty;10),\Gamma(19,19,\infty;11)$ and $\Gamma(19,19,\infty;k)$ for $k\geq 17$;

$\Gamma(20,20,\infty;11),\Gamma(20,20,\infty;12)$ and $\Gamma(20,20,\infty;k)$ for $k\geq 18$;

$\Gamma(21,21,\infty;11),\Gamma(21,21,\infty;12),\Gamma(21,21,\infty;13)$ and $\Gamma(21,21,\infty;k)$ for $k\geq 19$;

$\Gamma(22,22,\infty;12),\Gamma(22,22,\infty;13),\Gamma(22,22,\infty;14)$ and $\Gamma(22,22,\infty;k)$ for $k\geq 19$;

$\Gamma(23,23,\infty;12),...,\Gamma(23,23,\infty;15)$ and $\Gamma(23,23,\infty;k)$ for $k\geq 20$;

$\Gamma(24,24,\infty;13),...,\Gamma(24,24,\infty;16)$ and $\Gamma(24,24,\infty;k)$ for $k\geq 21$;

$\Gamma(25,25,\infty;13),...,\Gamma(25,25,\infty;17)$ and $\Gamma(25,25,\infty;k)$ for $k\geq 22$;

$\Gamma(26,26,\infty;14),...,\Gamma(26,26,\infty 19)$ and $\Gamma(26,26,\infty;k)$ for $k\geq 23$;

$\Gamma(27,27,\infty;14),...,\Gamma(27,27,\infty;21)$ and $\Gamma(27,27,\infty;k)$ for $k\geq 24$;

$\Gamma(28,28,\infty;15),...,\Gamma(28,28,\infty;23)$ and $\Gamma(28,28,\infty;k)$ for $k\geq 25$;

$\Gamma(29,29,\infty;k)$ for any $k(\geq 15)$;

$\Gamma(n,n,\infty;k)$ for any $n(>29)$ and $k(\geq [n/2]+1)$.

To find more non-discrete complex hyperbolic triangle groups of type $(n,n,\infty;k)$, we use another complex hyperbolic version of Jørgensen's inequality (see [6] and [28]).

Lemma 4.1. *Let $g\in \mathrm{PU}(2,1)$ be a regular elliptic element of order $m\geq 7$ that preserves a Lagrangian plane (i.e. trace(g) is real). Suppose that g fixes a point $z\in \mathrm{H}^2_{\mathbf{C}}$. Let h be any element of $\mathrm{PU}(2,1)$ with $h(z)\neq z$. If*

$$\cosh\left(\frac{d(h(z),z}{2}\right)\sin\left(\frac{\pi}{m}\right)<\frac{1}{2},$$

then $<g,h>$ is not discrete.

Taking $g=i_1i_2$ and $h=i_3$ in Lemma 4.1, we obtain

THEOREM 4.2. *Let Γ be a complex hyperbolic triangle group of type $(n, n, \infty; k)$ with $n \geq 7$. Let*

$$a_n = -1 + 8\cos^4(\pi/n) - 6\cos^2(\pi/n) - \sin(\pi/n)$$

and

$$b_n = -1 + 8\cos^4(\pi/n) - 6\cos^2(\pi/n) + \sin(\pi/n).$$

If $a_n < \cos(2\pi/k) < b_n$, then Γ is not discrete.

By Theorem 4.2, we see that the following groups are not discrete:

$\Gamma(11, 11, \infty; 6)$; $\Gamma(12, 12, \infty; 7)$;

$\Gamma(13, 13, \infty; 7)$; $\Gamma(14, 14, \infty; 8)$;

$\Gamma(15, 15, \infty; 8)$ and $\Gamma(15, 15, \infty; 9)$;

$\Gamma(16, 16, \infty; 9)$ and $\Gamma(16, 16, \infty; 10)$;

$\Gamma(17, 17, \infty; 9), \Gamma(17, 17, \infty; 10)$, and $\Gamma(17, 17, \infty; 11)$;

$\Gamma(18, 18, \infty; 10), \Gamma(18, 18, \infty; 11)$, and $\Gamma(18, 18, \infty; 12)$;

$\Gamma(19, 19, \infty; k)$ for $10 \leq k \leq 13$;

$\Gamma(20, 20, \infty; k)$ for $11 \leq k \leq 15$;

$\Gamma(21, 21, \infty; k)$ for $11 \leq k \leq 16$;

$\Gamma(22, 22, \infty; k)$ for $12 \leq k \leq 18$;

$\Gamma(23, 23, \infty; k)$ for $12 \leq k \leq 20$;

$\Gamma(24, 24, \infty; k)$ for $13 \leq k \leq 22$;

$\Gamma(25, 25, \infty; k)$ for $13 \leq k \leq 25$;

$\Gamma(26, 26, \infty; k)$ for $14 \leq k \leq 29$;

$\Gamma(27, 27, \infty; k)$ for $14 \leq k \leq 33$;

$\Gamma(28, 28, \infty; k)$ for $15 \leq k \leq 40$.

Now we show a different way to find non-discrete groups. It is well-known that if a group has an elliptic element of infinite order, then this group is not discrete.

Lemma 4.3. *Let g be an element of $\Gamma(n, n, \infty; k)$. If trace(g) is real and contained in $(-1, 3)$, then g is regular elliptic and trace$(g) = 1 + 2\cos\phi\pi$. Moreover, g has finite order if and only if ϕ is a rational number.*

Conway and Jones list all possible trigonometric Diophantine equations with up to four terms in [3]. We omit the detail, which can be found in [16]. To find elliptic element of infinite order in a group, we use the following result due to Parker, which extends the result of Conway and Jones.

Lemma 4.4. *Suppose that we have at most six distinct rational multiples of π lying strictly between 0 and $\pi/2$, for which some rational linear combination of their cosines is zero but no proper subset has this property, then the appropriate linear combination is proportional to one of the following:*

$$0 = \sum_{k=0}^{2} \cos(\phi + \frac{2k\pi}{3}), \ \ \phi \in (0, \pi), \ \ \phi \neq \frac{m\pi}{6};$$
$$0 = \sum_{k=0}^{4} \cos(\phi + \frac{2k\pi}{5}), \ \ \phi \in (0, \pi), \ \ \phi \neq \frac{n\pi}{10};$$
$$0 = \sum_{k=1}^{2} \cos(\phi + \frac{2k\pi}{3}) - \sum_{k=1}^{4} \cos(\phi + \frac{2k\pi}{5}), \ \ \phi \in (0, \pi), \ \ \phi \neq \frac{m\pi}{6}, \ \ \phi \neq \frac{n\pi}{10};$$
$$0 = \cos\frac{\pi}{3} - \cos\frac{\pi}{5} + \cos\frac{2\pi}{5};$$
$$0 = \cos\frac{\pi}{3} - \cos\frac{\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7};$$
$$0 = \cos\frac{\pi}{3} - \cos\frac{\pi}{11} + \cos\frac{2\pi}{11} - \cos\frac{3\pi}{11} + \cos\frac{4\pi}{11} - \cos\frac{5\pi}{11};$$
$$0 = \cos\frac{\pi}{3} - \cos\frac{\pi}{5} + \cos\frac{\pi}{15} - \cos\frac{4\pi}{15};$$

$$0 = \cos\frac{\pi}{3} + \cos\frac{2\pi}{5} - \cos\frac{2\pi}{15} + \cos\frac{7\pi}{15};$$

$$0 = \cos\frac{\pi}{3} - \cos\frac{\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{2\pi}{21} + \cos\frac{5\pi}{21};$$

$$0 = \cos\frac{\pi}{3} - \cos\frac{\pi}{7} - \cos\frac{3\pi}{7} + \cos\frac{\pi}{21} - \cos\frac{8\pi}{21};$$

$$0 = \cos\frac{\pi}{3} + \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{4\pi}{21} - \cos\frac{10\pi}{21};$$

$$0 = \cos\frac{\pi}{3} - \cos\frac{\pi}{7} + \cos\frac{\pi}{21} - \cos\frac{2\pi}{21} + \cos\frac{5\pi}{21} - \cos\frac{8\pi}{21};$$

$$0 = \cos\frac{\pi}{3} + \cos\frac{2\pi}{7} - \cos\frac{2\pi}{21} - \cos\frac{4\pi}{21} + \cos\frac{5\pi}{21} - \cos\frac{10\pi}{21};$$

$$0 = \cos\frac{\pi}{3} - \cos\frac{3\pi}{7} + \cos\frac{\pi}{21} - \cos\frac{4\pi}{21} - \cos\frac{8\pi}{21} - \cos\frac{10\pi}{21};$$

$$0 = \cos\frac{\pi}{5} - \cos\frac{2\pi}{5} - \cos\frac{\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7};$$

$$0 = \cos\frac{\pi}{5} - \cos\frac{2\pi}{5} - \cos\frac{\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{2\pi}{21} + \cos\frac{5\pi}{21};$$

$$0 = \cos\frac{\pi}{5} - \cos\frac{2\pi}{5} - \cos\frac{\pi}{7} - \cos\frac{3\pi}{7} + \cos\frac{2\pi}{21} - \cos\frac{8\pi}{21};$$

$$0 = \cos\frac{\pi}{5} - \cos\frac{2\pi}{5} + \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{4\pi}{21} - \cos\frac{10\pi}{21}.$$

Assume that $i_2 i_1 i_2 i_3$ is a regular elliptic element in $\Gamma(n, n, ; \infty; k)$. Then $trace(i_2 i_1 i_2 i_3)$ is written as

$$trace(i_2 i_1 i_2 i_3) = 20s^2 - 16s^2\cos\theta - 1 = 1 + 2\cos\phi\pi,$$

which yields that

$$\cos\phi\pi = 10s^2 - 8s^2\cos\theta - 1,$$

where ϕ is a real number. We obtain

$$\cos\phi\pi = -8s^4 + 10s^2 - 2 + \cos\frac{2\pi}{k} = -\cos\frac{4\pi}{n} + \cos\frac{2\pi}{n} + \cos\frac{2\pi}{k}.$$

It is seen that in each group of type $(5,5,\infty;3)$, $(7,7,\infty;4)$, $(9,9,\infty;5)$, $(11,11,\infty;6)$, $(12,12,\infty;7)$, $(13,13,\infty;7)$ or $(14,14,\infty;8)$, $i_2i_1i_2i_3$ is regular elliptic. Lemma 4.4 tells us that for $(n,k)=(5,3)$, $(7,4)$, $(9,5)$, $(11,6)$, $(12,7)$, $(13,7)$ and $(14,8)$, there are no rational numbers $\phi's$ satisfying

$$\cos\phi\pi = -\cos\frac{4\pi}{n} + \cos\frac{2\pi}{n} + \cos\frac{2\pi}{k}.$$

It follows that in the groups above, $i_2i_1i_2i_3$ is a regular elliptic element of infinite order. Therefore, the groups $\Gamma(5,5,\infty;3)$, $\Gamma(7,7,\infty;4)$, $\Gamma(9,9,\infty;5)$, $\Gamma(11,11,\infty;6)$, $\Gamma(12,12,\infty;7)$, $\Gamma(13,13,\infty;7)$ and $\Gamma(14,14,\infty;8)$ are not discrete.

Next consider elements $i_1i_2i_1i_2i_3i_2$ and $i_3i_1i_3i_1i_2i_1$. In the same manner as above, we see that in $\Gamma(8,8,\infty;5)$, $i_1i_2i_1i_2i_3i_2$ is a regular elliptic element of infinite order. Hence $\Gamma(8,8,\infty;5)$ is not discrete. Moreover, $i_3i_1i_3i_1i_2i_1$ is a regular elliptic element of infinite order in the following groups.

$\Gamma(6,6,\infty;5)$; $\Gamma(7,7,\infty;5), \Gamma(7,7,\infty;6)$; $\Gamma(8,8,\infty;7)$;

$\Gamma(9,9,\infty;6), \Gamma(9,9,\infty;7), \Gamma(9,9,\infty;8)$; $\Gamma(10,10,\infty;6)$;

$\Gamma(11,11,\infty;7), \Gamma(11,11,\infty;8), \Gamma(11,11,\infty;9)$;

$\Gamma(12,12,\infty;8), \Gamma(12,12,\infty;9), \Gamma(12,12,\infty;10)$;

$\Gamma(13,13,\infty;8), \Gamma(13,13,\infty;9), \Gamma(13,13,\infty;10), \Gamma(13,13,\infty;11)$;

$\Gamma(14,14,\infty;9), \Gamma(14,14,\infty;10), \Gamma(14,14,\infty;11)$;

$\Gamma(15,15,\infty;10), \Gamma(15,15,\infty;11), \Gamma(15,15,\infty;12)$;

$\Gamma(16,16,\infty;11), \Gamma(16,16,\infty;12), \Gamma(16,16,\infty;13)$;

$\Gamma(17,17,\infty;12), \Gamma(17,17,\infty;13), \Gamma(17,17,\infty;14)$;

$\Gamma(18,18,\infty;13), \Gamma(18,18,\infty;14), \Gamma(18,18,\infty;15)$;

$\Gamma(19,19,\infty;14), \Gamma(19,19,\infty;15), \Gamma(19,19,\infty;16)$;

$\Gamma(20,20,\infty;16), \Gamma(20,20,\infty;17)$;

$\Gamma(21, 21, \infty; 17), \Gamma(21, 21, \infty; 18)$.

Thus it is seen that the groups above are not discrete.

5. Proof of Theorem 1.1

We have only to summarize what we have showed so far. Here we give the outline of our proof only in the following four cases, but we can treat the other cases in the same way as in one of them.

First we consider the case where $n = 13$. Note that $\delta_{13} < \alpha_{13} < \beta_{13}$, which is different from the three cases below. Theorem 3.4 shows that $\Gamma(13, 13, \infty; 7)$ is not discrete. By Theorem 3.2 we see that $\Gamma(13, 13, \infty; k)$ is not discrete for $12 \leq k \leq 38$. It follows from Lemma 4.4 that i_{313121} is a regular elliptic element of infinite order for $8 \leq k \leq 11$. Hence $\Gamma(13, 13, \infty; k)$ is not discrete for $7 \leq k \leq 38$..

If $n = 14$, then $\alpha_{14} < \delta_{14} < \beta_{14}$. Theorem 3.2 shows that $\Gamma(14, 14, , \infty; k)$ is not discrete for $k \geq 12$. Theorem 3.4 implies that $\Gamma(14, 14, , \infty; 8)$ is not discrete. Consider the traces of i_{313121} in $\Gamma(14, 14, \infty; 9), \Gamma(14, 14, \infty; 10)$ or $\Gamma(14, 14, \infty; 11)$. By Lemma 4.4, we see that in these three groups i_{313121} is an elliptic element of infinite order. Therefore $\Gamma(14, 14, \infty; 9), \Gamma(14, 14, \infty; 10)$ and $\Gamma(14, 14, \infty; 11)$ are not discrete. Hence $\Gamma(14, 14, \infty; k)$ is not discrete for any $k \geq 8$.

For $n = 22$, $\alpha_{22} < \delta_{22} < \beta_{22}$. Theorems 3.2 and 3.4 show that $\Gamma(22, 22, \infty; k)$ is not discrete for $k \geq 19$. It follows from Theorem 4.2 that $\Gamma(22, 22, \infty; k)$ is not discrete for $12 \leq k \leq 18$. Hence $\Gamma(22, 22, \infty; k)$ is not discrete for any $k \geq 12$.

Finally we consider the case where $n \geq 29$. In this case, $\alpha_n < \delta_n < \gamma_n < \beta_n$. It follows from Theorems 3.2 and 3.4 that $\Gamma(n, n, \infty; k)$ is not discrete for $n \geq 29$.

Thus we have Theorem 1.1.

Remark 5.1. *It is known that the following groups are discrete.*
$\Gamma(3, 3, \infty; k)$ *for any* k*;*
$\Gamma(4, 4, \infty; 3), \Gamma(4, 4, \infty; 4), \Gamma(4, 4, \infty; 6), \Gamma(4, 4, \infty; \infty)$;
$\Gamma(6, 6, \infty; 4), \Gamma(6, 6, \infty; 6), \Gamma(6, 6, \infty; \infty)$
(see [14], [22]).

6. Problems

Schwartz has given a conjectural overview on complex hyperbolic triangle groups in [27]. We can find many conjectures and open problems on complex hyperbolic triangle groups in [21], [23] and [26]. As we are particularly concerned with complex hyperbolic triangle groups of type (n, n, ∞), we give some problems only on them.

(1) Complete the list of Theorem 1.1.
(2) Suppose neither $i_1 i_2 i_3$ nor $i_1 i_2 i_1 i_3$ is elliptic in $\Gamma(n, n, \infty)$.
Is this group discrete?
(3) Find a new discrete complex hyperbolic triangle group of type (n, n, ∞).

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