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Variations of complex and hyperbolic structures on Riemann surfaces –a comparative viewpoint–

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Abstract. A Riemann surface of higher genus has two important geometric structures; the complex structure and the hyperbolic metric. The Teichmüller space of Riemann surfaces hence can be regarded as a catalogue of both complex structures and hyperbolic metrics. In this article, we make a comparative study of these two characterizations of the Teichmüller space, by utilizing a natural L^2-product and a natural symplectic form defined on the space of complex structures, both of which behave nicely under the diffeomorphism group action.

1. Introduction

Otto Teichmüller was instrumental in making the theory of moduli on Riemann surfaces geometric. As it is well known, Riemann initiated the deformation theory of the Riemann surface, and it was then reinterpreted as the deformation theory of Fuchsian groups, initiated by Klein, Lie, Poincaré, and the investigation had matured as the representation theory. On the other hand, the global geometry of the moduli space of Riemannn surface had been left undeveloped for more than fifty years, until Teichmüller came with the idea of utilizing the theory of quasi-conformal mappings and holomorphic quadratic differentials. Shortly after the Second World War, A. Weil [15], along with L. Ahlfors, was aware of the importance of Teichmüller's work, and encouraged the development of the field. Ahlfors and L. Bers laid the foundation using the one-dimensional complex analysis. Note that the quasi-conformality is devoid of the metric structures on Riemann surfaces, only dependent of the complex/conformal structure of the surface. This approach was radically challenged in the 1970s, when W. Thurston appeared and demonstrated the effectiveness of the hyperbolic geometry in understanding the moduli theory of Riemann surface.

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It is ironic that the uniformization theorem of Poincaré and Koebe was left alone over many decades in this context. The hyperbolic geometry then (early 1970s) was considered as a "gadget", a term coined by V. Poénaru [13], something curious, but not a subject of essential importance and depth. The rest is history, as they say, and we are all aware of the subsequent fertile ground the hyperbolic geometry has provided to geometry and topology, perhaps most notably in higher dimensions in the context of the Geometrization Theorem conjectured by Thurston, completed by R. Hamilton and G. Perelman.

The goal of this article is to give a concise and clear perspective to the Teichmüller theory, with the convoluted history partially described above in mind. What is new here is based on a recent exposition of N. A'Campo [1] who has presented the Teichmüller space as a submanifold within the space of complex structures. It utilizes a symplectic structure defined on the deformation space of complex structures. The exposition is organized so that the hyperbolic geometry, and the consequent Weil-Petersson geometry of the Teichmüller space [18] is first presented, and then, in contrast, A'Campo's symplectic construction is explained.

2. Teichmüller Spaces of closed Riemann surfaces

Let Σ be a compact surface without boundary of genus $g \geq 1$ (when $g = 0$ the situation is very simple.) By the existence theorem of an isothermal coordinate system by Korn and Lichtenstein, any Riemannian metric g can be identified with a Riemann surface, namely a Riemannian surface is a Riemann surface. The universal covering space of the surface is either the whole plane or the upper half space, and thus the surface can be uniquely equipped with a Euclidean metric when $g = 1$ or a hyperbolic metric when $g > 1$. This is the statement of the Uniformization Theorem. Hence we can think of the space $\mathcal{M}_K$, $(K \equiv 0, -1)$ of constant curvature metrics as a subset of the space of smooth metrics $\mathcal{M}$ on Σ, the latter space being fibered by the elements of $\mathcal{M}_K$ so that each fiber consist of the metrics conformal to a constant curvature/uniformized metric $G \in \mathcal{M}$.

The Teichmüller space is then defined as the quotient space

$$P : \mathcal{M}_K \to \mathcal{M}_K/\mathrm{Diff}_0\Sigma =: \mathcal{T}_g$$

where the equivalence relation is given as

$$G_1 \sim G_2 \Leftrightarrow G_2 = \varphi^* G_1$$

for some φ in $\mathrm{Diff}_0\Sigma$. Here $\mathrm{Diff}_0\Sigma$ is the identity component of the orientation-preserving diffeomorphism group $\mathrm{Diff}\Sigma$. Recall that the map $\varphi : (\Sigma, \varphi^* G) \to (\Sigma, G)$ is an isometry. Note that in defining the identity element of $\mathrm{Diff}_0\Sigma$ one requires a reference Riemann surface (Σ_0, G_0) such that it acts as the domain of $\mathrm{Id} : \Sigma_0 \to \Sigma$. Namely (Σ_0, G_0) gives homotopy markings on the target surface.

By an important theorem of Earle-Eells [5], it is known that the identity component $\mathrm{Diff}_0\Sigma \subset \mathrm{Diff}\Sigma$ consists of diffeomorphisms homotopic to the identity map.

The moduli space $\mathfrak{M}_g$ is defined as

$$\tilde{P} : \mathcal{M}_K \to \mathcal{M}_K / \mathrm{Diff}\Sigma =: \mathfrak{M}_g$$

where the equivalence relation is given as

$$G_1 \sim G_2 \Leftrightarrow G_2 = \varphi^* G_1$$

for some φ in $\mathrm{Diff}\Sigma$. Thus the Teichmüller space projects down to the moduli space with the fibers identified with the discrete infinite group $\mathrm{Diff}\Sigma/\mathrm{Diff}_0\Sigma$, called mapping class group, or Teichmüller modular group. We denote this group by $\mathrm{Map}(\Sigma)$. We define now for a later use the full diffeomorphism group $\widehat{\mathrm{Diff}}\Sigma$ which, in addition to the elements of $\mathrm{Diff}\Sigma$, also contains the orientation-reversing diffeomorphisms of Σ. Then the quotient group $\widehat{\mathrm{Diff}}\Sigma/\mathrm{Diff}_0\Sigma$ is called the extended mapping class group $\widehat{\mathrm{Map}(\Sigma)}$.

3. Riemannian Structures of L^2-pairing

3.1. L^2-pairing and its Levi-Civita connection

3.1.1. L^2-pairing of deformation tensors

The tangent space $T_G\mathcal{M}$ of the space $\mathcal{M}$ at a metric G is the space of smooth symmetric $(0,2)$-tensors on Σ. This linear space has a natural L^2-pairing defined as follows.

$$\langle h_1, h_2 \rangle_{L^2(G)} = \int_\Sigma \langle h_1(x), h_2(x) \rangle_{G(x)} d\mu_G(x)$$

where the h_i's are symmetric $(0,2)$-tensors indicating the directions of deformation of G along the path $G + \varepsilon h_i + o(\varepsilon)$. The integrand can be rewritten, using a local coordinate chart, as

$$\begin{aligned} \langle h_1(x), h_2(x) \rangle_{G(x)} &= \sum_{1 \le i,j,k,l \le 2} G^{ij} G^{kl} (h_1)_{ik} (h_2)_{jl} \\ &= \mathrm{Tr}\Big((G^{-1} \cdot h_1) \cdot (G^{-1} \cdot h_2) \Big) \end{aligned}$$

where $A \cdot B$ denotes matrix multiplication and $\operatorname{Tr} A$ is the trace of the matrix A. This quantity is well defined, meaning it is invariant under change of coordinate charts. In particular it can be simplified by choosing a geodesic normal coordinate system where $G_{ij}(p) = \delta_{ij}$ at its center p as

$$\langle h_1(p), h_2(p)\rangle_{G(p)} = \sum_{j,k} (h_1)^j{}_k(p)(h_2)^k{}_j(p) \ \ (= \operatorname{Tr}(h_1 \cdot h_2))$$

the trace of the product of 2×2 matrices. From now on, we will use the Einstein notation of indices, omitting the summation symbols.

3.2. Tangential conditions and the Weil-Petersson metric

When G is a uniformizing metric of its conformal class, then the tangent space $T_G\mathcal{M}$ decomposes into the deformation of G preserving the constant curvature condition, and its complement. This can be formally stated as follows.

In dimension two, the Riemann curvature tensor is completely determined by one scalar function, the sectional curvature K. Then the Ricci curvature tensor is of the form

$$R_{ij} = KG_{ij}$$

namely G is an Einstein metric. The well-known variational formula (see [3]) of the Ricci tensor under a deformation $G + \varepsilon h$ at $\varepsilon = 0$ gives, after taking its trace:

$$G^{ij}\dot{R}_{ij} = -\triangle_G \mathrm{Tr}_G h + \delta_G \delta_G h.$$

Hence we have the following variational formula for the sectional curvature under the deformation of G in the direction of h:

$$\begin{aligned} \dot{K} &= G^{ij}\dot{R}_{ij} + \dot{G}^{ij}R_{ij} \\ &= G^{ij}\dot{R}_{ij} - h^{ij}KG_{ij} \\ &= -\triangle \operatorname{Tr}_G h + \delta_G\delta_G h - K \operatorname{Tr}_G h. \end{aligned}$$

We denote the quantity $-(\triangle_G + K)\operatorname{Tr}_G h + \delta_G\delta_G h$ by $\mathcal{L}_G h$, where the differential operator $\mathcal{L}_G$ is sometimes called Lichnerowicz operator. Hence if the deformation tensor h is tangential to $\mathcal{M}_K$, then h satisfies the following linear equation, which is the curvature-preserving condition

$$\mathcal{L}_G h = 0.$$

Having characterized the tangential condition to $\mathcal{M}_K$, we additionally require the deformation tensor h to be L^2-perpendicular to the diffeomorphism group $\mathrm{Diff}_0\Sigma$ action. Consider a one-parameter family of diffeomorphisms $\varphi_t : \Sigma \to \Sigma$ with $\varphi_0 = \mathrm{Id}|_\Sigma$ and let $\frac{d}{dt}\varphi_t|_{t=0} = X$ be a vector field on Σ. Recall that the Lie derivative $L_X G$ of the tensor G in the direction X is defined by

$$L_X G = \frac{d}{dt}\varphi_t^* G\Big|_{t=0}.$$

Take a chart which gives a geodesic normal coordinate centered at p. Then

$$L_X G(p) = X_{i;j} + X_{j;i}$$

as $G_{ij} = \delta_{ij}$ and $G_{ij;k} = 0$ at p. The condition that a symmetric $(0, 2)$-tensor h is L^2-perpendicular to the diffeomorphism group $\mathrm{Diff}_0\Sigma$ action is described as

$$0 = \langle h, L_X G\rangle_{L^2(G)}$$

for all $X \in \mathfrak{X}(\Sigma)$. The right hand side can be rewritten, with respect to a geodesic normal coordinate, as

$$\begin{aligned}
\langle h, L_X G\rangle_{L^2(G)} &= \int_\Sigma \langle h(x), L_X G(x)\rangle_{G(x)} d\mu_G(x) \\
&= \int_\Sigma h_{ij}(X_{i;j} + X_{j;i})\ d\mu_G(x) \\
&= 2\int_\Sigma h_{ij} X_{i;j}\ d\mu_G(x) \\
&= -2\int_\Sigma h_{ij;j} X_i\ d\mu_G(x) \\
&= -2\langle \delta_G h, X\rangle_{L^2(G)},
\end{aligned}$$

where integration by parts. or equivalently the divergence theorem, has been used. There is no boundary contribution as the surface Σ is closed. Therefore, for the tensor h to be L^2-perpendicular to the diffeomorphism group action, h is required to be *divergence-free*; $\delta_G h = 0$. Note that $\delta_G h$ is here regarded as a tensor of $(1, 0)$-type, that is, a vector field. In the normal coordinate system, the divergence-free condition is the same as $(\delta_G h)_i = \sum_i h_{ij;j} = 0$.

Now let h be a deformation tensor tangential to $\mathcal{M}_{-1}$ at a hyperbolic metric G. Then h satisfies the Lichnerowicz equation $\mathcal{L}_G h = 0$;

$$-(\triangle_G + K)\,\mathrm{Tr}_G\, h + \delta_G\delta_G h = 0$$

In addition, we require h to be perpendicular to the diffeormorphism action, which implies $\delta_G h = 0$, which in turn says that h satisfies $-(\triangle_G + K)\,\mathrm{Tr}_G\, h = 0$. When $K = 0, -1$ which are the cases we are interested in, the linear partial differential equation

$$-(\triangle_G + K)\,\mathrm{Tr}_G\, h = 0$$

has only the trivial solution on the closed surface, forcing an additional condition $\mathrm{Tr}_G\, h = 0$.

Therefore, we have so far characterized the conditions that a tangential vector to the Teichmüller space $\mathcal{T}_g = \mathcal{M}_K/\mathrm{Diff}_0\Sigma$ needs to satisfy; namely the *trace-free* condition

$$\mathrm{Tr}_G\, h = 0$$

which is $h_{ii} = 0$ in a normal coordinate system, and the *divergence-free condition*, also called the *transverse* condition

$$\delta_G h = 0.$$

We can now define the Weil-Petersson metric on Teichmüller space.

Definition 3.1 (Weil-Petersson metric [7]). *The L^2-pairing of $T_G\mathcal{M}$ restricted to the trace-free, divergence-free tensors is called Weil-Petersson metric on the Teichmüller space $\mathcal{T} = \mathcal{M}_K/\mathrm{Diff}_0\Sigma$.*

As a 2×2 matrix, the tangential tensor $h \in T_G\mathcal{T}$ can be expressed as

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & -h_{11} \end{pmatrix}$$

with respect to a geodesic normal coordinate system centered at a point p in Σ. The integrand of the Weil-Petersson pairing evaluated at p becomes $2(h_{11}^2 + h_{12}^2)$. Then the divergence-free condition is equivalent to the Cauchy-Riemann equation for $(h_{11} - ih_{12})(z)$ at the origin. We next look into this situation more closely.

3.3. Weil-Petersson metric and Weil-Petersson cometric

First from the discussion in modeling the Teichmüller space as a homogeneous space of $\mathrm{QS}(\Gamma)$ for the Fuchsian group Γ, without loss of generality, by using a Möbius transformation we may assume any given point p to be the origin O of the Poincaré disc. Let $z = x + iy$ be the standard Euclidean coordinate system at the origin. Note that this coordinate system matches with the geodesic normal coordinate system at $O(=p)$, namely $G = \lambda(z)(dx^2 + dy^2)$ with $\lambda(O) = 1$ and $\partial\lambda|_O = 0$, as the first derivatives of $4/(1 - |z|^2)^2$ at $z = 0$ all vanish, which in turn makes all the Christoffel symbols vanish. Then the function $(h_{11} - ih_{12})(z)$, where these indices denote the isothermal coordinates x and y, is holomorphic in z at the origin.

We recall that the cotangent space of Teichmüller space $T^*_{[G]}\mathcal{T}$ at a conformal structure $[G]$ has been identified with the space $\mathrm{QD}(\Sigma)$ of holomorphic quadratic differentials on the Riemann surface $(\Sigma, [G])$. Thus the correspondence between the tangent vectors and the cotangent vectors is

$$h_{11}\, dx \otimes dx + h_{12}\, dx \otimes dy + h_{12}\, dy \otimes dx + (-h_{11})\, dy \otimes dy \longleftrightarrow (h_{11} - ih_{12})(z)dz^2,$$

the former with respect to a geodesic normal coordinate chart, and the latter with an isothermal coordinate chart. The Weil-Petersson cometric defined for the elements of $\mathrm{QD}(\Sigma)$ has the form

$$\langle h_1^*, h_2^* \rangle_{L^2(G)} = \int_\Sigma \phi(z)\overline{\psi}(z)\frac{|dz|^2}{\rho^2(z)}$$

where $h_1^*(z) = \phi(z)dz^2$ and $h_2^*(z) = \psi(z)dz^2$ locally, and the hyperbolic metric G with respect to the isothermal coordinate z is given as $\rho^2(z)|dz|^2$. It is clear from the preceding argument that the two L^2-parings coincide, when restricted to the respective deformations of trace-free divergence-free tensors, and of holomorphic quadratic differentials.

3.4. L^2-decomposition theorem of Hodge-type

We consider the L^2-decomposition of the tangent space $T_G\mathcal{M}$. After having characterized the tangent vectors to the Teichmüller space $\mathcal{M}_{-1}/\mathrm{Diff}_0\Sigma$, it seems unnecessary to further investigate the linear structure. However, the precise formulation of the L^2-decomposition becomes crucial in formulating the nonlinear strucutre, namely the curvature of the spaces. The following statement is an adaptation to dimension two of the theorem by Fischer-Marsden [6] concerning the decomposition of the deformation space of a constant scalar curvature metric in higher (> 2) dimensions. It should be remarked that in the 1980s, Fischer and Tromba [7, 8, 14] undertook the task of rewriting Teichmüller theory from a Riemannian geometric viewpoint. In particular, they laid out the decomposition theory of the deformation tensors in $T_G\mathcal{M}_{-1}$. Below, we develop a theory where the decomposition of the bigger linear space $T_G\mathcal{M} = T_G\mathcal{M}_{-1} \oplus (T_G\mathcal{M}_{-1})^\perp$ is addressed.

We have already identified the adjoint operator of the divergence operator δ_G with the Lie derivative of G up to a constant;

$$\langle h, L_X G\rangle_{L^2(G)} = -2\langle \delta_G h, X\rangle_{L^2(G)}$$

which in turn can be stated as

$$\delta_G^* : X \mapsto -\frac{1}{2} L_X G$$

for $X \in \mathfrak{X}(\Sigma)$, the space of smooth vector fields on Σ.

We can also write down the adjoint operator of the Lichnerowicz operator $\mathcal{L}_G$ by noting the following:

$$\begin{aligned}\langle \mathcal{L}_G^* f, h\rangle_{L^2(G)} &= \langle f, \mathcal{L}_G h\rangle_{L^2(G)} \\ &= \int_\Sigma f(x)\big[(-\triangle_G - K)\operatorname{Tr}_G h + \delta_G\delta_G h\big](x)\ d\mu_G(x) \\ &= \int_\Sigma \langle\{(-\triangle_G - K)f\}G + \operatorname{Hess}_G f, h\rangle_{G(x)}\ d\mu_G(x).\end{aligned}$$

Hence

$$\mathcal{L}_G^* f = (-\triangle_G f - Kf)G + \operatorname{Hess}_G f.$$

For the following decomposition theorem [18], we restrict ourselves to the case $K \equiv -1$, i.e. when the surfaces are uniformized by hyperbolic metrics.

Theorem 3.2. *Suppose that G is a hyperbolic metric on Σ and that h is a smooth symmetric $(0,2)$-tensor defined over Σ. Then there is a unique L^2-orthogonal decomposition of h as a tangent vector in $T_G\mathcal{M}$,*

$$h = P_G(h) + L_X G + \mathcal{L}^* f,$$

where $P_G(h)$ is the projection of h onto $T_G\mathcal{T}$, $L_X G$ is a Lie derivative and $\mathcal{L}_G^* f$ is a tensor perpendicular to $\mathcal{M}_{-1}$. Here the vector field X solves the following equation uniquely

$$\delta_G\delta_G^* X = -\delta_G h$$

and is smooth, the function f solves the following equation uniquely

$$\mathcal{L}_G \mathcal{L}_G^* f = \mathcal{L}_G h$$

and is smooth. Consequently $P_G(h)$ is uniquely determined to be a smooth tensor given by

$$P_G(h) = h - L_X G - \mathcal{L}_G.$$

Each of the three terms belongs to each of the mutually L^2-orthogonal components

$$T_G \mathcal{M} = T_G \mathcal{T} \oplus_{L^2(G)} T_G \mathrm{Diff}_0 \Sigma \oplus_{L^2(G)} (T_G \mathcal{M}_{-1})^{\perp}.$$

We remark that this decomposition can be called of Hodge type for it identifies the tangential directions to Teichmüller space with the intersection of the kernel of the differential operator δ_G and the kernel of $\mathcal{L}_G$; for both of those there are associated elliptic operators $\delta_G \delta_G^*$ and $\mathcal{L}_G \mathcal{L}_G^*$.

Proof of Theorem 3.2. The differential operators $\delta_G \delta_G^*$ and $\mathcal{L}_G \mathcal{L}_G^*$ are both elliptic, self-adjoint, and with trivial kernel (and hence trivial co-kernel). The triviality of the kernel of $\delta_G \delta_G^*$ follows from first noting that $0 = \langle \delta_G \delta_G^* X, X \rangle_{L^2(G)} = \langle \delta_G^* X, \delta_G^* X \rangle_{L^2(G)}$ implies $\delta_G^* X = 0$ and then from the non-existence of Killing vector fields on Σ due to the negative curvature. The triviality of the kernel of $\mathcal{L}_G \mathcal{L}_G^*$ follows as $0 = \langle \mathcal{L}_G \mathcal{L}_G^* f, f \rangle_{L^2(G)} = \langle \mathcal{L}_G^* f, \mathcal{L}_G^* f \rangle_{L^2(G)}$ implies $\mathcal{L}_G^* f = 0$. By taking the trace of the equation $\mathcal{L}_G^* f = 0$, we obtain $-\triangle_G f + 2f = 0$ which implies $f \equiv 0$. This shows, by the standard theory of linear equations of elliptic type [9], that one can solve each of the two equations uniquely to specify the vector field $X = X(h)$ and the function $f = f(h)$, given the data h.

In showing the L^2-orthogonality, we need the following two lemmas, which trigger a series of orthogonal relations.

Lemma 3.1. *For any vector field Y on Σ, we have $\mathcal{L}_G L_Y G = 0$.*

This follows from the simple observation that $L_Y G$ is a deformation tensor induced by a one-parameter family of isometries $\phi_t^* G$ with $\dot{\phi}_0 = Y$, in particular preserving the curvature constraint, hence an element of $T_G \mathcal{M}_{-1}$, which is the kernel of the differential operator $\mathcal{L}_G$.

Lemma 3.2. *For any smooth function ϕ on Σ, we have $\delta_G \mathcal{L}_G^* f = 0$.*

Proof of Lemma 3.2. First choose a geodesic normal coordinate chart centered at p, $\{x^i\}$ so that $G = \delta_{ij}$ and $G_{ij;k} = 0$ for all i, j ad k where ";" stands for the covariant derivative. Then

$$\begin{aligned}\delta_G \mathcal{L}_G^* f &= \delta_G\{(-\triangle_G f + f)G + \mathrm{Hess}_G f\}\\ &= -\{\triangle_G f + f\}_j \delta_{ij} + f_{ij;j}\\ &= -\{\triangle_G f + f\}_j \delta_{ij} + f_{jj;i} + R_{ij} f_j\\ &= 0\end{aligned}$$

where the Ricci identity is used to interchange the order of the covariant derivatives for the second equality, and $R_{ij} = -\delta_{ij}$ on the hyperbolic surface Σ. □

We remark that an immediate consequence of the second lemma is that tensors of type $L_Y G$ and type $\mathcal{L}_G^* \phi$ are mutually L^2-perpendicular for an arbitrary vector field Y and an arbitrary function ϕ, due to the equality $\langle \delta_G \mathcal{L}_G^* \phi, -Y\rangle_{L^2(G)} = \langle \mathcal{L}_G^* \phi, L_Y G\rangle_{L^2(G)}$.

Hence we get the first orthogonality:

$$\langle L_X G, \mathcal{L}_G^* f\rangle_{L^2(G)} = 0.$$

By projecting h to $T_G\mathcal{T}$ and to $(T_G\mathcal{M}_{-1})^\perp$ respectively, we have

$$\begin{aligned}\langle P_G(h), \mathcal{L}_G^* f\rangle_{L^2(G)} &= \langle h - L_X G - \mathcal{L}_G^* f, \mathcal{L}_G^* f\rangle_{L^2(G)}\\ &= \langle \mathcal{L}_G h - \mathcal{L}_G L_X G - \mathcal{L}_G \mathcal{L}_G^* f, f\rangle_{L^2(G)}\\ &= \langle \mathcal{L}_G h - \mathcal{L}_G \mathcal{L}_G^* f, f\rangle_{L^2(G)}\\ &= 0.\end{aligned}$$

Finally the orthogonality between $P_G(h)$ and $L_X G$ can be checked by

$$\begin{aligned}\langle P_G(h), L_X G\rangle_{L^2(G)} &= \langle h - L_X G - \mathcal{L}_G^* f, L_X G\rangle_{L^2(G)}\\ &= \langle \delta_G h - \delta_G L_X G - \delta_G \mathcal{L}_G^* f, -X\rangle_{L^2(G)}\\ &= \langle \delta_G h + \delta_G \delta_G^* X, -X\rangle_{L^2(G)}\\ &= 0.\end{aligned}$$

We have used above the fact that f and X solve the elliptic system

$$\mathcal{L}_G \mathcal{L}_G^* f = \mathcal{L}_G h, \;\; \delta_G \delta_G^* X = -\delta_G h$$

uniquely.

Hence the L^2-decompositon has been achived. □

4. Space of complex structures

4.1. Complex Sturctures and J-fields

On the surface Σ, an almost complex structure J defines an orientation preserving linear endomorphism on each tangent space

$$J : T_p\Sigma \to T_p\Sigma \quad \text{with } J \circ J = -\mathrm{Id}.$$

The chosen orientation here is left-oriented. In two dimensions, due to the existence of isothermal coordinate system z in a neighborhood of each point, one can identify J with the multiplication by $i : z \mapsto iz$, which we denote by m_i via

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } J \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

This is equivalent to saying that in two dimensions, an almost complex structure is a complex structure, namely it is integrable. The space of all the compex structures on Σ is denoted by $\mathcal{J}(T\Sigma)$. The goal of this section is to introduce a natural topology, a natural symplectic structure, and a natural L^2-symplectic structure, as well as an L^2-metric.

In doing so, we will see that with the natural diffeomorphism group action, the Teichmüller space appears as an embedded submanifold in $\mathcal{J}(T\Sigma)$, give under a symplectic splitting, which should be contrasted to the L^2-decomposition of the Weil-Petersson geometry.

$\mathcal{J}(T\Sigma)$ restricted to a point p, denoted by $\mathcal{J}(T_p\Sigma)$, gives a linear endomorphism $J_p : T_p\Sigma \to T_p\Sigma$ woth $J^2 = \mathrm{Id}$, which, with the isothemral coordinates, is identified with an element of $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$. Indeed we have the following characterization (see [1] for the proof.)

Proposition 4.1. *The set $\mathcal{J}(T_p\Sigma)$ is canonically isomorphic to the hyperbolic plane:*

$$\mathbb{H} = \{J \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2) \,|\, \mathrm{J} \circ \mathrm{J} = -\mathrm{Id}_{\mathbb{R}^2}, \mathrm{J} \textit{ left-oriented } \}$$

where the identification is given by

$$J = \begin{pmatrix} h & \frac{-h^2-1}{k} \\ k & -h \end{pmatrix} \text{ with } h, k \in \mathbb{R}, k > 0$$

and $z = \frac{h}{k} + i\frac{1}{k}$ *is a point on the Poincaré upper half-plane* $\mathbb{H}$.

We note here that $z = \frac{h}{k} + i\frac{1}{k}$ is the fixed point of

$$\begin{pmatrix} h & \frac{-h^2-1}{k} \\ k & -h \end{pmatrix}$$

regarded as a fractional linear transformation on the upper half-plane.

One can introduce a L^∞-distance function on $\mathcal{J}(T\Sigma)$ by

$$d_{\mathcal{J}(T\Sigma)}(J_1, J_2) = \sup_{p\in\Sigma} \mathrm{dist}_{\mathbb{H}^2}((\mathrm{J}_1)_\mathrm{p}, (\mathrm{J}_2)_\mathrm{p}),$$

which in turn introduces a topology to the set $\mathcal{J}(T\Sigma)$. In particular, given a pair of complex structures J_1, J_2, which we will call J-fields interchangeably, one can connect them via the constant-speed geodesics $J_p(t)$ in each $\mathcal{J}(T_p\Sigma) \cong \mathbb{H}^2$, and thus the space $\mathcal{J}(T\Sigma)$ of complex structures defined on Σ is path-connected as an infinite dimensional manifold. In fact, $\mathcal{J}(T\Sigma)$ is contractible, as the fiber-wise contractibility induces that of the whole space.

We can linearize this picture by considering the family of the tangent vectors $\{J_p(t)\}_{p\in\Sigma}$ as an endomorphism of the tangent bundle $H : \Sigma \to T\Sigma$. Indeed by looking at a one-parameter family of endomorphism deviating from a complex structure J

$$(J + \varepsilon H)^2 = -\mathrm{Id} + \varepsilon(H \circ J + J \circ H) + \varepsilon^2 H \circ H$$

H needs to satisfy the anti-commuting property $H_p \circ J_p + J_p \circ H_p = 0$ at each point p on Σ. Hence we have identified

Proposition 4.2. *The tangent space of* $\mathcal{J}(T\Sigma)$ *consists of* $(1,1)$*-tensors which anti-commute with* J*:*

$$T_J\mathcal{J}(T\Sigma) = \{H : \Sigma \to \mathrm{End(T\Sigma)} \,|\, \forall \mathrm{p} \quad \mathrm{H_p \circ J_p + J_p \circ H_p = 0}\}$$

Now define $J^\circ : T_J\mathcal{J}(T\Sigma) \to T_J\mathcal{J}(T\Sigma)$ by $J^\circ(H) = J \circ H$. First note that $J \circ (J \circ H) + (J \circ H) \circ J = 0$ and that $J^\circ \circ J^\circ = -\mathrm{Id}_{T_J\mathcal{J}(T\Sigma)}$. It can be shown in

[1] that

PROPOSITION 4.3. *The map J° is a natural complex structure on the manifold $\mathcal{J}(T\Sigma)$, namely the group of orientation preserving diffeomophisms of Σ acts by the J°-holomorphic automorphisms of $\mathcal{J}(T\Sigma)$.*

Here a diffeomorphism $\phi : \Sigma \to \Sigma$ is a J°-holomorphic automorphism if

$$d\phi \circ J^\circ = J^\circ \circ d\phi.$$

The most important example of J-field is given as $(\mathbb{R}^2, J)$ where J is constant over the manifold $\mathbb{R}^2$ and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where $\mathbb{R}^2$ is equipped with the standard Cartesian coordinates. This J-field can be defined on the quotient space of the flat torus $\Sigma = \mathbb{R}^2/\Gamma_0$ where $\Gamma_0 = \mathbb{Z} + i\mathbb{Z}$. As we have seen above, the set $\mathcal{J}(T_p\Sigma)$ is identified with the hyperbolic plane $\mathbb{H}^2$. This gives a representation of the space of J-fields on $\mathbb{R}$, and hence on the quotient space; the flat torus $\Sigma = \mathbb{R}^2/\Gamma_0$. Instead of moving J around, equivalently one can move the Deck transformation group Γ where Γ is a lattice given by $\mathbb{Z} + \tau\mathbb{Z}$ with $\operatorname{Im}\tau > 0$, which is perhaps more familiar and better known model of the moduli space of the flat tori.

4.2. Symplectic Structures on $\mathcal{J}(T\Sigma)$

Next we define a pre-symplectic structure ω° on $\mathcal{J}(T\Sigma)$. First note here that we are using the term "Riemann surface" (Σ, c) where c stands for a conformal structure and a "J-surface" (Σ, J) with a J-structure interchangeably. Given J in $\mathcal{J}(T\Sigma)$, let U_α be a complex coordinate chart on Σ so that J is identified as the multiplication

$$m_i : z = x + iy \mapsto iz = -y + ix, \text{ or equivalently } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We denote this complex structure by J_0 for this locally defined canonical coordinates. Then a tangent vector $H \in T_J\mathcal{J}(T\Sigma)$ is, locally on U_α expressed as

$$H_\alpha(z) = \begin{pmatrix} a(z) & b(z) \\ b(z) & -a(z) \end{pmatrix}$$

so that

$$J \circ H + H \circ J = \begin{pmatrix} b(z) & a(z) \\ -a(z) & -b(z) \end{pmatrix} + \begin{pmatrix} -b(z) & a(z) \\ a(z) & b(z) \end{pmatrix} = 0$$

One can define a path of J-field through J_0 with its velocity vector H by

$$E_H^\varepsilon = \frac{1}{\sqrt{1-\varepsilon^2(a^2+b^2)}} \begin{pmatrix} \varepsilon a(z) & -1+\varepsilon b(z) \\ 1+\varepsilon b(z) & -\varepsilon a(z) \end{pmatrix}$$

for sufficiently small ε. Furthermore one can extend the tangent vector H in $T_{J_0}\mathcal{J}(T\Sigma)$ to a tangent vector $\tilde{H}$ in a neighborhood $\mathcal{U}_{J_0}$ in $\mathcal{J}(T\Sigma)$ by

$$\tilde{H}^\varepsilon := H + \frac{1}{4}\operatorname{Tr}(H \circ E_H^\varepsilon + E_H^\varepsilon \circ H)E_H^\varepsilon,$$

We can now define a nondegenerate closed differential 2-form ω_J° on the Fréchet manifold $\mathcal{J}(T\Sigma)$ with values in $C^\infty(\Sigma)$ by

$$\omega_J^\circ(H,K) = -\operatorname{Tr} HJK$$

for tangent vector H and K in $T_J\mathcal{J}(T\Sigma)$, where HJK are the matrix multiplications under the local complex/isothermal coordinates.

The skew-symmetry of ω_J° follows from $\operatorname{Tr} AB = \operatorname{Tr} BA$ as well as the fact that the elements of $T_J\mathcal{J}(T\Sigma)$ anti-commute with J. The fact that it is a closed form is due to the Cartan formula

$$\begin{aligned} d\omega^\circ(\tilde{H},\tilde{K},\tilde{L}) = \tilde{H}\omega^\circ(\tilde{K},\tilde{L}) - \tilde{K}\omega^\circ(\tilde{H},\tilde{L}) + \tilde{L}\omega^\circ(\tilde{H},\tilde{K}) - \\ \omega^\circ([\tilde{H},\tilde{K}],\tilde{L}) + \omega^\circ([\tilde{H},\tilde{L}],\tilde{K}) - \omega^\circ([\tilde{K},\tilde{L}],\tilde{H}). \end{aligned}$$

and the fact that $JKHL + KHLJ = 0$. Here $\tilde{H}$ denotes an extension of H in a neighborhood of J in $\mathcal{J}(T\Sigma)$. To see the vanishing of $d\omega^\circ(\tilde{H},\tilde{K},\tilde{L})$, expand each terms into the form of traces among K, H, L at J, and check that they all cancel.

We also note that for

$$H = \begin{pmatrix} a(z) & b(z) \\ b(z) & -a(z) \end{pmatrix}, \qquad \omega_J^\circ(H, JH) = 2(a^2+b^2) > 0$$

for $H \neq 0$.

Here the 2-form ω° is $C^\infty(\Sigma)$-valued, and to make it $\mathbb{R}$-valued, one can integrate over the surface Σ:

$$\omega^{\circ,\mu}(H,K) := \int_\Sigma \omega^\circ(H(z),K(z))d\mu$$

where $d\mu$ is the hyperbolic volume form induced from the Unifomization Theorem. Then the symplectic form is indeed the Weil-Petersson Kähler form.

4.3. Space of J-fields as a symplectic product

We have identified, in the dimension two, the space of conformal structures on Σ with the space $J(\mathcal{T}\Sigma)$ of the J-structures on Σ. Let g be the genus of the surface Σ. Hence we can alternatively define the Teichmüller space of Σ to be

$$\mathcal{T}_g = \mathcal{J}(T\Sigma)/\mathrm{Diff}_0\Sigma_g$$

where the quotient represent the equivalent relation under the pull-back/push-forward action of the diffeomorphisms. Let the submersion $\mathcal{J}(T\Sigma) \to \mathcal{J}(T\Sigma)/\mathrm{Diff}_0\Sigma$ be denoted by Π. Consider the action of $\mathrm{Diff}_0\Sigma_g$ on J where ϕ_t is a one-parameter family of diffeomorphisms on Σ with $\phi_0 = \mathrm{Id}$, and $X = \dot{\phi}_0$ is a vector field. When the one-parameter family of $(1,1)$-tensors

$$(\phi_t)^*_* J = (\phi_{-t}) \circ J_{\phi_t} \circ \phi_t$$

is linearized at $t = 0$, we obtain the following:

PROPOSITION 4.4. *A tangent vector H to the* $\mathrm{Diff}_0\Sigma_g$*-fiber at J has the expression*

$$H = \frac{d}{dt}(\phi_t)^*_* J\Big|_{t=0} = L_X \circ J - J \circ L_X$$

where L_X is the Lie derivative for a vector field X on Σ acting on the space $\mathcal{X}(\Sigma)$ of vector fields on Σ.

Similarly let Y be JX, and ψ_t be the flow generated by Y on Σ. Define K is by

$$K := \frac{d}{dt}(\psi_t)^*_* J\Big|_{t=0} = L_{JX} \circ J - J \circ L_{JX}.$$

Then the following commutative relation between endomorphisms of $T\Sigma$;

$$J^\circ \circ (L_X \circ J - J \circ L_X) = L_{JX} \circ J - J \circ L_{JX}.$$

can be verified by writing down both sides of the equality in local isothermal coordinates. This equality, combined with the fact that the restriction of ω° to the $\mathrm{Diff}_0\Sigma_g$ orbit is a closed symplectic form with the non-degeneracy $\omega^\circ_J(H, JH) = 2(a^2+b^2) > 0$, implies the following

Theorem 4.5. *The fibers of* Π *are* $\mathrm{Diff}_0\Sigma_g$ *orbits, and it is a symplectic submanifold in* $(\mathcal{J}(T\Sigma), \omega^{\circ,\mu})$*, namely* $\omega^{\circ,\mu}$ *restricted to each* $\mathrm{Diff}_0\Sigma_g$ *orbit is a closed non-degenerate* 2*-form. In particular* J° *leaves invariant each of the tangent spaces to the* $\mathrm{Diff}_0\Sigma_g$ *orbits.*

Let the family of the tangent spaces of each $\mathrm{Diff}_0\Sigma_g$ orbits define a distribution δ, which is integrable.

4.4. Teichmüller space as a submanifold

Now let τ be a distribution on $\mathcal{J}(T\Sigma)$ given as the symplectic complement of the tangent space to $\mathrm{Diff}_0\Sigma_g$-orbit $(\mathrm{Diff}_0\Sigma)\cdot \mathrm{J}$ of J with respect to the symplectic form $\omega^{\circ,\mu}$;

$$\tau_J := \{H \in T_J\mathcal{J}(T\Sigma) \,|\, \omega^{\circ,\mu}(H, Z) = 0 \quad \forall Z \in T_J[(\mathrm{Diff}_0\Sigma)\cdot \mathrm{J}]\}$$

As $\omega^{\circ,\mu}$ is non-degenrate when restricted on $T_J[(\mathrm{Diff}_0\Sigma)\cdot \mathrm{J}]$, the subspace $\delta_J := T_J[\mathrm{Diff}_0\Sigma\cdot \mathrm{J}] \subset \mathrm{T_J}\mathcal{J}(\mathrm{T}\Sigma)$ is *symplectic*; namely the two subspaces δ_J and τ_J are supplementary in $T_J\mathcal{J}(T\Sigma)$ and transverse only at the origin

$$\delta_J \cap \tau_J = \{0\}.$$

It follows that at each J, the differential $(d\Pi)_J : T_J\mathcal{J}(T\Sigma) \to T_{\Pi(J)}\mathcal{T}$ restricted to the subspace τ_J is an isomorphism

$$(d\Pi)_J|_{\tau_J} : \tau_J \to T_{\Pi(J)}\mathcal{T}.$$

Theorem 4.6. *The distribution* τ *is integrable. Each leaf of the resulting foliation of* $\mathcal{J}(T\Sigma)$ *is a symplectic submanifold, diffeomorphic to the Teichmüller space, and the foliation is* $\mathrm{Diff}_0\Sigma_g$*-invariant. Furthermore the tangent space to each leaf, which is* τ *is* J°*-invariant.*

Proof. We recall that any pair of vector fields $\tilde{X}, \tilde{Y}$ on the Teichmüller space $\mathcal{T}$ are canonically identified with a pair of vector fields X, Y in the distribution τ via the isomorphisms $(d\Pi)_J|_{\tau_J} : \tau_J \to T_{\Pi(J)}\mathcal{T}$. The integrability is equivalent to showing

$$\omega^{\circ,\mu}(H,[X,Y])=0 \qquad \forall H\in T_J[\mathrm{Diff}_0\Sigma\cdot \mathrm{J}].$$

Recall that $H=\frac{d}{dt}(\phi_{-t})\circ J_{\phi_t}\circ\phi_t\Big|_{t=0}$ for some flow ϕ on Σ. This equality follows from

$$\begin{aligned}0&=d\omega^{\circ,\mu}(H,X,Y)\\&=H\omega^{\circ,\mu}(X,Y)-X\omega^{\circ,\mu}(H,Y)+Y\omega^{\circ,\mu}(H,X)-\\&\qquad\omega^{\circ,\mu}([H,X],Y)+\omega^{\circ,\mu}([H,Y],X)-\omega^{\circ,\mu}([X,Y],H).\end{aligned}$$

We claim that the first five terms of the six on the right hand side of the second equality are all zero, leaving $0=-\omega^{\circ,\mu}[X,Y],H)$. The first term vanishes, as the term $\omega^{\circ,\mu}(X,Y)$ is $\mathrm{Diff}_0\Sigma_g$-invariant, and H is a linearized $\mathrm{Diff}_0\Sigma_g$-action. The second and third terms vanish by the definition of the distribution of τ. The third and forth terms are zero, as the vector fields H and X (and Y) commute as X and Y are $\mathrm{Diff}_0\Sigma_g$-invariant, and H is a linearized $\mathrm{Diff}_0\Sigma_g$-action. □

Hence we have identified the Teichmüller space $\mathcal{T}$ as a leaf of the foliation defined by the integrable distribution τ in $\mathcal{J}(T\Sigma)$.

Corollary 4.7. *The pair $(J^\circ,\omega^{\circ,\mu})$ defines a natural Kähler structure on the space $\mathcal{J}(T\Sigma)$ of complex structures, and it also induces a natural Kähler structure on the Teichmüller space $\mathcal{T}$, which is identified as a symplectic submanifold in $\mathcal{J}(T\Sigma)$. Indeed the space $\mathcal{J}(T\Sigma)$ is recognized as a symplectic product decomposition that is symplectomorphic to $\mathrm{Diff}_0\Sigma_g\times\mathcal{T}_g$ by the pair of integrable foliations δ and τ.*

A remark given in [1] (Theorem 15.6) is that this corollary provides an alternative proof to the result by Earle-Eells [5] in 1969 which says that $\mathrm{Diff}_0\Sigma_g$ is contractible with respect to the smooth Fréchet topology. This follows readily from the facts that $\mathcal{J}(T\Sigma)$ and $\mathcal{T}_g$ are both contractible. The space $\mathcal{J}(T\Sigma)$ is contractible, as it is the space of sections of fiber bundle whose fibers are copies of the hyperbolic plane $\mathbb{H}^2$. We note also that historically $\mathcal{T}_g$ had been known to be contractible since O. Teichmüller showed that it is homeomorphic to the space of holomorphic quadratic differentials on a Riemann surface (Σ,z).

4.5. *J*-fields and Weil-Petersson geometry

Note that in the definition of the symplectic structure $\omega^{\circ,\mu}$, the L^2-product is implicitly present. Indeed the (pre)-symplectic structure ω° at J defines, pointwise in Σ, a positive definite inner product

$$\omega^\circ(H, JK) = -\operatorname{Tr} HJ(JK) = \operatorname{Tr} HK =: g(H, K)$$

for H, K in $T_J\mathcal{J}(T\Sigma)$, each of which is locally represented as a 2×2 matrix. We first uniformize the Riemannian metric g so that $\tilde{g} := e^u\tilde{g}$ is the hyperbolic metric in the conformal structure determined by J. Denote $e^u\omega^\circ(H, JK)$ by $\tilde{\omega}^\circ(H, K)$. By integrating over Σ, we obtain

$$\langle H, K\rangle_{L^2(\tilde{g})} := \tilde{\omega}^{\circ,\mu}(H, JK) := \int_\Sigma \tilde{\omega}^\circ(H, JK)d\mu$$

where μ coincides with the volume form of $\tilde{g}$. The restriction of this inner product to the subspace τ_J is the Weil-Petersson metric on $\mathcal{T}$, provided that the tangent vectors here are $(1,1)$-tensors, while in the previous section, the deformation tensors are $(0,2)$-tensors. This identification is also demonstrated in [14].

It should be remarked at this point that A. Tromba [14] has written a book on the L^2-geometry of the Teichmüller spaces, and there the map $\mathcal{M}_{-1} \to J(T\Sigma)$ was also analyzed. The L^2-geometric approach of Tromba's should be contrasted to the symplectic approach of A'Campo.

5. Summary: hyperbolic geometry vs. conformal geometry

We recall that the Teichmüller space was defined as a quotient space of $P : \mathcal{M}_{-1} \to \mathcal{T}$ where P is a Riemannian submersion with respect to the L^2-metric. In our current setting of $\Pi : \mathcal{J}(T\Sigma) \to \mathcal{T}$, the Teichmüller space is realized as an embedded submanifold, or rather a family of submanifolds, in the ambient space $\mathcal{J}(T\Sigma)$, and Π is a projection whose fibers are the $\mathrm{Diff}_0\Sigma_g$ orbits of each point in $\mathcal{T}$.

The uniformization theorem states that on a compact surface Σ_g of higher genus $g > 1$, for each conformal structure, or equivalently a complex structure, there exists a unique hyperbolic metric. Let the correspondence be denoted by $U : \mathcal{J}(T\Sigma)) \to \mathcal{M}_{-1}$. Thus we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{J}(T\Sigma) & \xrightarrow{\quad U \quad} & \mathcal{M}_{-1} \\ & {\scriptstyle \Pi}\searrow \qquad \swarrow{\scriptstyle P} & \\ & \mathcal{T}_g & \end{array}$$

The projection Π has $\mathrm{Diff}_0\Sigma$ fibers, and each fiber is a symplectic submanifold of $\mathcal{J}(T\Sigma))$, where the symplectic supplements of the tangential subspace to the

$\mathrm{Diff}_0\Sigma$-fibers form the tangent spaces of the Teichmüller space as a symplectic submanifold. Historically there are several constructions of Teichmüller spaces within ambient spaces, such as the Bers embeddings and Maskit embeddings (cf. [10]) in the context of Kleinian groups. Compared to those, the current approach is distinguished in the sense that the ambient space $J(T\Sigma)$ is reflective of the local geometry of the Riemann surfaces.

On the other hand, the quotient map P also has $\mathrm{Diff}_0\Sigma$ fibers, and each fiber defines the vertical directions for the Riemannian submersion P, where the horizontal directions constitute the trace-free transverse tensors, which can be identified with the tangent space of the Teichmüller space, even though the resulting distribution is not integrable. Even though the Teichmüller space is not realized as a submanifold of $\mathcal{M}_{-1}$, recall that it was crucial that the one dimensional distributions are always integrable, and thus each Weil-Petersson geodesic can be lifted to an L^2-geodesic in $\mathcal{M}_{-1}$.

In conclusion, the two representations of the Teichmüller space are useful in terms of contrasting the hyperbolic geometry and the conformal geometry of Riemann surfaces. In juxtaposing them, we see how the local geometries on each Riemann surface are interacting, and by integrating the various geometric quantities over the suface, we can relate the L^∞-theory of conformal geometry to the L^2-geometry based on the hyperbolic metrics on the surfaces. It is well-known that the L^2-geometry are closely related the the elliptic variational framework, in particular the harmonic map theory in the context of Weil-Petersson geometry, which has provided many applications such as the works of Y. Minsky [11], M. Wolf [17], and the author [18, 19]. On the other hand, less well-known yet no less interesting is L^2-variational theory of the conformal geometry, called the Gerstenhaber-Ruach theory (cf. see [4] for references.) The current discussion in this article could be a starting point for such a viewpoint.

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