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Analytical methods for inverse obstacle problems

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Abstract.

This expository paper is concerned with methodology on the reconstruction issue for inverse obstacle problems governed by partial differential equations and consists of three sections. First an introduction of representative classical analytical methods, which are the linear sampling and factorization methods, the probe and singular sources methods and the enclosure method is given. Second the recent development of the enclosure method in time domain is described. It is focused on showing three types of applications of the method to inverse obstacle problems governed by wave equations. Finally, as a most recent topic, an integrated theory of the probe and singular sources methods is presented.

1. Introduction

In this paper we consider

Problem. Extract information about geometry of *discontinuity* from observation data.

The term "discontinuity" appears in nondestructive testing and evaluation. It gives an effect on the propagation of various signals (electric current, heat, various waves etc.) in a given body or medium, such as cavity, crack, inclusion, obstacle.

The problem mentioned above comes from nondestructive testing and evaluation (e.g., [13], [5]), radar, sonar, electrical impedance tomography, microwave imaging (e.g., [57]), thermography, etc. One can see also in nature, for example, dolphin, bat, their echo location and object recognition ability using sound wave, for example, see pp. 140-215 of [3].

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Many of such problems are mathematically formulated by using solutions of various partial differential equations which are called the *governing equations*. They are called "inverse obstacle problems" as can be seen, for example, in the title of Isakov's review paper [49].

By the way, looking back on the author's published works since 1987, the main interest of the author was seeking analytical methods for inverse obstacle problems governed by partial differential equations, in particular, its reconstruction issue. In particular, it was fortune that the author was able to place himself and act in the midst of dramatic changes of the research direction of the problems. More precisely, from the late 1990s to the end of the 20th century (1996-2000) there was an appearance of direct *analytical methods* for inverse obstacle problems, in particular, through inverse obstacle scattering problems in frequency domain and the Calderón problem [6] whose governing equations are given by elliptic PDEs. Those methods do not need iterative computation such as traditional optimization methods together with linearization procedures.

A list of classical analytical methods taken up in this paper is as follows.

Linear Sampling Method of Colton-Kirsch [7]

Probe Method of Ikehata [14, 15, 17]

Factorization Method of Kirsch [50, 51, 52]

Enclosure Method of Ikehata [16, 18]

Singular Sources Method of Potthast [58]

Those are briefly discussed in the following subsections from author's perspective.

1.1. Linear Sampling and Factorization Methods

In this section, we describe briefly the essence of the linear sampling and factorization methods. Originally they were introduced for inverse obstacle scattering problems in the whole space or an exterior domain. Here we restrict ourself to a typical and most important inverse obstacle scattering problem in three dimensions (cf. [10]) and present their ideas from the author's point of view.

We denote by D an unknown obstacle placed in the whole space $\mathbb{R}^3$. We assume that D is given by an open subset of $\mathbb{R}^3$ with smooth boundary such that $\mathbb{R}^3 \setminus \overline{D}$

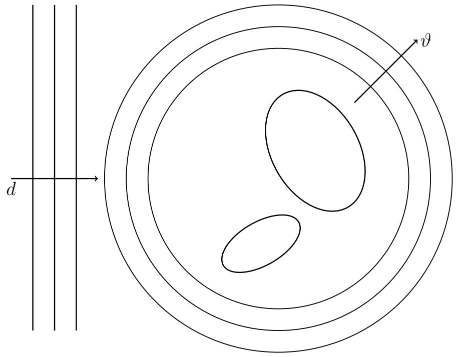

Figure 1. Two ellipses represent the connected components of obstacle D.

is connected. Fix $k > 0$. Given an incident direction $d \in S^2$ let $w = w(x; d)$ satisfy

$$\begin{cases} \Delta w + k^2 w = 0, & x \in \mathbb{R}^3 \setminus \overline{D}, \\ \dfrac{\partial w}{\partial \nu} = -\dfrac{\partial}{\partial \nu}(e^{ikx\cdot d}), & x \in \partial D, \\ r\left(\dfrac{\partial w}{\partial r} - ikw\right) \to 0, r = |x| \to \infty, \end{cases}$$

where ν the unit outward vector field on ∂D and $i = \sqrt{-1}$. The inhomogeneous Neumann boundary condition means the obstacle surface has vanishing admittance and such obstacle is called the sound-hard obstacle. The function $e^{ikx\cdot d}$ denotes an incident plane wave. The existence and uniqueness of w are well known (Integral equation approach or variational approach). Besides, it is well known that scattered wave w has the asymptotic expansion as $r \to \infty$:

$$w(r\vartheta; d) = \frac{e^{ikr}}{r} F(\vartheta, d) + O(\frac{1}{r^2}), \vartheta \in S^2,$$

where the coefficient $F(\vartheta, d)$ is called the far field pattern of w.

The obstacle scattering problem using a plane wave is to establish the way of the calculation of the $F(\vartheta, d)$ along the flow as indicated below(see also Figure 1 for a conceptual illustration of obstacle scattering):

$$e^{ikx\cdot d} \longrightarrow D \longrightarrow w(r\vartheta, d) \longrightarrow_{r\to\infty} F(\vartheta, d)$$

Contrary to this, the problem to be used for the explanation of the classical analytical methods is

Problem 1. Fix known k. Reconstruct D from $F(\vartheta, d)\,(\forall\vartheta\,\forall d)$.

It is well known that we have the expression

$$F(\vartheta, d) = \frac{1}{4\pi}\int_{\partial D}\frac{\partial}{\partial\nu(x)}\,(e^{-ikx\cdot\vartheta}\,)\,u(x;d)\,dS(x),$$

where $u(x;d) = e^{ikd\cdot x} + w(x;d)$ denotes the total wave. This shows intuitively that the far-field pattern depends on the obstacle *nonlinearly.* Thus Problem 1 is a nonlinear problem even the governing equation is linear.

The uniqueness issue of Problem 1 has been established in Kirsch-Kress [54] by using a contradiction argument.

By the way, the observation data $\{F(\vartheta, d)\,|\,\vartheta, d \in S^2\}$ determine the operator

$$F : L^2(S^2) \ni f \mapsto \int_{S^2} F(\,\cdot\,, d)f(d)\,dS(d) \in L^2(S^2)$$

and vice versa.

The meaning of operator F is as follows. The Herglotz wave function can be considered as a continuous superposition of plane waves, that is

$$Hf(x) = \int_{S^2} e^{ikx\cdot d}f(d)\,dS(d) \approx \sum_d e^{ikx\cdot d}f(d),\ \ x \in \mathbb{R}^3.$$

Send this from infinity. Then the far field pattern of scattered wave by obstacle D is given by

$$Ff(\vartheta) = \int_{S^2} F(\vartheta, d)f(d)\,dS(d) \approx \sum_d F(\vartheta, d)f(d), \vartheta \in S^2.$$

The following diagram sums up the flow:

$$Hf(x) \longrightarrow D \longrightarrow \int_{S^2} w(r\vartheta; d)f(d)dS(d) \longrightarrow_{r\to\infty} Ff(\vartheta)$$

Here we introduce the key concept in the linear sampling method. That is the spherical wave with a singularity at an arbitrary point $x \in \mathbb{R}^3$ given by

$$\Phi(y,x) = \frac{e^{ik|y-x|}}{4\pi|y-x|}, y \in \mathbb{R}^3 \setminus \{x\}.$$

This is the unique fundamental solution of the Helmholtz equation with the radiation condition

$$r\left(\frac{\partial}{\partial r}\Phi(r\vartheta,x) - ik\Phi(r\vartheta,x)\right) \to 0$$

as $r \to \infty$.

The spherical wave $\Phi(\,\cdot\,,x)$ takes the form

$$\Phi(r\vartheta,x) = \frac{e^{ikr}}{4\pi r}e^{-ik\vartheta\cdot x} + O(r^{-2}), r \to \infty.$$

Thus the far field pattern of $\Phi(\,\cdot\,,x)$ is given by

$$\frac{e^{-ik\vartheta\cdot x}}{4\pi}, \vartheta \in S^2.$$

Linear Sampling Method

First we introduce an operator from obstacle surface to far field.

Given $h(y),\ y \in \partial D$ let $w = w(y;h), y \in \mathbb{R}^3 \setminus \overline{D}$ solve

$$\begin{cases} (\Delta + k^2)w = 0, & y \in \mathbb{R}^3 \setminus \overline{D}, \\ \dfrac{\partial w}{\partial \nu} = h(y), & y \in \partial D, \\ r\left(\dfrac{\partial w}{\partial r} - ikw\right) \to 0, r \to \infty. \end{cases}$$

Let Gh denote the far field pattern of w. Note that $G : h \mapsto Gh$ is linear. It is easy to see (almost trivial)

$$x \in D \Leftrightarrow \text{Equation } Gh(\vartheta) = e^{-ik\vartheta\cdot x}, \vartheta \in S^2 \text{ is solvable}$$

$$\Leftrightarrow e^{-ik\vartheta\cdot x} \in \operatorname{Range} G$$

(if $x \in D$ choose $h = -4\pi\partial_\nu\Phi(y,x)$; if $x \notin D$ contradiction by the Rellich theorem).

Here we have

$$Ff = -G(\partial_\nu(Hf)|_{\partial D})$$

and thus

$$\text{Range}\, F \subset \text{Range}\, G.$$

Thus

$$x \notin D \Longrightarrow e^{-ik\vartheta\cdot x} \notin \text{Range}\, F$$

or

$$e^{-ik\vartheta\cdot x} \in \text{Range}\, F \Longrightarrow x \in D$$

i.e.,

$$\mathcal{A} \equiv \left\{x \in \mathbb{R}^3 \,|\, \exists f\, Ff = e^{-ik\vartheta\cdot x}\right\} \subset D.$$

Unfortunately, in general the symbol $\subset$ cannot be replaced with symbol $=$ *.

Instead, given $\epsilon > 0$ and $x \in \mathbb{R}^3$ consider the set

$$\mathcal{F}_\epsilon(x) = \left\{f \in L^2(S^2) \,|\, \|Ff - e^{-ik\vartheta\cdot x}\|_{L^2(S^2)} < \epsilon\right\}.$$

Any member of $\mathcal{F}_\epsilon(x)$ "solves" approximately the far-field equation

$$Ff(\vartheta) = e^{-ik\vartheta\cdot x}, \vartheta \in S^2$$

with an $L^2(S^2)$-error at most ϵ. However, the existence is not a trivial matter.

Here we introduce

Assumption. The k^2 is not an eigenvalue of $-\Delta$ in D with the Neumann boundary condition.

It is known that this assumption yields: the range of F is dense in $L^2(S^2)$. Thus, for all $\epsilon > 0$ and $x \in \mathbb{R}^3$ we have

$$\mathcal{F}_\epsilon(x) \neq \phi.$$

*And some case $\mathcal{A} = \phi$, see [23] for an example in two dimensions.

Besides Colton-Kirsch in [7] state more about the set $\mathcal{F}_\epsilon(x)$, that is the following result.

THEOREM 1.1. *Under Assumption described above, given $\epsilon > 0$ there exists a family $(f_{x,\epsilon})$ of $L^2(S^2)$ indexed with $x \in D$ such that, for each $x \in D$*

$$f_{x,\epsilon} \in \mathcal{F}_\epsilon(x)$$

and for each $a \in \partial D$

$$\lim_{D \ni x \to a} \|f_{x,\epsilon}\|_{L^2(S^2)} = \infty.$$

This means: given $\epsilon > 0$ the set

$$\cup_{x \in \mathbb{R}^3} \mathcal{F}_\epsilon(x) \ (\subset L^2(S^2)\,)$$

contains surely information about geometry of D.

The idea of the Linear Sampling Method consists of two steps (a) and (b) indicated below.
(a) Fix $\epsilon > 0$. For each *sampling point* $x \in \mathbb{R}^3$ find $f_x \in \mathcal{F}_\epsilon(x)$, i.e.,

$$\|Ff_x - e^{-ikx\cdot\vartheta}\|_{L^2(S^2)} < \epsilon.$$

(b) Plot the function

$$x \longmapsto \|f_x\|_{L^2(S^2)}.$$

The background of the idea which is based on Theorem 1.1 is the belief: the points where the values of function of (b) are large, could be the surface of obstacle. They confirmed experimentally (numerical).

However, needless to say, there should be points to be clarified mathematically from the beginning of their appearance.

• Their original existence proof of the families $(f_{x,\epsilon})_{x\in\Lambda,\epsilon>0}$ in the cases $\Lambda = \mathbb{R}^3 \setminus D$ and $\Lambda = D$ are different. Note that D is assumed to be unknown!

• There was no proof that the numerically constructed density $f_{x,\epsilon}$ for $x \in D$ behaves exactly, as $x \to \partial D$ the same as the member of family $(f_{x,\epsilon})_{x\in D,\epsilon>0}$ ensured in Theorem 1.1.

The rigorous justification has been done 10 years later with the help of Kirsch's factorization method (Arens [1], Arens-Lechleiter [2]) described below. They gave a unified numerical construction method of the family $(f_{x,\epsilon})_{x\in\mathbb{R}^3}$ which behaves the same as predicted in Theorem 1.1 when $D \ni x \to \partial D$.

Factorization Method

Let us go back to the formula

$$D = \left\{ x \in \mathbb{R}^3 \,|\, e^{-ik\vartheta\cdot x} \in \mathrm{Range}\, G \right\}.$$

So if one can calculate Rang G by using F only, one gets the reconstruction formula of D itself! Factorization method connects "two different things" by analyzing a factorization of the far field operator and enables us to do the calculation. More precisely, Kirsch in [50] has established the following result.

THEOREM 1.2. *Assume that k^2 not an eigenvalue of $-\Delta$ in D with homogeneous Neumann boundary condition. Then, we have*

$$Range\, G = Range\, (F^*F)^{\frac{1}{4}}.$$

Thus

$$D = \left\{ x \in \mathbb{R}^3 \,|\, e^{-ik\vartheta\cdot x} \in \; Range\,(F^*F)^{\frac{1}{4}} \right\}.$$

The result says that: for an arbitrary h on ∂D, the Gh which is a function on S^2 can be calculated from observation data F even both of h and D are unknown. Besides, by Picard's Theorem [8], as a corollary it holds

$$D = \left\{ x \in \mathbb{R}^3 \,|\, K(x) \equiv \sum_{n=1}^{\infty} \frac{|(\Psi_n, e^{-ikx\cdot\vartheta})_{L^2(S^2)}|^2}{|\sqrt{|\lambda_n|}|^2} < \infty \right\},$$

where Ψ_n, λ_n complete orthonormal eigenfunctions, corresponding eigenvalues of operator F, which is compact and normal.

Thus we have

Conclusion. Instead of the far field equation

$$Ff(\vartheta) = e^{-ik\vartheta\cdot x}, \vartheta \in S^2,$$

the solvability of modified far field equation

$$(F^*F)^{\frac{1}{4}} f(\vartheta) = e^{-ik\vartheta\cdot x}, \vartheta \in S^2,$$

gives the complete characterization of unknown obstacle.

Since the introduction of Factorization method, depending on the problems, based on the idea of the method, various modified far field equations have been introduced. See, the books [53, 8] together with their references and e.g., Furuya's recent works [11, 12].

1.2. Probe and Singular Sources Methods

In [14], [17](1998, 1999) and [58], [59](2000, 2001), Ikehata and Potthast gave the solution to Problem 1, independently, with different methods which are called Ikehata's Probe Method and Potthast's Singular Sources Method. The common character of the original version of the Probe Method and Singular Sources Method is to compute a field defined outside unknown obstacle by using observation data. The field blows up on the surface of obstacle. By looking at the calculated field, unknown objects emerge.

Singular Sources Method

Let $\mathcal{E}_x(y)$ denote the scattered wave by obstacle D with the incident spherical wave $\Phi(y,x)$ having singularity at $y = x \in \mathbb{R}^3 \setminus \overline{D}$. That is, the wave $\mathcal{E}_x = \mathcal{E}$ solves the exterior problem

$$\begin{cases} \Delta\mathcal{E}(y) + k^2\mathcal{E}(y) = 0, \quad y \in \mathbb{R}^3 \setminus \overline{D}, \\ \dfrac{\partial\mathcal{E}}{\partial\nu}(y) = -\dfrac{\partial}{\partial\nu}\Phi(y,x), y \in \partial D, \\ r\left(\dfrac{\partial}{\partial r}\mathcal{E} - ik\mathcal{E}\right) \to 0, \quad r = |y| \to \infty. \end{cases}$$

The indicator function of Singular Sources Method is given by the function

$$\mathbb{R}^3 \setminus \overline{D} \ni x \longmapsto \mathcal{E}_x(x) = \mathcal{E}_x(y)|_{y=x}.$$

Potthast's singular sources method developed in [58] and [59] is summarized as

THEOREM 1.3. (i) *One can calculate $\mathcal{E}_x(x)$ for $x \in \mathbb{R}^3 \setminus \overline{D}$ from $F(\vartheta, d)$ given at all directions ϑ and d.*
(ii) *It holds that, for an arbitrary point a on ∂D*

$$\lim_{x \to a} |\mathcal{E}_x(x)| = \infty.$$

Comments on proof.

On (i). It is based on, roughly speaking, the computation formula of the indicator function for the singular sources method: for an arbitrary fixed $x \in \overline{B} \setminus \overline{D}$, it holds that

$$\mathcal{E}_x(x) = \lim_{n \longrightarrow \infty} \int_{S^2} \left(\lim_{l \longrightarrow \infty} \frac{1}{4\pi} \int_{S^2} F(-\vartheta, d) g_l(\vartheta, x) dS(\vartheta) \right) g_n(d, x) dS(d),$$

where $g = g_l(\cdot, x), g_n(\cdot, x)$ satisfy roughly speaking

$$Hg(z) \approx \Phi(z, x), \ z \in G_x,$$

B large open ball with $\overline{D} \subset B$, G_x a bounded domain depending on x satisfying $\overline{D} \subset G_x$, $\mathbb{R}^3 \setminus \overline{G_x}$ connected and $x \in \mathbb{R}^3 \setminus \overline{G_x}$. The G_x on which one can find the density g as above is called a sampling domain associated with x.
On (ii). This is a consequence of a point wise expression of the solution $\mathcal{E}_x(y)$ in a neighbourhood of ∂D by using the method of integral equation.

Probe Method

Fix a known bounded domain Ω such that $\overline{D} \subset \Omega$ and $\mathbb{R}^3 \setminus \overline{\Omega}$ connected. Assume k^2 neither the eigenvalue of $-\Delta$ in $\Omega \setminus \overline{D}$ with homogeneous Dirichlet boundary condition on $\partial\Omega$ and Neumann boundary condition on ∂D nor Dirichlet eigenvalue of $-\Delta$ in Ω. Then, given $x \in \Omega \setminus \overline{D}$ there exists the unique solution $w_x = w$ of

$$\begin{cases} \Delta w(y) + k^2 w(y) = 0, \ \ y \in \Omega \setminus \overline{D}, \\ \\ \dfrac{\partial w}{\partial \nu}(y) = -\dfrac{\partial}{\partial \nu} \Phi(y, x), y \in \partial D, \\ \\ w(y) = 0, \qquad\qquad\qquad y \in \partial\Omega. \end{cases}$$

The indicator function of Probe Method is as follows:

$$\Omega \setminus \overline{D} \ni x \longmapsto I(x) \equiv \|\nabla \Phi(\,\cdot\,, x)\|^2_{L^2(D)} - k^2 \|\Phi(\,\cdot\,, x)\|^2_{L^2(D)}$$
$$+ \|\nabla w_x\|^2_{L^2(\Omega \setminus \overline{D})} - k^2 \|w_x\|^2_{L^2(\Omega \setminus \overline{D})}.$$

In [14] and [17] the author established the following result.

THEOREM 1.4. (i) *One can calculate $I(x)$ for $x \in \Omega \setminus \overline{D}$ from $F(\vartheta, d)$ given at all directions ϑ and d.*

(ii) *It holds that, for an arbitrary point a on ∂D*

$$\lim_{x\to a} I(x) = \infty.$$

Comments on proof.

On (i). The proof consists of three steps.

1. $F \Rightarrow S_D$ **on** $\partial\Omega$

Here, S_D denotes the single layer integral operator on $\partial\Omega$ given by

$$S_D f(y) = \int_{\partial\Omega} \Phi_D(y,x) f(x) dS(x),\ y \in S$$

and set $S_D = S_0$ if $D = \phi$. The integral kernel of S_D is given by

$$\Phi_D(y,x) = \Phi(y,x) + \mathcal{E}_x(y),\ y \in \mathbb{R}^3 \setminus \overline{D},\ x \in \mathbb{R}^3 \setminus \overline{D},\ x \neq y.$$

The point is to calculate $\Phi_D(y,x)$ for $(x,y) \in \partial\Omega$ with $x \neq y$ from F. For this do the similar procedures as (i) of Singular Sources Method in a neighbourhood of $\partial\Omega$ (however, not close to ∂D). See [14] for the procedures.

Hereafter we take two different key steps from Singular Sources Method.

2. $S_D \Rightarrow \Lambda_0 - \Lambda_D$

This is due to the connection formula [17]

$$S_0^{-1} - S_D^{-1} = \Lambda_0 - \Lambda_D,$$

where Λ_D denotes the Dirichlet-to-Neumann map $u|_{\partial\Omega} \mapsto \frac{\partial u}{\partial \nu}|_{\partial\Omega}$, with $\Delta u + k^2 u = 0$ in $\Omega \setminus \overline{D}$ and $\frac{\partial u}{\partial \nu} = 0$ on ∂D; $\Lambda_D = \Lambda_0$ if $D = \phi$.

The third step is the core of the probe method.

3. $\Lambda_0 - \Lambda_D \Rightarrow I(x)$

The procedure is based on the fact: given an arbitrary needle σ with a tip at x, roughly speaking, which is a path connecting a point on $\partial\Omega$ and x (see Subsection 3.1 for the definition) and a sequence $\{v_n\}$ of solutions of the Helmholtz equation in the whole Ω such that $v_n \to \Phi(\,\cdot, x)$ in $H^1_{\text{loc}}(\Omega \setminus \sigma)$ it holds

$$I(x) = \lim_{n\to\infty} \int_{\partial\Omega} (\Lambda_0 - \Lambda_D) v_n|_{\partial\Omega} \cdot \overline{v_n|_{\partial\Omega}}\, dS$$

if $\sigma \cap \overline{D} = \phi$.

On (ii). This needs a technical argument based on PDE [17]. See also the recent article [40] for another direct approach.

Historically, in [17] the author gave a reconstruction formula of unknown obstacles from the data

$$\{\Phi_D(y,x)\,|(x,y)\in(\partial\Omega)^2, x\neq y\}.$$

The result is formulated by using the original probe method [15] and the proof employs the connection formula mentioned above. The article [14] combines the procedure **1** mentioned above with the result of [17].

As can be seen from the comments on proofs of Theorems 1.3-1.4, both methods consist of numerical calculation procedures for their indicator functions as summarized below.

$$
\begin{array}{lll}
 & \nearrow \quad \text{Sampling domain} & \mathcal{E}_x(x),\ x\in\mathbb{R}^3\setminus\overline{D}\\
F(\vartheta,d)\ \forall\vartheta\,\forall d & & \\
 & \searrow \quad \text{Needle} & I(x),\ x\in\Omega\setminus\overline{D}
\end{array}
$$

1.3. Enclosure Method for Inverse Obstacle Scattering

First let us describe the formulation of obstacle scattering problem in two dimensions. Let D be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary such that $\mathbb{R}^2\setminus\overline{D}$ is connected. Let $k>0$. Given $d\in S^1$ let $w=w(x;d)$ solve

$$
\begin{cases}
\Delta w+k^2w=0, & x\in\mathbb{R}^2\setminus\overline{D},\\[2mm]
\dfrac{\partial w}{\partial\nu}=-\dfrac{\partial}{\partial\nu}(e^{ikx\cdot d}), & x\in\partial D,\\[2mm]
\sqrt{r}\left(\dfrac{\partial w}{\partial r}-ikw\right)\to 0, r=|x|\to\infty. &
\end{cases}
$$

The $w=w(r\vartheta;d)$, $\vartheta\in S^1$ as $r\to\infty$ has the expression

$$w(r\vartheta;d)=\frac{e^{ikr}}{\sqrt{r}}F(\vartheta,d)+O(r^{-\frac{3}{2}}), r\to\infty.$$

The function $F(\,\cdot\,,d)$ on S^1 is called the far field pattern of w .

Problem 2. Fix k and d. Assume $D \neq \phi$. Extract information about geometry of D from $\{F(\vartheta, d) \,|\, \vartheta \in S^1\}$.

Note that unlike Problem 1, the incident direction d is fixed.
The indicator function of Enclosure Method presented here is defined by the formula

$$I_N(\tau, \omega) = \int_{S^1} F(-\varphi, d) g_N(\varphi; \tau, \omega) ds(\varphi),$$

where $N = 1, 2, \cdots$, $\tau > 0$, $\omega = (\omega_1, \omega_2) \in S^1$ and the density g_N is given by

$$g_N(\varphi; \tau, \omega) = \frac{1}{2\pi} \sum_{|m| \leq N} \left\{ \frac{ik\varphi}{(\tau + \sqrt{\tau^2 + k^2})\omega} \right\}^m.$$

Note that we identify the real unit vector $\varphi = (\varphi_1, \varphi_2) \in S^1$ with the complex number $\varphi_1 + \sqrt{-1}\,\varphi_2$ and denote it by the same symbol φ. So the φ and ω in each term on the right side of g_N have that meaning.

The g_N is the truncation of a divergent Fourier series which is a formal "solution" of the integral equation

$$\int_{S^1} e^{iky \cdot \varphi} g(\varphi) ds(\varphi) = e^{y \cdot z}, \; y \in \mathbb{R}^2$$

and

$$z = \tau\omega + i\sqrt{\tau^2 + k^2}\,\omega^{\perp}, \; \omega^{\perp} = (\omega_2, -\omega_1).$$

The left-hand side of this integral equation is the Herglotz wave function with density g in two dimensions. Since z satisfies $z \cdot z = -k^2$, the function $e^{y \cdot z}$ on the right-hand side satisfies the Helmholtz equation and is called the complex plane wave solution or the complex geometrical optics solution. The solution at $k = 0$ has been used in the linearization of the Calderón problem proposed in [6].

The following result gives us a direct extraction formula of some information about the geometry of an unknown obstacle from the far field pattern $F(\,\cdot\,, d)$ for an arbitrary fixed d and has been established in [21] by the author-himself.

First we assume that D is given by the union of finitely many interiors of polygons $D_1, \cdots, D_M$ with unknown counting number M such that $\overline{D_i} \cap \overline{D_j} = \phi$ if $i \neq j$. Let ω be regular with respect to D, that is, the set $\{x \cdot \omega = h_D(\omega)\} \cap \partial D$ consists of a single point, where $h_D(\omega) = \sup_{x \in D} x \cdot \omega$ called the support function of D (see Figure 2 for an illustration of the situation).

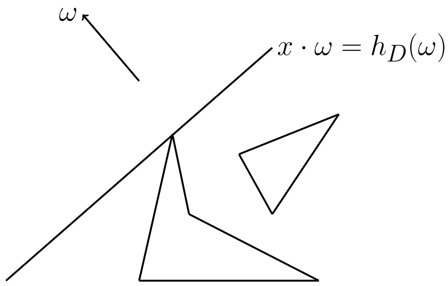

Figure 2. An illustration of regular direction ω. Two polygons are the connected components of polygonal D.

THEOREM 1.5. *Let B_R be an open disc centered at the origin of coordinates such that $\overline{D} \subset B_R$. Let $\{\tau(N)\}_{N=1,2,\cdots}$ be an arbitrary sequence of positive numbers satisfying, as $N \to \infty$*

$$\tau(N) = \frac{c}{eR}N + O(1)$$

with an arbitrary fixed constant $c \in]0,\ \beta_0[$ and β_0 satisfying $2\beta_0 + e\log\beta_0 = 0$. Then, it holds that

$$\lim_{N\to\infty} \frac{1}{\tau(N)} \log|I_N(\tau(N), \omega)| = h_D(\omega).$$

Comments on proof.

• The idea of the transformation by Vekua [62], [63] combined with Fourier series expansion enables us to find g_N.

• The $\tau = \tau(N)$ is chosen in such a way that, as $N \to \infty$

$$\int_{S^1} e^{iky\cdot\varphi} g_N(\varphi;\tau,\omega)ds(\varphi) \approx e^{y\cdot z}, y \in \overline{B_R}.$$

• A result in [19] is the core part. That is, as $\tau \to \infty$, it holds that, for all regular ω

$$\lim_{\tau\to\infty} \frac{1}{\tau} \left| \int_{\partial B_R} \left(\frac{\partial v_\tau}{\partial \nu} u - \frac{\partial u}{\partial \nu} v_\tau \right) ds(y) \right| = h_D(\omega),$$

where $v_\tau(y) = e^{y\cdot z}$ and $u(y) = w(y) + e^{ikx\cdot d}$. The proof of this formula is not a trivial matter.

• Another core is a connection formula of F and the Cauchy data of u on ∂B_R, see (2.9) in [9].

Remarkably enough, no assumption on k is imposed unlike the Linear Sampling, Factorization, Singular Sources and Probe Methods! A numerical work based on this formula, has been done in Ikehata-Niemi-Siltanen [48].

Finally let us give a bird view that indicates the central object in each method.

A Bird View

Linear Sampling Method, Factorization Method

Far field pattern of $\Phi(\,\cdot\,,x)$

Singular Sources Method, Probe Method

$\Phi(\,\cdot\,,x)$ itself

Enclosure Method

$e^{y\cdot z}$ with $z\cdot z=-k^2$

2. Time Domain Enclosure Method

The observation data of the original enclosure method come from partial differential equations being independent of time variable. However, there are a lot of inverse obstacle problems governed by partial differential equations in time domain, e.g., heat and wave equations, the Navier system, the Maxwell system, the Stokes system. Those are motivated by nondestructive testing of material, noninvasive evaluation of a body and the solutions may have applications to them. Thus it is quite natural to consider the following problem: how to treat observation data generated by the solutions of PDEs in time domain? This is a question the author has been pondering ever since first discovering the Enclosure Method.

In [22], by considering heat and wave equations in one-space dimension, the author proposed a set of general ideas applicable to inverse obstacle problems in time domain, which should be called the Time Domain Enclosure Method. Since then, we have a various applications in three-space dimensions, see Section 4 of [42] for the total and detailed survey on the Time Domain Enclosure Method and references therein until 2020.

In this section, we present some of prototype results for three types of inverse obstacle problems. Those are governed by wave equations in an exterior domain, the whole space and a bounded domain of $\mathbb{R}^3$ and show various ideas. It should be emphasized that those problems are just a few examples and there should be a lot of problems to be solved further by using the time domain enclosure method.

2.1. Exploring Surface of Obstacle

First we describe the prototype problem to be used for the explanation of the idea of exploring the surface of obstacle.

Let $D \subset \mathbb{R}^3$ be a bounded open set of $\mathbb{R}^3$ with $\partial D \in C^2$ and satisfy that $\mathbb{R}^3 \setminus \overline{D}$ is connected. A function $\gamma \in L^\infty(\partial D)$ satisfies $\gamma \ge 0$. The symbol $\nu = \nu_q$ denotes the unit outward normal vector at $q \in \partial D$. Let B be an open ball with a sufficiently small radius and satisfy $\overline{B} \cap \overline{D} = \phi$. We denote by χ_B the characteristic function of B.

Fix $0 < T < \infty$ and let $u = u_B(x,t)$, $(x,t) \in (\mathbb{R}^3 \setminus \overline{D})\times\,]0,\, T[$ solve

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, & (x,t) \in\, (\mathbb{R}^3 \setminus \overline{D})\times\,]0,\, T[, \\ u(x,0) = 0, & x \in \mathbb{R}^3 \setminus \overline{D}, \\ \dfrac{\partial u}{\partial t}(x,0) = \chi_B(x),\, x \in \mathbb{R}^3 \setminus \overline{D}, \\ \dfrac{\partial u}{\partial \nu} - \gamma(x)\dfrac{\partial u}{\partial t} = 0,\, (x,t) \in \partial D\times\,]0,\, T[. \end{cases}$$

Problem 3. Fix a large T (to be specified later). Extract information about geometry of D together with value of γ from observation data $u_B(x,t)$, $x \in B$, $0 < t < T$.

To explain of the role of γ look at the energy integral

$$\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}^3 \setminus \overline{D}} (|\partial_t u|^2 + |\nabla u|^2)\, dx,\, 0 < t < T.$$

Then, a formal computation yields

$$\mathcal{E}'(t) = -\int_{\partial D} \gamma(x)|\partial_t u|^2\, dS \le 0.$$

Thus the energy of solution may lose on the surface of obstacle because of γ. In this sense, the coefficient γ expresses the state of the surface of obstacle D. In the Maxwell system, it is corresponding to the surface admittance and $\gamma \equiv 1$ the admittance of the background medium. In fact, in the case when $\gamma \equiv 1$ and the space dimension is 1, one can not recognize the existence of D by the observation data [25].

In this subsection, the indicator function takes the form

$$I_B(\tau) = \int_B (w - v)\, dx, \tau > 0,$$

where

$$w(x) = w_B(x, \tau) = \int_0^T e^{-\tau t} u(x, t)\, dt, x \in \mathbb{R}^3 \setminus \overline{D}, \tau > 0$$

and $v = v_B(\,\cdot\,, \tau) \in H^1(\mathbb{R}^3)$ is the unique solution of the modified Helmholtz equation with inhomogeneous term

$$(\Delta - \tau^2)v + \chi_B(x) = 0, x \in \mathbb{R}^3.$$

It's explicit form is given by

$$v_B(x, \tau) = \frac{1}{4\pi} \int_B \frac{e^{-\tau|x-y|}}{|x-y|}\, dy, x \in \mathbb{R}^3.$$

The indicator function given above has a formal similarity to original Enclosure Method [16] as pointed out in [25]. More precisely, fix an arbitrary T and choose Ω a bounded domain of $\mathbb{R}^3$ in such a way that $\partial\Omega \in C^2$, $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected, $\overline{B} \cap \Omega = \phi$ and $\overline{D} \subset \Omega$. Then, as $\tau \to \infty$ we have

$$I_B(\tau) = \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu}\, w - \frac{\partial w}{\partial \nu}\, v \right) dS + O(\tau^{-1} e^{-\tau T}),$$

where ν denotes the unit outward normal vector to the surface $\partial\Omega$ and $v = v_B$ and $w = w_B$ satisfy

$$\begin{cases} (\Delta - \tau^2)v = 0, & x \in \Omega, \\ (\Delta - \tau^2)w = O(\tau e^{-\tau T}), & x \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w}{\partial \nu} - \tau\gamma(x)w = O(e^{-\tau T}), x \in \partial D. & \end{cases}$$

Before describing the first result we impose two conditions on γ.

- Write $\gamma << 1$ if $\exists C' > 0\ \ \gamma(x) \le 1 - C'$ a.e. $x \in \partial D$.
- Write $\gamma >> 1$ if $\exists C' > 0\ \gamma(x) \ge 1 + C'$ a.e. $x \in \partial D$.

Now we are ready to state the first result of this subsection, which has been established in [25].

Theorem 2.1. *Let $T > 2dist(D, B)$. We have*
(i) *if $\gamma << 1$, then for all sufficiently large τ we have $I_B(\tau) > 0$;*
(ii) *if $\gamma >> 1$, then for all sufficiently large τ we have $I_B(\tau) < 0$.*
Besides, in both cases it holds

$$\lim_{\tau\to\infty} \frac{\log |I_B(\tau)|}{2\tau} = -dist(D, B).$$

The last formula yields $d_{\partial D}(p) = \inf_{y\in\partial D}\ |y - p|$ since we have

$$d_{\partial D}(p) = \inf_{y\in\partial D} |y - p| = \text{dist}\,(D, B) + \eta.$$

Statements (i) and (ii) of Theorem 2.1 give us a qualitative criterion whether $\gamma >> 1$ or $\gamma << 1$ provided γ satisfies one of the conditions $\gamma >> 1$ and $\gamma << 1$ provided γ satisfies one of $\gamma >> 1$ or $\gamma << 1$. It is natural to go one step further or ask the question: can one explain its reason quantitatively ? In [31] we have already given the answer as follows.

First we introduce notation. Set $B_r(x) = \{y \in \mathbb{R}^3\,|\,|y-x| < r\}$ and $B = B_\eta(p)$, $p \in \mathbb{R}^3 \setminus \overline{D}$. Denote by $\Lambda_{\partial D}(p)$ the set of all points on ∂D that are most closest to point p, that is

$$\Lambda_{\partial D}(p) = \{q \in \partial D\,|\,|q - p| = d_{\partial D}(p)\,\}.$$

Hereafter, for simplicity, we call solution u_B the wave generated at p if $B = B_\eta(p)$ for a small known η

From Differential Geometry [56] one must recall the notion of shape operator and its property. For each $q \in \Lambda_{\partial D}(p)$ let $S_q(\partial D)$ denote the shape operator of surface ∂D at q with respect to the unit *outward* normal vector ν_q of ∂D at q. And the symbol $S_q(\partial B_{d_{\partial D}(p)}(p))$ denotes the shape operator of sphere $\partial B_{d_{\partial D}(p)}(p)$ at q with respect to the unit *inward* normal vector ($= -\nu_q$) of $\partial B_{d_{\partial D}(p)}(p)$ at q. Note that the point $q \in \Lambda_{\partial D}(p)$ attains the minimum value of function $\partial D \ni y \longmapsto |y-p|$, both $\partial B_{d_{\partial D}(p)}(p)$ and ∂D has a common tangent plane at q and as the quadratic form on the common tangent space at q it holds

$$S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D) \ge 0.$$

Succeeding to Theorem 2.1, the author in [31] clarified the leading profile of the indicator function as stated below.

THEOREM 2.2. *Assume* $\partial D \in C^3$, $\gamma \in C^2(\partial D)$ *and* γ *satisfies* $\gamma(x) > 1(\forall x \in \partial D)$ *or* $0 < \gamma(x) < 1(\forall x \in \partial D)$. *Assume also that center point* p *of* B *satisfies:* $\Lambda_{\partial D}(p)$ *is finite and*

$$det\,(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D)) > 0\, \forall q \in \Lambda_{\partial D}(p). \qquad (\star\star)$$

Then it holds that, as $\tau \to \infty$

$$e^{2\tau\, dist\,(D,B)}\, I_B(\tau) = \frac{\pi}{2}\left(\frac{\eta}{d_{\partial D}(p)}\right)^2 \sum_{q\in\Lambda_{\partial D}(p)} k_q(p)\frac{1-\gamma(q)}{1+\gamma(q)}\tau^{-4} + o(\tau^{-4}),$$

where

$$k_q(p) = \frac{1}{\sqrt{det\,(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D)).}}$$

Note that in the case when $\gamma \equiv 0$, the corresponding result has been derived as a corollary of [27].

Here we restrict ourself only to describing the keys for proof of Theorems 2.1-2.2.

Energy Decomposition Formula ([25]). We have, as $\tau \to \infty$

$$I_B(\tau) = J(\tau) + E(\tau) + O(\tau^{-1}e^{-\tau T}),$$

where

$$J(\tau) = \int_{\partial D}\left(\frac{\partial v}{\partial \nu} - \tau\,\gamma v\right) v\, dS$$

and

$$E(\tau) = \int_{\mathbb{R}^3\setminus\overline{D}} (|\nabla(w-v)|^2 + \tau^2|w-v|)^2)\, dx + \tau\int_{\partial D}\gamma|w-v|^2\, dS.$$

Energy Asymptotic Formula ([31]). Let $T > \text{dist}(D,B)$. Assume: (i) γ has a positive lower bound; (ii) the set of $\Lambda_{\partial D}(p)$ consists of finite points and satisfies ($\star\star$); (iii) there exists a point $q \in \Lambda_{\partial D}(p)$ such that $\gamma(q) \neq 1$. Then, we have, as $\tau \to \infty$

$$E(\tau) \sim \int_{\partial D}\left(\frac{\partial v}{\partial \nu} - \tau\gamma v\right)\frac{1-\gamma}{1+\gamma} v\, dS.$$

Theorems 2.1-2.2 enable us to extract information about the geometry of unknown obstacle together with the state of the surface as shown below.

Application of Theorem 2.1 (determination of all the points on $\Lambda_{\partial D}(p)$)

Given $p\in\mathbb{R}^3\setminus\overline{D}$ and direction ω, using wave, one can determine whether there exist a point on $\Lambda_{\partial D}(p)$ when going from p to ω-direction straightly

(a) Find $d_{\partial D}(p)$ by using Theorem 2.1 (wave generation at p).

(b) Pick a point p' outside B however, still near to p on the line from p to ω-direction.

(c) Find $d_{\partial D}(p')$ by using the similar procedure as **(a)** (wave generation at p').

Note that we have always $d_{\partial D}(p')\ge d_{\partial D}(p)-\vert p-p'\vert$. Besides, we know $d_{\partial D}(p')=d_{\partial D}(p)-\vert p-p'\vert$ if and only if $p+d_{\partial D}(p)\,\omega\in\Lambda_{\partial D}(p)$.

Thus, repeating **(b)** and **(c)** for infinitely many directions ω, one can completely reconstruct the set $\Lambda_{\partial D}(p)$ for given point p outside D.

Application of Theorem 2.2 (Determination of local shape and admittance)

Given a point $q\in\Lambda_{\partial D}(p)$, using three aves, one can find both of value of γ and approximate shape of ∂D at q.

(a) Pick three points p_1, p_2, p_3 on the segment connecting p and q (see Figure 3 for an illustration of the situation). Then we have $d_{\partial D}(p_j)$ since this coincides with $\vert p-q\vert-\vert p-p_j\vert$. Besides, it holds that $\Lambda_{\partial D}(p_j)=\{q\}$, $j=1,2,3$ and

$$\det\left(S_q(\partial B_{d_{\partial D}(p_j)}(p_j))-S_q(\partial D)\right)>0, j=1,2,3.$$

(b) Generate wave at each p_j. Applying Theorem 2.2 to each wave generated, one gets

$$\mathcal{F}_j\equiv\frac{1}{\sqrt{\det\left(S_q(\partial B_{d_{\partial D}(p_j)}(p_j))-S_q(\partial D)\right)}}\frac{1-\gamma(q)}{1+\gamma(q)}, j=1,2,3.$$

Let $H=H_{\partial D}(q)$, $K=K_{\partial D}(q)$ denote mean curvature, Gauss curvature of ∂D at q with respect to ν_q. It holds, for $\lambda_j=d_{\partial D}(p_j)^{-1}$, $j=1,2,3$

$$\det\left(S_q(\partial B_{d_{\partial D}(p_j)}(p_j))-S_q(\partial D)\right)=\lambda_j^2-2H\lambda_j+K.$$

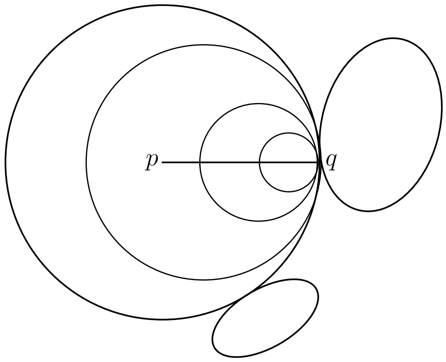

Figure 3. The largest circle denotes the sphere $|x-p| = d_{\partial D}(p)$. Other three circles denote the spheres centered at p_j with radius $d_{\partial D}(p_j)$, $j=1,2,3$. Two ellipses represent the connected components of obstacle D.

Then from **(b)** we have linear system

$$\begin{pmatrix} -(\lambda_1\mathcal{F}_1^2-\lambda_2\mathcal{F}_2^2) & \mathcal{F}_1^2-\mathcal{F}_2^2 \\ \\ -(\lambda_2\mathcal{F}_2^2-\lambda_3\mathcal{F}_3^2) & \mathcal{F}_2^2-\mathcal{F}_3^2 \end{pmatrix} \begin{pmatrix} 2H \\ \\ K \end{pmatrix} = \begin{pmatrix} \mathcal{F}_2^2\lambda_2^2-\mathcal{F}_1^2\lambda_1^2 \\ \\ \mathcal{F}_3^2\lambda_3^2-\mathcal{F}_2^2\lambda_2^2 \end{pmatrix}.$$

(c) Find $H = H_{\partial D}(q)$ and $K = K_{\partial D}(q)$ by solving the linear system above numerically with a regularization method.
(d) Find $\frac{1-\gamma(q)}{1+\gamma(q)}$ and thus $\gamma(q)$ from **(c)** and, for example, $\mathcal{F}_1$ of **(b)**.

This is a method of approaching and cricking three times to a point $q \in \Lambda_{\partial D}(p)$ which yields the curvatures $H(q)$, $K(q)$ and thus an approximate shape of ∂D at q and $\gamma(q)$.

Some additional remarks are in order.

• The classical result due to Majda [55] in the context of the Lax-Phillips scattering theory observes data at infinity and taken over the whole time $T=\infty$. Only Gauss curvature can be extracted under the strong convexity condition on the shape.

• Our result employs only near field data taken over finite time interval and provides two methods to extract (i) all the nearest points on the surface of obstacle from a given point p (ii) not only Gauss curvature but also mean curvature at an arbitrary given nearest point on the surface of obstacle from p.

• When the place where data is sent and received is away from obstacle, that is, when $d_{\partial D}(p) \to \infty$, Theorem 2.2 tells us that information about mean curvature is hidden. This indirectly explains why Majda's result yields Gaussian curvature only.

• Theorems 2.1-2.2 have been extended by the author to an analogous inverse obstacle problem governed by the Maxwell system in [29], [30] and [34].

2.2. Detecting Something Added

This subsection introduces an idea for the general problem: how do we detect when something has been added to a known possibly complex background medium? A typical situation is through-wall imaging [4] via electromagnetic waves. We formulae this as an inverse obstacle problem governed by the wave equation with a variable coefficient and present a result by using the Time Domain Enclosure Method.

Fix $0 < T < \infty$. Let B open ball with a sufficiently small radius. Let $u = u_B(x,t)$, $(x,t) \in \mathbb{R}^3 \times]0,\, T[$ solve

$$\begin{cases} \alpha(x)\dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, (x,t) \in \mathbb{R}^3 \times]0,\, T[, \\ \\ u(x,0) = 0, \qquad\qquad x \in \mathbb{R}^3, \\ \\ \dfrac{\partial u}{\partial t}(x,0) = \chi_B(x), \quad x \in \mathbb{R}^3, \end{cases}$$

where $\alpha \in L^\infty(\mathbb{R}^3)$ satisfies $\operatorname{ess.inf}_{x\in\mathbb{R}^3} \alpha(x) > 0$ and takes the form

$$\alpha(x) = \begin{cases} \alpha_0(x), \qquad\qquad x \in \mathbb{R}^3 \setminus D, \\ \\ \alpha_0(x) + h(x), x \in D, \end{cases}$$

D is a bounded open set of $\mathbb{R}^3$ with Lipschitz boundary, $\alpha_0 \in L^\infty(\mathbb{R}^3)$ satisfies $\exists m_0 > 0$, $\exists M_0 > 0$ $m_0^2 \le \alpha_0(x) \le M_0^2$ a.e. $x \in \mathbb{R}^3$ and $h \in L^\infty(D)$.
Here we impose two conditions on h:

• If $\exists C > 0$ $h(x) \ge C$ a.e. $x \in D$, write $\alpha >> \alpha_0$.

• If $\exists C > 0$ $-h(x) \ge C$ a.e. $x \in D$, write $\alpha << \alpha_0$.

The set D is a model of something added to the background medium described by α_0 with unknown jump h over D. The governing equation appears as a simplified model of propagation of electromagnetic wave.
In what follows if $D \ne \phi$, we assume one of two conditions $\alpha >> \alpha_0$ and $\alpha << \alpha_0$.

Problem 4. Let $\overline{B} \cap \overline{D} = \phi$. Fix a large T (to be specified later). Assume α_0 is known, both of D and h are unknown. Determine whether $D = \phi$ or not by using the observation data $u_B(x,t)$, $x \in B$, $0 < t < T$. Besides, when $D \neq \phi$, extract information about geometry of D and property of h, that is whether $\alpha >> \alpha_0$ or $\alpha << \alpha_0$ from observation data.

Define the indicator function by

$$I_B(\tau) = \int_B \alpha_0(x)(w - v)\,dx, \tau > 0,$$

where

$$w = w(x,\tau) = \int_0^T e^{-\tau t}\,u_B(x,t)\,dt, x \in \mathbb{R}^3$$

and $v = v(\,\cdot\,,\tau) \in H^1(\mathbb{R}^3)$ the unique solution of

$$\Delta v - \alpha_0(x)\tau^2 v + \alpha_0(x)\chi_B = 0, x \in \mathbb{R}^3.$$

Note that, to compute solution v numerically it needs full knowledge of α_0.

The following theorem is taken from [28].

THEOREM 2.3. (i) *If $D = \phi$, then, for all T we have* $\lim_{\tau\to\infty} e^{\tau T}\,I_B(\tau) = 0$. (ii) *If $D \neq \phi$, then, for all T satisfying $T > 2M_0\,dist(D,B)$ we have*

$$\lim_{\tau\to\infty} e^{\tau T}\,I_B(\tau) = \begin{cases} -\infty, & \alpha >> \alpha_0, \\ \\ \infty, & \alpha << \alpha_0. \end{cases}$$

Besides, it holds

$$0 < l \le dist(D,B) \le L < \infty,$$

where

$$\begin{cases} l = -\dfrac{1}{M_0}\liminf_{\tau\to\infty}\dfrac{\log|I_B(\tau)|}{2\tau}, \\ \\ L = -\dfrac{1}{m_0}\limsup_{\tau\to\infty}\dfrac{\log|I_B(\tau)|}{2\tau}. \end{cases}$$

In particular, the result, which gives upper and lower bounds on the Euclidean distance between a sound source and an obstacle using quantities that can be calculated from observational data, suggests a new direction for research into the inverse obstacle problem. As a closely related application of the enclosure method to the detection of an unknown obstacle behind a known impenetrable obstacle, we have [35].

Comments on proof. The key points are two facts (a) and (b) below.

(a) As $\tau\to\infty$, we have

$$\begin{cases} I_B(\tau)\le\tau^2\displaystyle\int_{\mathbb{R}^3}\frac{\alpha_0}{\alpha}(\alpha_0-\alpha)v^2\,dx+O(\tau^{-1}e^{-\tau T}),\\[2ex] I_B(\tau)\ge\tau^2\displaystyle\int_{\mathbb{R}^3}(\alpha_0-\alpha)v^2\,dx+O(\tau^{-1}e^{-\tau T}).\end{cases}$$

(b) We have

$$\begin{cases} v(x)\le\displaystyle\int_{\mathbb{R}^3}\alpha_0(y)f(y)\frac{e^{-m_0\tau|x-y|}}{4\pi|x-y|}\,dy,\\[2ex] v(x)\ge\displaystyle\int_{\mathbb{R}^3}\alpha_0(y)f(y)\,\frac{e^{-M_0\tau|x-y|}}{4\pi|x-y|}dy,\end{cases}$$

where $f(y)=\chi_B(y)(\ge 0)$ and recall $m_0^2\le\alpha_0(y)\le M_0^2$.

Remark 2.4. Theorem 2.3 in which $I_B(\tau)$ is replaced with another indicator function $I_B(\tau)_e$ defined by

$$I_B(\tau)_e=\int_B\alpha_0(x)(w-v_e)\,dx,\tau>0,$$

is also valid, where

$$v_e=v_e(x,\tau)=\int_0^T e^{-\tau t}V(x,t)\,dt,\tau>0$$

and V solves

$$\begin{cases} \alpha_0(x)\dfrac{\partial^2 V}{\partial t^2} - \Delta V = 0, (x,t) \in \mathbb{R}^3 \times\,]0,\, T[, \\ V(x,0) = 0, \qquad\qquad x \in \mathbb{R}^3, \\ \dfrac{\partial V}{\partial t}(x,0) = \chi_B(x), \quad x \in \mathbb{R}^3. \end{cases}$$

This version enables us to compute $I_B(\tau)_e$ experimentally without knowing α_0 outside B. Just generate wave V at B, observe on B and compute v_e via integral above. Thus it could be possible to make use of monitoring in a region of interest to catch a change from α_0 to α!

Before closing this subsection, for further research, we present a conjecture. It is concerned with an inverse obstacle problem governed by the Maxwell system. The problem is: can one detect something through-wall or placed in a complicated background medium by using electromagnetic waves in the time domain? We formulate the problem using the full Maxwell system in the whole space

$$\begin{cases} \epsilon\dfrac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{H} = -\sigma \boldsymbol{E} + \boldsymbol{J}, (x,t) \in \mathbb{R}^3 \times\,]0,\, T[, \\ \mu\dfrac{\partial \boldsymbol{H}}{\partial t} + \nabla \times \boldsymbol{E} = \boldsymbol{0}, \qquad\qquad (x,t) \in \mathbb{R}^3 \times\,]0,\, T[, \\ \boldsymbol{E}|_{t=0} = \boldsymbol{H}|_{t=0} = \boldsymbol{0}, \qquad\quad x \in \mathbb{R}^3, \end{cases}$$

where electric field $\boldsymbol{E} = \boldsymbol{E}(x,t) \in \mathbb{R}^3$, magnetic field $\boldsymbol{H} = \boldsymbol{H}(x,t) \in \mathbb{R}^3$, $(x,t) \in \mathbb{R}^3 \times [0,\, T]$; electric permittivity $\epsilon = \epsilon(x) \in \mathbb{R}$, magnetic permeability $\mu = \mu(x) \in \mathbb{R}$, electric conductivity $\sigma = \sigma(x) \in \mathbb{R}$; current density $\boldsymbol{J}(x,t) = f(t)\chi_B(x)\boldsymbol{a}$, constant unit vector $\boldsymbol{a}$, $f \in H^1(0,\, T)$ with $f(0) = 0$ and $f \not\equiv 0$; B a fixed open ball.

Here we introduce a set of basic assumptions.

• The ϵ, μ and σ belong to $L^\infty(\mathbb{R}^3)$ and satisfy ess. $\inf_{x\in\mathbb{R}^3} \epsilon(x) > 0$, ess. $\inf_{x\in\mathbb{R}^3} \mu(x) > 0$ and ess. $\inf_{x\in\mathbb{R}^3} \sigma(x) \geq 0$.

• The ϵ, μ, and σ have the expressions

$$\epsilon_r(x) \equiv \frac{\epsilon(x)}{\epsilon_0(x)} = \begin{cases} 1, & x \in \mathbb{R}^3 \setminus D, \\ 1 + e(x), & x \in D, \end{cases}$$

$$\mu_r(x) \equiv \frac{\mu(x)}{\mu_0(x)} = \begin{cases} 1, & x \in \mathbb{R}^3 \setminus D, \\ \\ 1 + m(x), & x \in D, \end{cases}$$

$$\sigma(x) = \begin{cases} \sigma_0(x), & x \in \mathbb{R}^3 \setminus D, \\ \\ \sigma_0(x) + h(x), & x \in D, \end{cases}$$

where D is a bounded open set of $\mathbb{R}^3$, $e, m, h \in L^\infty(D)$, $\epsilon_0 = \epsilon_0(x), \mu_0 = \mu_0(x), \sigma_0 = \sigma_0(x) \in L^\infty(\mathbb{R}^3)$ and satisfy $\text{ess.}\inf_{x\in\mathbb{R}^3} \epsilon_0(x) > 0$, $\text{ess.}\inf_{x\in\mathbb{R}^3} \mu_0(x) > 0$, $\text{ess.}\inf_{x\in\mathbb{R}^3} \sigma_0(x) \geq 0$. The $(\epsilon_0, \mu_0, \sigma_0)$ denotes the permittivity, permeability and conductivity of the background medium occupied the whole space and D is a model something added to the medium. In what follows we denote by $\boldsymbol{E} = \boldsymbol{E}_0$ if $(\epsilon, \mu, \sigma) = (\epsilon_0, \mu_0, \sigma_0)$.

In order to describe the conjecture, we introduce the indicator function defined as follows. Let $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ be linearly independent unit vectors and fix an open ball B such that $\overline{B} \cap \overline{D} = \phi$. Define the indicator function by

$$\boldsymbol{I}(\tau) = I(\tau, T) \equiv I_1(\tau) + I_2(\tau),$$

where

$$I_j(\tau) = \tilde{f}(\tau) \int_B \boldsymbol{a}_j \cdot ((\boldsymbol{W}_e)_j - (\boldsymbol{W}_0)_j) dx, \tau > 0,$$

$$(\boldsymbol{W}_e)_j(x, \tau) = \int_0^T e^{-\tau t} \boldsymbol{E}_j(x, t) dt, (\boldsymbol{W}_0)_j(x, \tau) = \int_0^T e^{-\tau t} (\boldsymbol{E}_0)_j(x, t) dt,$$

$$\boldsymbol{J} = f(t) \chi_B(x) \boldsymbol{a}, \boldsymbol{a} = \boldsymbol{a}_j$$

and

$$\tilde{f}(\tau) = \int_0^T e^{-\tau t} f(t) dt.$$

It is assumed that there exists a real number γ such that

$$\liminf_{\tau \to \infty} \tau^\gamma |\tilde{f}(\tau)| > 0.$$

This is a standing condition of $\boldsymbol{J}$ at $t=0$ not to vanish at $t=0$ with infinite order. Note that $\boldsymbol{E}_j$ and $(\boldsymbol{E}_0)_j$ are generated by the input current $\boldsymbol{J}$ with $\boldsymbol{a}=\boldsymbol{a}_j$.

The form of the indicator function as the sum of $I_1(\tau)$ and $I_2(\tau)$ goes back to that of [29]. Roughly speaking, both the two reflected waves generated by $\boldsymbol{J}$ with $\boldsymbol{a}=\boldsymbol{a}_j$, $j=1,2$ can not be weak at the same time and so one of $I_1(\tau)$ and $I_2(\tau)$ would catch a strong reflected signal from obstacle surface. So we take their sum.

Besides, we impose two conditions on the jump of ϵ_r and μ_r over D listed below.

$$\textbf{(A.I)}\ \exists C_1>0\ \left(1-\frac{1}{\epsilon_r(x)}\right)+(1-\mu_r(x))\geq C_1 \text{ a.e. } x\in D.$$

$$\textbf{(A.II)}\ \exists C_2>0\ (1-\epsilon_r(x))+\left(1-\frac{1}{\mu_r(x)}\right)\geq C_2 \text{ a.e. } x\in D.$$

As far as the author knows, these conditions are new. The author proposed the following conjecture at online talk on Dec. 16th. 2021, Akademgorodok, Novosibirsk, Russia, in Conference Eurasia 2021, which corresponds to Theorem 2.3 and is still unsolved.

Conjecture. Let $T>2E_0M_0\,\mathrm{dist}(D,B)$ and positive constants m_0, M_0, e_0 and E_0 satisfy

$$e_0\leq\epsilon_0(x)\leq E_0,\ \ m_0\leq\mu_0(x)\leq M_0 \text{ a.e. } x\in\mathbb{R}^3.$$

Assume that one of conditions (A.I) and (A.II) is satisfied with (ϵ_r,μ_r). Then it holds

$$0<L\leq\mathrm{dist}\,(D,B)\leq U<\infty,$$

where

$$\begin{cases} L=-\dfrac{1}{\sqrt{M_0E_0}}\liminf_{\tau\to\infty}\dfrac{\log|\boldsymbol{I}(\tau)|}{2\tau}, \\[2ex] U=-\dfrac{1}{\sqrt{m_0e_0}}\limsup_{\tau\to\infty}\dfrac{\log|\boldsymbol{I}(\tau)|}{2\tau}. \end{cases}$$

In [41], we have already shown that if $\epsilon_0\equiv e_0=E_0$, $\mu_0\equiv m_0=M_0$, that is the uniform background case, then the conjecture is true as

$$L = U = \text{dist}\,(D, B) = -\frac{1}{\sqrt{\epsilon_0 \mu_0}} \lim_{\tau \to \infty} \frac{\log |\boldsymbol{I}(\tau)|}{2\tau}.$$

For the general case, the author in [41] confirmed the validity of a result analogous to (a) in comment on proof of Theorem 2.3. However, the approach by using (b)-type estimate in that of Theorem 2.3 still problematic because of system.

2.3. Designing Neumann Data

In this last subsection we present an idea that generates a *good input* to extract information about the geometry of unknown obstacle embedded in a known background medium.

Let Ω be a bounded domain (connected open) of $\mathbb{R}^3$ with $\partial\Omega \in C^2$. Let D be a nonempty open subset of $\mathbb{R}^3$ with $\partial D \in C^2$ and satisfy $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected.

Fix $0 < T < \infty$. Given $f = f(x,t), (x,t) \in \partial\Omega \times\,]0,\, T[$ let $u = u_f(x,t), (x,t) \in (\Omega \setminus \overline{D}) \times\,]0,\, T[$ solve

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, (x,t) \in\ (\Omega \setminus \overline{D}) \times\,]0,\, T[, \\[2mm] u(x,0) = 0, \qquad x \in \Omega \setminus \overline{D}, \\[2mm] \dfrac{\partial u}{\partial t}(x,0) = 0, \quad x \in \Omega \setminus \overline{D}, \\[2mm] \dfrac{\partial u}{\partial \nu} = 0, \qquad (x,t) \in \partial D \times\,]0,\, T[, \\[2mm] \dfrac{\partial u}{\partial \nu} = f(x,t), \quad (x,t) \in \partial\Omega \times\,]0,\, T[. \end{cases}$$

We call this system the inner system (IS) which is the governing equation of inverse obstacle problem mentioned below.

Problem 5. Fix a large T (to be specified later). Find a suitable Neumann data f such that the observation data $u_f(x,t)$, $x \in \partial\Omega$, $0 < t < T$ corresponding to f yields information about the geometry of D.

A method of generating suitable Neumann data for Enclosure Method, which

is a solution to Problem 5, is so simple and as follows.
Let $B = B_\eta(p)$ be an open ball with radius η centered at p and satisfy $\overline{B} \cap \overline{\Omega} = \phi$.

Let $v = v(x,t)$, $(x,t) \in \mathbb{R}^3 \times]0,\, T[$ solve

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \Delta v = 0, & (x,t) \in \mathbb{R}^3 \times]0,\, T[, \\ v(x,0) = 0, & x \in \mathbb{R}^3, \\ \dfrac{\partial v}{\partial t}(x,0) = (\eta - |x-p|)\chi_B(x), x \in \mathbb{R}^3 & \end{cases}$$

and set

$$f(x,t) = \frac{\partial v}{\partial \nu}(x,t), x \in \partial\Omega, 0 < t < T.$$

We call the governing equation of v the outer system (OS).

Using the corresponding solution u_f, we define

$$I_B(\tau) = \int_{\partial\Omega} (w - w_0)\partial_\nu w \, dS, \tau > 0,$$

where

$$\begin{cases} w(x) = w(x,\tau) = \displaystyle\int_0^T e^{-\tau t} u_f(x,t) dt, \; x \in \Omega \setminus \overline{D}, \\ w_0(x) = w_0(x,\tau) = \displaystyle\int_0^T e^{-\tau t} v(x,t) dt, x \in \mathbb{R}^3. \end{cases}$$

Both the values of w and $\partial_\nu w$ on $\partial\Omega$ can be calculated from those of f and u_f on $\partial\Omega \times]0,\, T[$.

The following theorem established in [32] suggests a new direction of the time domain enclosure method.

THEOREM 2.5. (i) *Let T satisfy*

$$T > 2\, dist(D,B) - dist(\Omega, B).$$

Then, for all sufficiently large τ we have $I_B(\tau) > 0$ and

$$\lim_{\tau \longrightarrow \infty} \frac{1}{\tau} \log I_B(\tau) = -2\, dist(D,B).$$

(ii) *We have* $0-\infty$ *criterion*

$$\lim_{\tau\longrightarrow\infty} e^{\tau T} I_B(\tau) = \begin{cases} \infty, T > 2\,dist\,(D,B), \\ \\ 0, \;\; T < 2\,dist\,(D,B). \end{cases}$$

This theorem shows the specially designed input Neumann data has the effect of making outer boundary transparent. If the transmitter is placed close to the outer surface and sends a specially designed wave, then the outer surface can be ignored. It should be emphasized that the Neumann data do not have a separation of variables type form, that is, a product of function of x and that of t, such as considered in [24].

Ideas of choosing another type of initial data in OS have been proposed in [36] using the time reversal invariance of the wave equation and [39] choosing a shell-type support of the initial data. These ideas yield the smallest sphere that encloses unknown obstacle with an arbitrary center point.

We propose the guiding principle: given IS, one can choice an appropriate OS.

It is not necessary to choose the same type of equation of OS as that of IS. Some experimental attempts have been proposed in [37]. In the article, IS is the heat equation and OS is the wave equation with growing propagation speed. In [38], IS is the heat equation and OS is a wave equation. The time-reversal invariance of the wave equation helps to generate a *good* Neumann data (input heat flux) in the sense that it never blow up on the outer surface.

And we have other published articles on classical partial differential equations listed below.

Table 1. This table presents the OS and IS for some references.

	OS	IS
[33], [38]	wave eq/heat eq	Thermo elasticity system
[46]	heat eq	Stokes system

Finally, it should be noted that, in Ikehata-Kian [47] the idea of this subsection has been applied also to an inverse problem governed by the time fractional diffusion equation in a bounded domain of $\mathbb{R}^3$ over the time interval $]0,\,\infty[$

$$\left(\frac{\partial}{\partial t}\right)^{\alpha(x)} u - \Delta u = 0,$$

where $\left(\frac{\partial}{\partial t}\right)^{\alpha(x)}$ denotes the Caputo fractional derivative of order $\alpha(x) \in]0,\, 1[$ with respect to t.

3. Integrating Probe and Singular Sources Methods

This section focuses on the author's most recent study on the Probe and Singular Sources Methods. As we saw in the first section, the common character of both methods is summarized as follows.

• Calculating a "field" defined outside an unknown obstacle from the observation data

• Blowing up of the "field" on the surface of the obstacle

Such a field called the indicator function. Recently in [43] the author proposed an integrated theory of the Probe and Singular Sources Methods (IPS) by considering a prototype inverse obstacle problem in a bounded domain.

The problem is as follows. Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary. It is a mathematical model of the domain occupied by a known background medium. We denote by D the unknown obstacle embedded in Ω D embedded in Ω. We assume that D is a non-empty open set of $\mathbb{R}^3$ with smooth boundary such that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected.

Given a function f on $\partial\Omega$ let $u = u(x)$ solve

$$\begin{cases} \Delta u = 0, x \in \Omega \setminus \overline{D}, \\ \\ \dfrac{\partial u}{\partial \nu} = 0, x \in \partial D, \\ \\ u = f, \quad x \in \partial\Omega. \end{cases}$$

In this section, the observation data is given by the Dirichlet-to-Neumann map Λ_D defined by

$$\Lambda_D : f \longmapsto \frac{\partial u}{\partial \nu}|_{\partial\Omega}.$$

Write $\Lambda_D = \Lambda_0$ if $D = \phi$. The prototype inverse obstacle problem to be discussed here is

Problem 6. Reconstruct D from Λ_D.

The contents of this section are mainly based on two papers [43] and [44] and consist of three parts.

• Introducing the singular sources method combined with the notion of the probe method

• Finding a third indicator function whose two ways decomposition yields the indicator functions in the Probe and Singular Sources Methods

• Finding the completely integrated version of the Probe and Singular Sources Methods and its byproduct

3.1. Singular Sources Method via Probe Method

The Probe Method has two sides which are called the Side A and, Side B firstly appeared in [20].

First recall the Side A of Probe Method. To describe Side A we need three concepts listed below.

Needle Given $x \in \Omega$, a non-self intersecting piecewise linear curve $\sigma : [0,\, 1] \longrightarrow \overline{\Omega}$ is called a needle with tip at x if $\sigma(0) \in \partial\Omega$, $\sigma(t) \in \Omega$ for $0 < t \le 1$ and $\sigma(1) = x$. We write $\sigma \in N_x$.

Singular solution Given $x \in \Omega$ let

$$G(y-x) = \frac{1}{4\pi|y-x|}, y \in \mathbb{R}^3 \setminus \{x\}.$$

Needle sequence Given $x \in \Omega$ and $\sigma \in N_x$, a sequence $\{v_n\}$ of solutions of the Laplace equation $\Delta v = 0$ in Ω is called a needle sequence for (x, σ) if $v_n \to G(\,\cdot - x)$ in $H^1_{\text{loc}}(\Omega \setminus \sigma)$.

In what follows, we write $\int_{\partial\Omega} fg\, dS =< f, g >$. The Side A of Probe Method is summarized as follows, which is taken from [20].

Theorem 3.1. (i) *Let $x \in \Omega \setminus \overline{D}$ and $\sigma \in N_x$. If $\sigma \cap \overline{D} = \phi$, then for all needle sequences $\{v_n\}$ for (x, σ), it holds that*

$$\lim_{n\to\infty} < (\Lambda_0 - \Lambda_D) f_n, f_n >= I(x),$$

where $f_n = v_n|_{\partial\Omega}$.

(ii) *For each $a \in \partial D$, it holds that*

$$\lim_{x\to a} I(x) = \infty.$$

(iii) *For each* $\epsilon > 0$, *it holds that*

$$\sup_{x \in \Omega \setminus \overline{D}, dist(x, \partial D) > \epsilon} |I(x)| < \infty.$$

The function $I(x)$ of independent variables $x \in \Omega \setminus \overline{D}$ satisfying (i) to (iii) is called the indicator function for the Probe Method. It follows from (i) and integration by parts that the indicator function takes the form

$$I(x) = \|\nabla w_x\|^2_{L^2(\Omega \setminus \overline{D})} + \|\nabla G(\,\cdot\, - x)\|^2_{L^2(D)},\ x \in \Omega \setminus \overline{D},$$

where the function $w_x = w_x(y) = w(y)$, $y \in \Omega \setminus \overline{D}$ is the unique solution of

$$\begin{cases} \Delta w = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w}{\partial \nu} = -\dfrac{\partial}{\partial \nu} G(y - x), & y \in \partial D, \\ w = 0, & y \in \partial \Omega. \end{cases}$$

The w_x is called the reflected solution and appears in the energy integral expression of $I(x)$ as above. See also subsection 1.2, Probe Method. This is the case when $k = 0$.

The Singular sources method via probe method has been proposed in [43] as follows.

THEOREM 3.2. (i) *Let* $x \in \Omega \setminus \overline{D}$ *and* $\sigma \in N_x$. *If* $\sigma \cap \overline{D} = \phi$, *then for all needle sequences* $\{v_n\}$ *for* (x, σ), *it holds that*

$$-\lim_{n \to \infty} < (\Lambda_0 - \Lambda_D) f_n, g_n >= w_x(x),$$

where $f_n = v_n|_{\partial\Omega}$ *and* $g_n = (G(\,\cdot\, - x) - v_n)|_{\partial\Omega}$.
(ii) *For each* $a \in \partial D$, *it holds that*

$$\lim_{x \to a} w_x(x) = \infty.$$

(iii) *For each* $\epsilon > 0$, *it holds that*

$$\sup_{x \in \Omega \setminus \overline{D}, dist(x, \partial D) > \epsilon} |w_x(x)| < \infty.$$

Theorem 3.2 realizes the spirit of Singular Sources Method since the restriction of reflected solution $w_x(y)$ to the diagonal set $y=x$ plays the same role as $I(x)$.

The difference between Theorems 3.1 and 3.2 is just the computation formula of each indicator function:

$$\begin{cases} -\lim_{n\to\infty} <(\Lambda_0-\Lambda_D)f_n, g_n>= w_x(x), \\ \\ \lim_{n\to\infty} <(\Lambda_0-\Lambda_D)f_n, f_n>= I(x), \\ \\ f_n=v_n|_{\partial\Omega},\, g_n=(G(\,\cdot\,-x)-v_n)|_{\partial\Omega}. \end{cases}$$

This is the meaning of "Introducing the Singular Sources Method combined with the notion of the Probe Method".

3.2. Integrated Theory (IPS)

In [43] the author introduced a new concept which integrates both the Probe and Singular Sources Methods.

The IPS function Given $x\in\Omega\setminus\overline{D}$, let $W=W_x=W_x(y)$, $y\in\Omega\setminus\overline{D}$ be the unique solution of

$$\begin{cases} \Delta W=0, & y\in\Omega\setminus\overline{D}, \\ \\ \dfrac{\partial W}{\partial\nu}=-\dfrac{\partial}{\partial\nu}G(y-x), & y\in\partial D, \\ \\ W=G(y-x), & y\in\partial\Omega. \end{cases}$$

The restriction of $W_x(y)$ to $y=x$ is called the IPS function.

Clearly, the IPS function has a natural decomposition which is called

Outer decomposition of IPS It holds that

$$W_x(x)=w_x(x)+w_x^1(x), x\in\Omega\setminus\overline{D},$$

where $w^1 = w^1_x = w^1_x(y)$, $y \in \Omega \setminus \overline{D}$ solves

$$\begin{cases} \Delta w^1 = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w^1}{\partial \nu} = 0, & y \in \partial D, \\ w^1 = G(y-x), y \in \partial\Omega. \end{cases}$$

From this we have immediately, as $x \to \partial D$

$$W_x(x) = w_x(x) + O(1).$$

Our finding in [43] is another decomposition of IPS function

Inner decomposition of IPS It holds that

$$W_x(x) = I(x) + I^1(x), x \in \Omega \setminus \overline{D},$$

where

$$I^1(x) = \|\nabla w^1_x\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla G(\,\cdot\, - x)\|^2_{L^2(\mathbb{R}^3\setminus\overline{\Omega})}.$$

This yields, as $x \to \partial D$

$$W_x(x) = I(x) + O(1).$$

Therefore, as $x \to \partial D$, modulo O(1)

$$W_x(x) \sim w_x(x) \sim I(x).$$

This is the meaning of "Finding a third indicator function whose two ways decomposition yields the indicator functions in Probe and Singular Sources Methods".

Finally, we explain the meaning of "Finding the completely integrated version of the probe and singular sources methods".

Recall a special singular solution of the Laplace equation.

Green's function for Ω Given $x \in \Omega$, let $H(\,\cdot\,,x)$ be the unique solution of

$$\begin{cases} \Delta H = 0, & y \in \Omega, \\ \\ H = -G(y-x), & y \in \partial\Omega. \end{cases}$$

Define

$$G(y,x) = G(y-x) + H(y,x), y \in \Omega.$$

For each fixed $x \in \Omega$, the $G(\,\cdot\,,x)$ satisfies

$$\Delta_y G(y,x) + \delta(y-x) = 0, y \in \Omega$$

in the sense of distribution and $G(\,\cdot\,,x) = 0$ on $\partial\Omega$. Note that Δ_y denotes the Laplacian with respect to the variable $y = (y_1, y_2, y_3)$.

Now replace $G(\cdot\, - x)$ with $G(\,\cdot\,,x)$ in the definition of w_x, w_x^1 and W_x and denote them by w_x^*, $(w_x^1)^*$, W_x^*, respectively. Then, the governing equation of $w_x^* = w^*$ becomes

$$\begin{cases} \Delta w^* = 0, & y \in \Omega \setminus \overline{D}, \\ \\ \dfrac{\partial w^*}{\partial \nu} = -\dfrac{\partial}{\partial \nu} G(y,x), & y \in \partial D, \\ \\ w^* = 0, & y \in \partial\Omega. \end{cases}$$

We have $(w_x^1)^* = 0$ and thus $W_x^*(x) = w_x^*(x)$.

The indicator function for the completely integrated version of the Probe and Singular Sources Methods is defined by

$$I^*(x) = \|\nabla w_x^*\|^2_{L^2(\Omega \setminus \overline{D})} + \|\nabla G(\,\cdot\,,x)\|^2_{L^2(D)}, x \in \Omega \setminus \overline{D}.$$

The property of $I^*(x)$ is described as below, which is taken from [43].

Theorem 3.3. (i) *Given $x \in \Omega \setminus \overline{D}$ and $\sigma \in N_x$ let $\{v_n(\,\cdot\,;x)\}$ be an arbitrary needle sequence for (x,σ). If $\sigma \cap \overline{D} = \phi$, then we have*

$$\lim_{n \to \infty} < (\Lambda_0 - \Lambda_D) g_n, g_n >= I^*(x),$$

where $g_n = (G(\,\cdot\, - x) - v_n)|_{\partial\Omega}$.

(ii) *We have, for each* $a \in \partial D$

$$\lim_{x \to a} I^*(x) = \infty.$$

(iii) *We have, for each* $\epsilon > 0$

$$\sup_{x\in\Omega\setminus\overline{D},\ dist\,(x,\partial D)>\epsilon, dist(x,\partial\Omega)>\epsilon} I^*(x) < \infty.$$

(iv) *For all* $x \in \Omega \setminus \overline{D}$ *we have*

$$I^*(x) = w_x^*(x).$$

Comments on proof. It holds that

$$g_n = -(v_n + H(\,\cdot\,, x))|_{\partial\Omega},$$

$$\Delta_y(v_n(y) + H(y, x)) = 0,\ \ y \in \Omega,$$

and

$$v_n(y) + H(y, x) \to G(y, x) \text{ in } H^1_{\text{loc}}(\Omega \setminus \sigma).$$

Thus everything is reduced to Probe Method in which just replace $G(\,\cdot\, - x)$ with $G(\,\cdot\,, x)$.

REMARK 3.4. Besides, it holds that, for all $x \in \Omega \setminus \overline{D}$

$$I^*(x) = I(x) + 2(I^1(x) - w_x^1(x)) + < (\Lambda_0 - \Lambda_D)(G(\,\cdot\, - x)|_{\partial\Omega}), G(\,\cdot\, - x)|_{\partial\Omega} > .$$

Thus, as $x \to \partial D$

$$I^*(x) = I(x) + O(1).$$

Note (iv) of Theorem 3.3 and the inner decomposition of IPS function yield also

$$w_x^*(x) = w_x(x) + (I^1(x) - w_x^1(x)) + < (\Lambda_0 - \Lambda_D)(G(\,\cdot\, - x)|_{\partial\Omega}), G(\,\cdot\, - x)|_{\partial\Omega} > .$$

Green's function $G(\,\cdot\,, x)$ eliminated!

3.3. Side B of IPS and Solution to Old Question

Let $x \in \Omega$. Let $\sigma \in N_x$ and $\{v_n\}$ be an arbitrary needle sequence for (x, σ). In IPS we have three sequences (called indicator sequence)

$$I_n(x, \sigma, \{v_n\}) = \begin{cases} < (\Lambda_0 - \Lambda_D) f_n, f_n > \text{ (Probe Method)} \\ \\ - < (\Lambda_0 - \Lambda_D) f_n, g_n > \text{ (Singular Sources Method)} \\ \\ < (\Lambda_0 - \Lambda_D) g_n, g_n > \text{ (Completely Integrated)} \end{cases}$$

where $f_n = v_n|_{\partial\Omega}$, $g_n = (G(\,\cdot\, - x) - v_n)|_{\partial\Omega}$.
Side B asks

• Fix $I_n(x, \sigma, \{v_n\})$ and study the behaviour as $n \to \infty$.

The result below should be called the Side B of IPS, which has been established by combining [43] and an idea of [44].

Theorem 3.5. *Let $x \in \Omega$ and $\sigma \in N_x$. Assume that one of the two cases* (i) *and* (ii) *is satisfied:*
(i) *$x \in \overline{D}$;*
(ii) *$x \in \Omega \setminus \overline{D}$ and $\sigma \cap D \neq \phi$.*
Then, for any needle sequence $\{v_n\}$ for (x, σ) we have

$$\lim_{n\to\infty} I_n(x, \sigma, \{v_n\}) = \infty.$$

See Figure 4 for an illustration of the cases (i) and (ii) of Theorem 3.5.

And we have an explicit characterization of $\overline{D}$ itself.

Corollary 3.6. *Corollary 3.6. A point $x \in \Omega$ belongs to the set $\overline{D}$ if and only if for any needle $\sigma \in N_x$ and needle sequence $\{v_n\}$ for (x, σ) we have*

$$\lim_{n\to\infty} I_n(x, \sigma, \{v_n\}) = \infty.$$

In the proof, the blowing up property of needle sequences described below is essential and firstly established in [20].

Proposition 3.7. *Given an arbitrary point $x \in \Omega$ and needle $\sigma \in N_x$ let $\{v_n\}$ be an arbitrary needle sequence for (x, σ).*
(i) *Le V be an arbitrary finite cone with vertex at x. Then, we have*

$$\lim_{n\to\infty} \|\nabla v_n\|^2_{L^2(V\cap\Omega)} = \infty.$$

(ii) *Let $z \in \Omega$ be an arbitrary point on $\sigma \setminus \{x\}$ and B open ball centered at z. Then, we have*

$$\lim_{n\to\infty} \|\nabla v_n\|^2_{L^2(B\cap\Omega)} = \infty.$$

REMARK 3.8. Proposition 3.7 says that the energy of needle sequence concentrates on needle and blows up! It made us to create a new side of the original probe method [15], which we call now the Side B.

In the proof of Theorem 3.5 the case when

$$I_n(x,\sigma,\{v_n\}) = -<(\Lambda_0-\Lambda_D)f_n, g_n> \text{ (Singular Sources method)}$$

is most interesting one. Having Theorem 3.5 and Corollary 3.6, IPS enables us to answer the old question (see also subsection 4.1 of [43] for a discussion).

- Does Singular Sources Method have Side B as Probe Method?

Now we can say the answer is yes! The key idea is so simple and was found in [44] which considers IPS of the Stokes system. Fix $x \in \Omega$. We have

$$f_n(y) + g_n(y) = G(y-x),\ y \in \partial\Omega.$$

Thus

$$<(\Lambda_0-\Lambda_D)(G(\,\cdot\,-x))|_{\partial\Omega}, G(\,\cdot\,-x)|_{\partial\Omega}>$$

$$=<(\Lambda_0-\Lambda_D)f_n, f_n> + <(\Lambda_0-\Lambda_D)g_n, g_n>$$

$$+<(\Lambda_0-\Lambda_D)f_n, g_n> + <(\Lambda_0-\Lambda_D)g_n, f_n>.$$

By the symmetry of DN map, this yields

$$-<(\Lambda_0-\Lambda_D)f_n, g_n)> = \frac{1}{2}(<(\Lambda_0-\Lambda_D)f_n, f_n> + <(\Lambda_0-\Lambda_D)g_n, g_n>$$

$$-<(\Lambda_0-\Lambda_D)(G(\,\cdot\,-x)|_{\partial\Omega}), G(\,\cdot\,-x)|_{\partial\Omega}>).$$

From this and the behaviour of two sequences $\{<(\Lambda_0-\Lambda_D)f_n, f_n>\}$ and $\{<(\Lambda_0-\Lambda_D)g_n, g_n>\}$, we see the validity of the Side B of Singular Sources Method (formulated by Probe Method). The old question was finally solved purely algebraically. The author thinks this is the most emphasized part of the theoretical advantage of IPS.

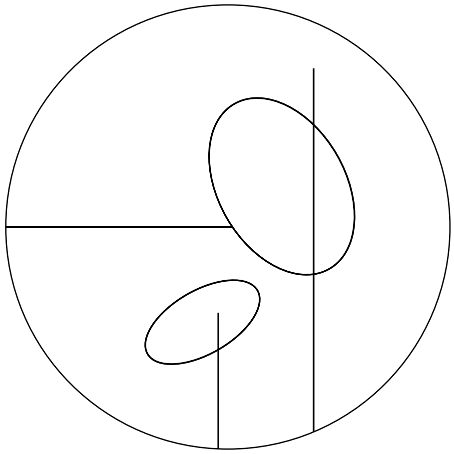

Figure 4. The circle denotes $\partial\Omega$. Two ellipses represent the connected components of obstacle D. The three segments are needles and the tips of them satisfy (i) or (ii) of Theorem 3.5.

3.4. Further Development

In [44] by developing IPS for the Stokes system, the author has considered

• How to treat the complexity caused by system of equations.

In particular, as a byproduct, we have established, in particular, the corresponding theorem for the Stokes system to Theorem 3.5.

In the article [45], by considering an inverse obstacle problem governed by the Helmholtz equation we have established

• A way from IPS to Probe and Singular Sources Methods.

• Creation of Method of Complimenting Function as a technique in IPS.

Pursuing IPS for various equations like the Maxwell's, Navier's ones, and the heat equation in time domain, etc., to be continued.

3.5. Other References

Finally we mention the articles [60] and [61].

• They state a relationship between the probe and singular sources method in terms of the singular sources method, however, no IPS view together with Side B. And see [42] for a total review on Probe and Enclosure Methods from the author's point of view together with [26] for an earlier review.

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