

Asymptotic properties of solutions to elastic equations in a perturbed half-space in \mathbf{R}^3

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Abstract. We give an overview of our recent results for an elastic wave equation in a locally perturbed half-space in \mathbf{R}^3 . We derive asymptotic expansions at infinity of stationary solutions, formulate the radiation condition and prove the uniqueness of scattering solutions including body waves as well as Rayleigh waves.

1. Introduction

The elastic equation describes the motion of displacement vectors of solids (see e.g. [2]). If the material occupies the whole space \mathbf{R}^3 , there occur two types of spherical waves with two propagation speeds: the primary wave (P-wave) and the secondary wave (S-wave). If the material is homogeneous at infinity, the motion splits into these two waves. However, if there is a boundary, different phenomenon appears. Let us consider the half-space case $\mathbf{R}_+^3 = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3; x_3 > 0\}$, which is already hard to study in spite of its appearance. There occur two types of waves. One is the body wave, as in the case \mathbf{R}^3 , having a spherical shape and propagating inside the body. The other is the *surface wave*, called the Rayleigh wave, which is exponentially decreasing in the direction vertical to the surface and propagating like a two-dimensional spherical wave along the surface (see [8]). One can calculate these waves explicitly by passing to the partial Fourier transform in the reduced wave equation, and by projecting to the eigenspaces associated with these waves. One can then observe these waves separately. However, it is far from obvious to derive these waves simultaneously in the asymptotic expansion at infinity of solutions to the reduced wave equation. The main difficulty is to find an appropriate topology which can capture all of these waves. The body waves move three-dimensionally while the Rayleigh waves propagate two-dimensionally. The

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same problem also occurs in the elastic wave equation in a stratified medium, for which there exist Stonely waves propagating only along the interface.

In the series of works [10], [11], [12], we studied this problem with the following strategy. We use the classical method of stationary scattering theory. Dermenjian-Guillot [4] constructed the plane wave solutions to the unperturbed elastic equation in the half-space, and proved the limiting absorption principle for the perturbed operator in weighted L^2 spaces. Dermenjian-Gaitan [5] derived an eigenfunction expansion theorem which makes it possible to represent all functions by these generalized plane waves. Our arguments are based on these results. As a new step, we employ the Besov type space \mathcal{B}^* introduced by Agmon-Hörmander [1]. We represent the resolvent by the generalized plane waves and derive its asymptotic expansion at infinity by applying the stationary phase method. All of the above body waves and surface waves are then captured in the expansion simultaneously without any restriction on the direction of propagation. Here, we encounter new difficulties. The wave equation in a half space or a stratified medium has the phenomena of reflection or refraction of waves at the boundary or the interface, which give rise to singularities for the second order derivatives of generalized eigenfunctions along some cone or critical angle. However, by a close application of the classical stationary phase method, we can overcome this difficulty.

Many works have been devoted to the spectral and scattering theory for symmetric systems (see e.g. [15], [18], [20], and as rather recent results [21], [19]). Let us mention in particular that in [6], the radiation condition and the uniqueness of solutions to elastic equations have been announced for the case of \mathbf{R}_+^2 with free boundary. In [13], the authors studied the elastic equation in a wedge domain in \mathbf{R}^2 , the limiting absorption principle and the radiation condition and proved the uniqueness theorem. The function spaces they adopted are different from the weighted L^2 -spaces, Besov type spaces used in this paper.

The radiation condition is a type of boundary condition at infinity that ensures the uniqueness of solutions to partial differential equations. It is generally expected that such a condition should be derived naturally from the asymptotic behavior of the solution (or the resolvent) at spatial infinity. However, in cases involving Rayleigh waves, only local asymptotic information had been available, and a complete description had remained elusive. In our previous work, we successfully established a full characterization of the asymptotic behavior of the solution, even in the presence of Rayleigh waves. This allowed us, for the first time, to formulate a radiation condition that naturally reflects the solution's (or resolvent's) behavior at infinity, including the contribution of surface waves. This radiation condition is expressed in the \mathcal{B}^* space, which is a natural function space for characterizing scat-

tering solutions to the elastic wave equation in a half-space. The norm of this space precisely captures the full asymptotic profile, including Rayleigh waves. Therefore, describing the radiation condition within this space is not only natural but also essential.

In Section 2, we illustrate the main problems for the case of the wave equation in an exterior domain. We explain the limiting absorption principle, Besov type spaces, radiation condition and eigenfunction expansion theorems. In Section 3, we explain them for the case of elastic equation in \mathbf{R}^3 . In Section 4, we start our main theme of elastic equation in a half-space. The asymptotic expansion of the resolvent and the stationary solutions are derived in the topology of \mathcal{B}^* including all of body waves and Rayleigh waves. In Section 5, we formulate the radiation condition and state the uniqueness theorem.

2. Reduced wave equation in an exterior domain

2.1. Rellich type theorem

We first consider the case of $H = -\Delta$ defined in a domain Ω exterior to a bounded open set Ω in \mathbf{R}^n , $n \geq 2$, with Neumann boundary condition on $\partial\Omega$. To study the wave equation $\partial_t^2 u = -Hu$ in Ω , we reduce it to the behavior of time-periodic solutions, i.e. solutions of the form $u(t, x) = e^{-it\sqrt{\lambda}}v(x)$, which leads us to the Helmholtz equation

$$(1) \quad (-\Delta - \lambda)v = 0 \quad \text{in } \Omega, \quad \partial_\nu v = 0 \quad \text{on } \partial\Omega,$$

ν being the unit normal to $\partial\Omega$. We assume that the energy λ is positive. In the case of $\Omega = \mathbf{R}^n$, there exists a plane wave solution $v_p(x) = e^{i\sqrt{\lambda}\omega \cdot x}$, $\omega \in S^{n-1}$. However, the following space plays an important role. Let \mathcal{B} be the set of functions f such that

$$(2) \quad \|f\|_{\mathcal{B}}^2 = \sup_{R>1} \frac{1}{R} \int_{|x|<R} |f(x)|^2 dx < \infty.$$

We use the notation $f \simeq g$ to mean

$$(3) \quad f \simeq g \iff \lim_{R \rightarrow \infty} \sup_{R>1} \frac{1}{R} \int_{|x|<R} |f(x) - g(x)|^2 dx = 0.$$

Then, in addition to the plane wave, one can also find another solution

$$(4) \quad v_s(x) = \int_{S^{n-1}} e^{i\sqrt{\lambda}\omega \cdot x} \psi(\omega) d\omega,$$

where $\psi(\omega) \in C^\infty(S^{n-1})$. This solution should be called a spherical wave, since, as can be shown by the stationary phase method, it behaves like

$$(5) \quad v_s(x) \simeq C(\lambda) \frac{e^{i\sqrt{\lambda}r}}{r^{(n-1)/2}} \psi(\hat{x}) + \overline{C(\lambda)} \frac{e^{-i\sqrt{\lambda}r}}{r^{(n-1)/2}} \psi(-\hat{x}),$$

as $r = |x| \rightarrow \infty$, where $\hat{x} = x/r$ and $C(\lambda)$ is a constant depending on λ . The first term v_+ and the second term v_- of the right-hand-side of (5) satisfy

$$\left(\frac{\partial}{\partial r} - i\sqrt{\lambda} \right) v_+ = o(r^{-(n-1)/2}), \quad \left(\frac{\partial}{\partial r} + i\sqrt{\lambda} \right) v_- = o(r^{-(n-1)/2}),$$

which are called the outgoing and incoming *radiation condition*, respectively. The solutions v_\pm behave like $O(r^{-(n-1)/2})$ as $r \rightarrow \infty$. The crucial fact is that this decay rate $O(r^{-(n-1)/2})$ is critical for the existence of solutions to the Helmholtz equation. In fact, we know the following theorem due to Rellich and Vekua.

THEOREM 2.1. *Assume that $u(x)$ satisfies $(-\Delta - \lambda)u = 0$ on $|x| > R$, for some constants λ and R . If $u(x) = o(r^{-(n-1)/2})$ as $r \rightarrow \infty$, then $u(x) = 0$ for $|x| > R$.*

Theorem 2.1 is the uniqueness result near infinity of solutions to the Helmholtz equation. It requires that u satisfies $(-\Delta - \lambda)u = 0$ only near infinity. This theorem plays an important role for the spectral theory of $-\Delta$. It proves the non-existence of eigenvalues embedded in the continuous spectrum of $-\Delta$ in an exterior domain. Moreover, it also provides us with an important link between the scattering waves at infinity and the those in a bounded part of the domain. By virtue of Theorem 2.1, one can show that the far-field patterns of the wave equation near infinity determine the near field patterns.

2.2. Limiting absorption principle

For a self-adjoint operator A in a Hilbert space \mathcal{H} , the inverse $(A - \lambda)^{-1}$ does not exist for $\lambda \in \sigma(A)$. However, in some cases, when λ is in the continuous spectrum of A , the limit

$$(A - \lambda \mp i0)^{-1} = \lim_{\epsilon \downarrow 0} (A - (\lambda \pm i\epsilon))^{-1}$$

exists as a bounded operator from \mathcal{X} to \mathcal{Y} , where \mathcal{X} and \mathcal{Y} are Banach spaces rigging A , namely they satisfy

$$\mathcal{X} \subset \mathcal{H} \subset \mathcal{Y}$$

where the inclusions are dense and continuous. For instance, the Laplacian $-\Delta$ in \mathbf{R}^3 is known to have this property, for $\lambda > 0$, by choosing $\mathcal{X} = L^1(\mathbf{R}^3)$ and $\mathcal{Y} = L^\infty(\mathbf{R}^3)$, observing the explicit form of the resolvent kernel $e^{i\sqrt{\lambda}|x-y|}/(4\pi|x-y|)$. This property, called the *Limiting absorption principle (LAP)*, is a powerful tool for the study of continuous spectrum of $-\Delta$, or more generally, differential operators with continuous spectrum.

As Banach spaces \mathcal{X}, \mathcal{Y} , the weighted L^2 spaces $L^{2,s}$,

$$(6) \quad L^{2,s} \ni f \iff \|f\|_s^2 = \int_{\Omega} |f(x)|^2 (1 + |x|^2)^s dx, \quad s \in \mathbf{R},$$

were usually adopted: $\mathcal{X} = L^{2,s}$, $\mathcal{Y} = L^{2,-s}$ with $s > 1/2$. The optimal choice was given as the Besov-type spaces by Agmon-Hörmander [1]. Let

$$R_j = 2^j \quad (j \geq 0), \quad \Omega_0 = \{x \in \Omega; |x| < 1\},$$

$$\Omega_j = \{x \in \Omega; 2^{j-1} < |x| < 2^j\} \quad (j \geq 1),$$

and \mathcal{X} be the space of functions $f \in L^2_{loc}(\Omega)$ such that

$$\|f\|_{\mathcal{X}} = \sum_{j=0}^{\infty} R_j^{1/2} \|f\|_{L^2(\Omega_j)} < \infty.$$

The space \mathcal{Y} is defined to be the set of functions such that

$$\|u\|_{\mathcal{Y}}^2 = \sup_{R>1} \frac{1}{R} \int_{\Omega \cap \{|x|<R\}} |u(x)|^2 dx < \infty.$$

It is known that \mathcal{Y} is identified with the dual space of \mathcal{X} and the \mathcal{Y} -norm is equivalent to the norm of the dual space of \mathcal{X} . With this in mind, we denote

$$\mathcal{X} = \mathcal{B}, \quad \mathcal{Y} = \mathcal{B}^*,$$

employing the norms $\|f\|_{\mathcal{X}} = \|f\|_{\mathcal{B}}$, $\|u\|_{\mathcal{Y}} = \|u\|_{\mathcal{B}^*}$. Especially, if $u \in \mathcal{B}^*$ satisfies

$$\lim_{R \rightarrow \infty} \frac{1}{R} \int_{\Omega \cap \{|x|<R\}} |u(x)|^2 dx = 0,$$

we denote $u \in \mathcal{B}_0^*$.

Let us consider the Neumann Laplacian $H = -\Delta$ in an exterior domain $\Omega \subset \mathbf{R}^n$. Let $R(z) = (H - z)^{-1}$ for $z \in \mathbf{C} \setminus [0, \infty)$. Then, for any $\lambda > 0$ and $f, g \in \mathcal{B}$, there

exists a limit

$$\lim_{\epsilon \downarrow 0} (R(\lambda \pm i\epsilon)^{-1}f, g) =: (R(\lambda \pm i0)f, g).$$

For any compact interval $I \subset (0, \infty)$, there exists a constant $C > 0$ such that

$$(7) \quad \|R(\lambda \pm i0)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \quad \lambda \in I, \quad f \in \mathcal{B}.$$

2.3. Eigenfunction expansion

The Fourier transformation in \mathbf{R}^n

$$\mathcal{F}_0 : f \rightarrow \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ix \cdot \xi} f(x) dx$$

gives a spectral representation of $-\Delta$ in \mathbf{R}^n . It is an integral transformation by $e^{-ix \cdot \xi}$, a generalized eigenfunction of $-\Delta$, which is unitary : $L^2(\mathbf{R}^n) \rightarrow L^2(\mathbf{R}^n)$, diagonalizes H i.e. $(\mathcal{F}_0 H f)(\xi) = |\xi|^2 \mathcal{F}_0 f(\xi)$, and gives an inversion formula

$$f(x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{ix \cdot \xi} (\mathcal{F}_0 f)(\xi) d\xi.$$

One should also note that $\mathcal{F}_0 f$ is computed by observing the behavior at infinity of the resolvent: For $f \in \mathcal{B}$ and $\lambda > 0$,

$$(8) \quad (-\Delta - \lambda - i0)^{-1}f \simeq C(\lambda) \frac{e^{i\sqrt{\lambda}r}}{r^{(n-1)/2}} (\mathcal{F}_0 f)(\sqrt{\lambda}\omega), \quad r = |x| \rightarrow \infty, \quad \omega = x/r,$$

$$C(\lambda) = \sqrt{\frac{\pi}{2}} e^{-(n-3)\pi i/4} \lambda^{(n-3)/4}.$$

These facts are extended to $H = -\Delta$ in an exterior domain. Assuming that $\Omega \subset \{|x| > R\}$, we take a C^∞ function $\chi(x)$ such that $\chi(x) = 0$ for $|x| < R + 1$, $\chi(x) = 1$ for $|x| > R + 2$, and put

$$\varphi(x, \xi) = \chi(x) e^{ix \cdot \xi} + R(|\xi|^2 - i0)(\Delta + |\xi|^2)(\chi(x) e^{ix \cdot \xi}),$$

where $R(z) = (H - z)^{-1}$ and H is the Neumann Laplacian on Ω . Define the Fourier transformation \mathcal{F} by

$$(\mathcal{F}f)(\xi) = (2\pi)^{-n/2} \int_{\Omega} \overline{\varphi(x, \xi)} f(x) dx.$$

Then, $\mathcal{F} : L^2(\Omega) \rightarrow L^2(\mathbf{R}^n)$ is unitary, a diagonalization of H , and gives an inversion formula

$$f(x) = (2\pi)^{-n/2} \int_{\Omega} \varphi(x, \xi)(\mathcal{F}f)(\xi)d\xi, \quad f \in L^2(\Omega).$$

The generalized plane wave $\varphi(x, \xi)$ has the following property. Letting $\xi = \sqrt{\lambda}\omega$, $\lambda > 0$, $\omega \in S^{n-1}$, it has the following asymptotic expansion

$$\varphi(x, \sqrt{\lambda}\omega) \simeq e^{i\sqrt{\lambda}\omega \cdot x} + \frac{e^{i\sqrt{\lambda}r}}{r^{(n-1)/2}} a(\lambda; \theta, \omega), \quad r = |x|, \quad \theta = x/r,$$

and for a suitable choice of the constant $C_0(\lambda)$ the operator

$$(S(\lambda)\psi)(\theta) = \psi(\theta) + C_0(\lambda) \int_{S^{n-1}} a(\theta, \omega)\psi(\omega)d\omega, \quad \psi \in L^2(S^{n-1})$$

becomes a unitary operator on $L^2(S^{n-1})$. This $S(\lambda)$ is called Heisenberg's S-matrix.

One can further prove the following fact. For any $a^{(in)}(\omega) \in L^2(S^{n-1})$, there exist a unique $u \in \mathcal{B}^*$ and $a^{(out)}(\omega) \in L^2(S^{n-1})$ such that

$$(-\Delta - \lambda)u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

$$(9) \quad u \simeq \frac{e^{i\sqrt{\lambda}r}}{r^{(n-1)/2}} a^{(out)}(\omega) - \frac{e^{-i\sqrt{\lambda}r}}{r^{(n-1)/2}} a^{(in)}(\omega).$$

Moreover

$$(10) \quad a^{(out)} = S(\lambda)Ja^{(in)},$$

where $J\psi(\omega) = \psi(-\omega)$.

3. Spectral properties for elastic equations in \mathbf{R}^3

The stationary elastic wave equation in \mathbf{R}^3 is written as

$$(11) \quad (\mathcal{L} - \lambda)\mathbf{u} = \mathbf{f},$$

where the operator \mathcal{L} is defined by

$$(12) \quad \mathcal{L}\mathbf{u} = \left\{ -\frac{1}{\rho(x)} \sum_{j=1}^3 \frac{\partial}{\partial x_j} \sigma_{ij}(\mathbf{u}) \right\}_{1 \leq i \leq 3}.$$

Here $\mathbf{u}(x) = {}^t(u_1(x), u_2(x), u_3(x))$ is the vector displacement, $\rho(x) > 0$ is the density and $\sigma_{ij}(\mathbf{u})$ is the stress tensor of the form

$$(13) \quad \sigma_{ij}(\mathbf{u}) = \lambda(x)(\nabla \cdot \mathbf{u})\delta_{ij} + 2\mu(x)\mathcal{E}_{ij}(\mathbf{u}),$$

where δ_{ij} is the Kronecker's delta, $\lambda(x)$ and $\mu(x)$ are Lamé coefficients, and $\mathcal{E}_{ij}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ is the deformation tensor. We assume that $\lambda(x), \mu(x), \rho(x)$ are positive, and equal to constant λ_0, μ_0, ρ_0 outside a compact set. Then, the unperturbed equation is of the form

$$(14) \quad \mathcal{L}_0\mathbf{u} = \lambda\mathbf{u} \iff \mu_0\Delta\mathbf{u} + (\lambda_0 + \mu_0)\nabla(\nabla \cdot \mathbf{u}) + \rho_0\lambda\mathbf{u} = \mathbf{0}.$$

Then, there are two characteristic roots, c_P, c_S defined by

$$(15) \quad c_P = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}}, \quad c_S = \sqrt{\frac{\mu_0}{\rho_0}}.$$

Hence the equation (14) has two types of plane waves

$$e^{ic_P^{-1}\sqrt{\lambda}\omega \cdot x} \mathbf{a}, \quad e^{ic_S^{-1}\sqrt{\lambda}\omega \cdot x} \mathbf{b}, \quad \omega \in S^2$$

with constant vectors $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3$. Taking the spherical mean $\int_{S^2} e^{ic\sqrt{\lambda}\omega \cdot x} \psi(\omega) d\omega$, we find two types of spherical waves

$$(16) \quad \frac{e^{ic_P^{-1}\sqrt{\lambda}r}}{r} \mathbf{a}(\hat{x}), \quad \frac{e^{ic_S^{-1}\sqrt{\lambda}r}}{r} \mathbf{b}(\hat{x}),$$

called primary and secondary waves. As in the case of scalar wave equation, the following Rellich type theorem holds.

THEOREM 3.1. *Let $\mathbf{u} \in C^2(\mathbf{R}^3, \mathbf{C}^3)$ be a solution of (14). Decompose \mathbf{u} into $\mathbf{u}^{(P)} + \mathbf{u}^{(S)}$, where*

$$\mathbf{u}^{(P)} = \frac{1}{(c_S^{-2} - c_P^{-2})\lambda} (\Delta + c_S^{-2}\lambda)\mathbf{u}, \quad \mathbf{u}^{(S)} = \frac{1}{(c_P^{-2} - c_S^{-2})\lambda} (\Delta + c_P^{-2}\lambda)\mathbf{u}.$$

Suppose that as $r = |x| \rightarrow \infty$,

$$(17) \quad \mathbf{u}^{(j)} = O(r^{-1}), \quad \left(\frac{\partial}{\partial r} - ic_j^{-1} \sqrt{\lambda} \right) \mathbf{u}^{(j)} = O(r^{-2}),$$

where $j = P, S$. Then $\mathbf{u} \equiv \mathbf{0}$.

Namely, the solution \mathbf{u} is decomposed into two parts $\mathbf{u}^{(P)}$ (primary wave) and $\mathbf{u}^{(S)}$ (secondary wave), and we can use the Sommerfeld radiation condition for each part. One can also obtain an asymptotic expansion of solutions to (11). Let us consider the unperturbed case. Let $R_0(\lambda + i0)$ be the resolvent of L_0 . If $\mathbf{f} \in L^\infty(\mathbf{R}^3, \mathbf{C}^3)$ is compactly supported, $R_0(\lambda + i0)\mathbf{f}(x)$ has the following asymptotic behavior at infinity: Letting $r = |x|$ and $\hat{x} = x/r$,

$$(18) \quad R_0(\lambda + i0)\mathbf{f}(x) = \frac{e^{ic_P^{-1}\sqrt{\lambda}r}}{r} \mathbf{a}_P(\lambda, \hat{x}) + \frac{e^{ic_S^{-1}\sqrt{\lambda}r}}{r} \mathbf{a}_S(\lambda, \hat{x}) + O(r^{-2}),$$

$$(19) \quad \begin{aligned} \mathbf{a}_P(\lambda, \hat{x}) &= \frac{1}{4c_P^2\pi} \left(\hat{\mathbf{f}}(c_P^{-1}\sqrt{\lambda}\hat{x}) \cdot \hat{x} \right) \hat{x}, \\ \mathbf{a}_S(\lambda, \hat{x}) &= \frac{1}{4c_S^2\pi} \hat{\mathbf{f}}(c_S^{-1}\sqrt{\lambda}\hat{x}) - \left(\hat{\mathbf{f}}(c_S^{-1}\sqrt{\lambda}\hat{x}) \cdot \hat{x} \right) \hat{x}, \end{aligned}$$

where $\hat{\mathbf{f}}$ is the Fourier transform of \mathbf{f} .

By the well-known perturbation theory, see e.g. [9], one can prove LAP for the perturbed operator L and extend (18) to L . One can also prove the expansion of the form (9) for L and define the S-matrix as in (10).

Let $\mathbf{d}_P(\varphi)$, $\mathbf{d}_{SV}(\varphi)$, and $\mathbf{d}_{SH}(\varphi)$ denote the displacement vectors of the P -, SV -, and SH -waves, respectively, and let $\mathcal{F}_P(\lambda)$, $\mathcal{F}_{SV}(\lambda)$, and $\mathcal{F}_{SH}(\lambda)$ represent the Fourier transforms with respect to the operator L . The detailed definitions are omitted here and will be given in the next section in the case of a half-space.

THEOREM 3.2. *Let $\lambda > 0$. Then the following follows:*

For $\mathbf{f} \in \mathcal{B}$ and $\lambda \in (0, \infty)$, the boundary value of the resolvent of L admits the following asymptotic expansion

$$\begin{aligned} R(\lambda + i0)\mathbf{f} &\simeq C \frac{e^{i\sqrt{\lambda}r/c_P}}{r} (\mathcal{F}_P(\lambda)\mathbf{f})\mathbf{d}_P(\varphi) + C \frac{e^{i\sqrt{\lambda}r/c_S}}{r} (\mathcal{F}_{SV}(\lambda)\mathbf{f})\mathbf{d}_{SV}(\varphi) \\ &\quad + C \frac{e^{i\sqrt{\lambda}r/c_S}}{r} (\mathcal{F}_{SH}(\lambda)\mathbf{f})\mathbf{d}_{SH}(\varphi), \end{aligned}$$

where $C = e^{i\pi/4}(2\rho_0)^{-1/2}$.

Theorem 3.2 is essentially known. The proof is almost the same as (8).

It should be noted that the total motion of the S -wave is represented by a vector displacement lying within the wavefront plane, which can be decomposed into SV and SH components by introducing an appropriate coordinate system. However, from the point of view of applications (e.g., in elasticity or seismology), when considering S -waves propagating in the whole space, such a decomposition into SV and SH is generally regarded as less essential (see, for instance, [16, Sec. 2.4]). We describe it in this way with the comparison to the half-space case, which will be discussed in the next section, in mind.

4. Spectral properties for elastic equations in a half-space

Our aim is to prove the results in §3 to a locally perturbed upper half space $\mathbf{R}_+^3 = \{x \in \mathbf{R}^3; x_3 \geq 0\}$. We consider an elastic solid $\Omega \subset \mathbf{R}^3$ such that

$$\begin{aligned}\Omega \cap \{x \in \mathbf{R}^3; |x| > R\} &= \Omega \cap \{x \in \mathbf{R}_+^3; |x| > R\}, \\ \Omega &= (\mathbf{R}_+^3 \setminus K) \cup \Omega_-, \end{aligned}$$

where $R > 0$ is a fixed constant, $K \subset \mathbf{R}^3$ is a bounded closed set, and Ω_- is a bounded open set in $\mathbf{R}_-^3 = \{x \in \mathbf{R}^3; x_3 \leq 0\}$.

Let $W^{k,p}(\Omega)$ be the usual Sobolev space of order k in $L^p(\Omega)$. However, we often use $H^m(\Omega, \mathbf{C}^3) = W^{m,2}(\Omega, \mathbf{C}^3)$. Assume that the Lamé coefficients $\lambda(x)$ and $\mu(x)$, and the density $\rho(x)$ satisfy the following conditions:

(A1) $\lambda, \mu, \rho \in W^{1,\infty}(\Omega)$.

(A2) *There exist positive constants m and M satisfying*

$$0 < m \leq \lambda(x), \rho(x), \mu(x) \leq M \quad \text{on} \quad \bar{\Omega}.$$

(A3) *Let λ_0, μ_0 and ρ_0 be positive constants. Assume that*

$$\lambda(x) = \lambda_0, \quad \mu(x) = \mu_0, \quad \rho(x) = \rho_0 \quad \text{on} \quad B_R^c := \{x \in \mathbf{R}^3; |x| > R\}.$$

Using \mathcal{L} in (12), we consider the elastic operator $\mathcal{L}\mathbf{u}$ with a boundary condition $\sigma(\mathbf{u})\nu|_{\partial\Omega} = \mathbf{0}$, where ν is the exterior normal at $x \in \partial\Omega$. Here the trace $\sigma(\mathbf{u})\nu|_{\partial\Omega} =$

$\mathbf{0}$ means the following generalized sense:

$$(20) \quad \int_{\Omega} (\mathcal{L}\mathbf{u})_i \bar{v}_i \rho(x) dx - \int_{\Omega} \left\{ \lambda(x)(\nabla \cdot \mathbf{u})(\nabla \cdot \bar{\mathbf{v}}) + 2\mu(x)\mathcal{E}_{ij}(\mathbf{u})\mathcal{E}_{ij}(\bar{\mathbf{v}}) \right\} dx = 0$$

for $\mathbf{u} \in H^1(\Omega, \mathbf{C}^3) \cap L^2(\Omega, \mathcal{L}, \mathbf{C}^3)$ and $\mathbf{v} \in H^1(\Omega, \mathbf{C}^3)$, where

$$L^2(\Omega, \mathcal{L}, \mathbf{C}^3) = \{\mathbf{u} \in L^2(\Omega, \mathbf{C}^3); \mathcal{L}\mathbf{u} \in L^2(\Omega, \mathbf{C}^3)\}$$

As in [4], the elastic operator $L\mathbf{u} = \mathcal{L}\mathbf{u}$ in $L^2(\Omega, \mathbf{C}^3, \rho(x)dx)$ with domain

$$D(L) = \{\mathbf{u} \in H^1(\Omega, \mathbf{C}^3) \cap L^2(\Omega, \mathcal{L}, \mathbf{C}^3); \sigma(\mathbf{u})\nu|_{\partial\Omega} = \mathbf{0}\}$$

is a positive self-adjoint operator, with the spectrum $\sigma(L) = [0, \infty)$, the continuous spectrum $\sigma_{\text{cont}}(L) = [0, \infty)$ and without singular spectrum $\sigma_{\text{sc}}(L)$. In addition, L has no positive eigenvalues embedded in $(0, \infty)$ under the assumptions (A1), (A2), and (A3) (see Sini [19]).

The limiting absorption principle for $R(z) = (L - z)^{-1}$ was studied by Dermenjian-Guillot [4]. They computed the generalized eigenfunctions for L_0 and used them to study the boundary values $(L_0 - \lambda \mp i0)^{-1}$ in weighted L^2 spaces. The eigenfunction expansion for L in Ω was proven by Dermenjian-Gaitan [5]. We prefer to use the Besov type spaces \mathcal{B} and \mathcal{B}^* in the same way as in §2.2 with \mathbf{R}^n replaced by the upper-half space \mathbf{R}_+^3 .

THEOREM 4.1 ([11, Theorem 5.3]). *Let $\lambda > 0$. Then the following follows: For any $\lambda \in (0, \infty)$, the limit*

$$\lim_{\varepsilon \downarrow 0} (R(\lambda \pm i\varepsilon)\mathbf{f}, \mathbf{g}) := (R(\lambda \pm i0)\mathbf{f}, \mathbf{g}), \quad \forall \mathbf{f}, \mathbf{g} \in \mathcal{B}$$

exists. There exists a constant $C > 0$ such that

$$\|R(\lambda \pm i0)\mathbf{f}\|_{\mathcal{B}^*} \leq C\|\mathbf{f}\|_{\mathcal{B}}, \quad \lambda \in (0, \infty).$$

By the spherical mean, the generalized plane waves give rise to spherical waves: the primary wave, the secondary wave and their reflections at the boundary $\partial\Omega$. They are 3-dimensional spherical waves propagating inside Ω and called the *body wave* (cf. (16)). A new phenomenon is the appearance of *Rayleigh wave* which is a

wave propagating only along the boundary. It is represented as

$$(21) \quad \frac{e^{i\sqrt{\lambda}r_*/c_R}}{\sqrt{r_*}} e^{-\sqrt{\lambda}\tau x_3},$$

where for $x = (x_*, x_3)$, $r_* = |x_*|$, $\tau > 0$ is a constant and c_R is a unique solution of

$$(22) \quad \left(1 - \frac{k^2}{2c_S^2}\right) - \left(1 - \frac{k^2}{c_S^2}\right)^{1/2} \left(1 - \frac{k^2}{c_P^2}\right)^{1/2} = 0, \quad 0 < k < c_S < c_P.$$

Therefore, horizontally, the Rayleigh wave is a 2-dimensional spherical wave and vertically decays exponentially. The behavior of Rayleigh waves is very anisotropic compared to that of the Body waves, and we are interested in how to observe these waves simultaneously, namely the norm by which we can derive these waves asymptotically. The following theorem shows that the Besov type space \mathcal{B}^* in (2) suffices for this purpose. We define $f \simeq g$ by

$$(23) \quad f \simeq g \iff \lim_{R \rightarrow \infty} \frac{1}{R} \int_{\mathbf{R}_+^3 \cap \{|x| > R_0\}} |f(x) - g(x)|^2 dx = 0,$$

where $R_0 > 0$ is chosen large enough.

We put $r = |x|$, $\varphi = (\varphi_*, \varphi_3) = (\varphi_j)_{j=1,2,3} = x/r$, $r_* = |x_*|$, $\varphi_{**} = x_*/r_*$. We define the displacements by

$$\begin{aligned} \mathbf{d}_P &= \mathbf{d}_P(\varphi) = \varphi, \\ \mathbf{d}_{SV} &= \mathbf{d}_{SV}(\varphi) = {}^t \left(\frac{\varphi_1 \varphi_3}{|\varphi_*|}, \frac{\varphi_2 \varphi_3}{|\varphi_*|}, -|\varphi_*| \right), \\ \mathbf{d}_{SH} &= \mathbf{d}_{SH}(\varphi) = {}^t \left(-\frac{\varphi_2}{|\varphi_*|}, \frac{\varphi_1}{|\varphi_*|}, 0 \right), \\ \mathbf{d}_R^{(1)} &= \mathbf{d}_R^{(1)}(\varphi_{**}) = {}^t (i\varphi_{**}, -\tilde{c}_{RP}), \\ \mathbf{d}_R^{(2)} &= \mathbf{d}_R^{(2)}(\varphi_{**}) = {}^t (i\tilde{c}_{RS}\varphi_{**}, -1), \end{aligned}$$

where $\tilde{c}_{R\dagger} = \sqrt{1 - \frac{c_R^2}{c_\dagger^2}}$ ($\dagger = P, S$). Then, the vectors \mathbf{d}_P , \mathbf{d}_{SV} and \mathbf{d}_{SH} are orthonormal bases in \mathbf{R}^3 .

THEOREM 4.2 ([11, Theorem 5.8 and Remark 2.1]). *Let $\lambda > 0$. Then the following follows:*

- (1) *There exist bounded operators $\mathcal{F}_\#(\lambda)$ ($\# = P, SV, SH$) from \mathcal{B} to $L^2(\mathbf{S}_+^2)$ and*

$\mathcal{F}_R(\lambda)$ from \mathcal{B} to $L^2(\mathbf{S}^1)$.

- (2) For $\mathbf{f} \in \mathcal{B}$ and $\lambda \in (0, \infty)$, the boundary value of the resolvent of L admits the following asymptotic expansion

$$\begin{aligned} R(\lambda + i0)\mathbf{f} &\simeq C \frac{e^{i\sqrt{\lambda}r/c_P}}{r} (\mathcal{F}_P(\lambda)\mathbf{f})\mathbf{d}_P(\varphi) + C \frac{e^{i\sqrt{\lambda}r/c_S}}{r} (\mathcal{F}_{SV}(\lambda)\mathbf{f})\mathbf{d}_{SV}(\varphi) \\ &\quad + C \frac{e^{i\sqrt{\lambda}r/c_S}}{r} (\mathcal{F}_{SH}(\lambda)\mathbf{f})\mathbf{d}_{SH}(\varphi) \\ &\quad + \sum_{\ell=1}^2 \frac{e^{i\sqrt{\lambda}r_*/c_R}}{\sqrt{r_*}} e^{-\sqrt{\lambda}\tau_\ell x_3} E_\ell (\mathcal{F}_R(\lambda)\mathbf{f})\mathbf{d}_R^{(\ell)}(\varphi_{**}), \end{aligned}$$

where $C = e^{i\pi/4}(2\rho_0)^{-1/2}$ and, $\tau_\ell (> 0)$ and E_ℓ are constants depending only on c_P , c_S and c_R .

THEOREM 4.3 ([11, Theorem 1.5]). *Let $\lambda \in (0, \infty)$. Suppose that $\mathbf{u} \in \mathcal{B}^*$ satisfies $(L - \lambda)\mathbf{u} = \mathbf{0}$. Then there exists $\mathbf{f}^{(\pm)} = (f_P^{(\pm)}, f_{SV}^{(\pm)}, f_{SH}^{(\pm)}, f_R^{(\pm)}) \in \mathbf{h}(\lambda) = [L^2(\mathbf{S}_+^2)]^3 \times L^2(\mathbf{S}^1)$ such that*

$$\begin{aligned} \mathbf{u}(x) &\simeq \frac{e^{i\sqrt{\lambda}r/c_P}}{r} f_P^{(+)} \mathbf{d}_P(\varphi) + \frac{e^{i\sqrt{\lambda}r/c_S}}{r} f_{SV}^{(+)} \mathbf{d}_{SV}(\varphi) + \frac{e^{i\sqrt{\lambda}r/c_S}}{r} f_{SH}^{(+)} \mathbf{d}_{SH}(\varphi) \\ &\quad + \sum_{\ell=1}^2 \frac{e^{i\sqrt{\lambda}r_*/c_R}}{\sqrt{r_*}} e^{-\sqrt{\lambda}\tau_\ell x_3} E_\ell f_R^{(+)} \mathbf{d}_R^{(\ell)}(\varphi_{**}) \\ &\quad - \frac{e^{-i\sqrt{\lambda}r/c_P}}{r} f_P^{(-)} \mathbf{d}_P^{(-)}(\varphi) - \frac{e^{-i\sqrt{\lambda}r/c_S}}{r} f_{SV}^{(-)} \mathbf{d}_{SV}^{(-)}(\varphi) - \frac{e^{-i\sqrt{\lambda}r/c_S}}{r} f_{SH}^{(-)} \mathbf{d}_{SH}^{(-)}(\varphi) \\ &\quad - \sum_{\ell=1}^2 \frac{e^{-i\sqrt{\lambda}r_*/c_R}}{\sqrt{r_*}} e^{-\sqrt{\lambda}\tau_\ell x_3} \overline{E}_\ell f_R^{(-)} \mathbf{d}_R^{(\ell)}(-\varphi_{**}), \end{aligned}$$

where $\mathbf{d}_\#^{(-)}(\varphi) = \mathbf{d}_\#(-\varphi_*, \varphi_3)$ ($\# = P, SV, SH$) and, $\tau_\ell > 0$ and E_ℓ are constants depending only on c_P , c_S and c_R . Furthermore, there exists a bounded operator $\mathbf{S}(\lambda)$ on $\mathbf{h}(\lambda)$ such that $\mathbf{f}^{(+)} = \mathbf{S}(\lambda)\mathbf{f}^{(-)}$.

Theorems 4.3 together with Theorem 5.1 in the next section give a complete description of scattering in the stationary form. In fact, let f_- be a data having the same form as in the last 4 terms of the right-hand side of $\mathbf{u}(x)$ in Theorem 4.3. Then, there exists a unique $\mathbf{u}(x)$ satisfying $(L - \lambda)\mathbf{u} = 0$, having the asymptotic expansion in Theorem 4.3. In fact, letting $\mathbf{u} = R(\lambda + i0)F_+ - f_-$, where $F_+ = (L - \lambda)f_-$, we know the existence of \mathbf{u} . The uniqueness of such \mathbf{u} and the expansion

follows from Theorem 5.1 .

The proof of Theorem 4.2, on which Theorem 4.3 is based, is done by applying the stationary phase method to integrals for generalized eigenfunction of L over the semi-sphere. However, a serious problem arises here. Due to the reflection or refraction of waves at the boundary, the generalized plane waves have singularities along some cone. The waves are piecewise C^1 , but their second order derivatives are not continuous along this cone. This is a characteristic feature of the layered media and happens even in the simpler case of wave equation. By this reason, in [10], we studied the problem in Theorem 4.2 for the stratified media in \mathbf{R}^3 , i.e. for the operator $H = -c(x)^2\Delta_x$, where

$$c(x) = \begin{cases} c_+ & \text{for } x_3 > 0, \\ c_- & \text{for } x_3 < 0, \end{cases}$$

with $0 < c_- < c_+$. A careful application of classical stationary phase method by [3] and [17] works for this purpose, and they can also be applied to the elastic equation.

5. Radiation conditions for elastic equations in a half-space

In Theorem 4.3, we considered the existence of solutions to the elastic equation in a half-space with spherical wave like asymptotics. In this section, we consider its uniqueness. Our main concern is the Rayleigh wave. For $\delta > 0$, put

$$\mathbf{D}_\delta = \{x \in \mathbf{R}^3; x_3 < \delta|x_*|\}, \quad \mathbf{E}_\delta = \{x \in \mathbf{R}^3; x_3 > \delta|x_*|\}.$$

For a solution \mathbf{u} to $(L - \lambda)\mathbf{u} = 0$, we put

$$\mathbf{u}_L = (\mathbf{u} \cdot \mathbf{d}_P)\mathbf{d}_P, \quad \mathbf{u}_H = (\mathbf{u} \cdot \mathbf{d}_{SH})\mathbf{d}_{SH}, \quad \mathbf{u}_T = \mathbf{u} - \mathbf{u}_L, \quad \mathbf{u}_V = \mathbf{u} - \mathbf{u}_H.$$

THEOREM 5.1 ([12, Theorem 2.1]). *Let $\lambda > 0$, and $\mathbf{u} \in \mathcal{B}^* \cap H_{loc}^1(\Omega; \mathbf{C}^3)$ be a solution to (11) with boundary condition (20). Suppose that the following conditions (I), (II) are satisfied:*

(I) *For any $\delta > 0$,*

$$(24) \quad (D_0 - i\rho_0 c_P \sqrt{\lambda}) \mathbf{u}_L \in \mathcal{B}_0^*(\Omega \cap \mathbf{B}_{R_0}^e \cap \mathbf{E}_\delta, \mathbf{C}^3),$$

$$(25) \quad (D_0 - i\rho_0 c_S \sqrt{\lambda}) \mathbf{u}_T \in \mathcal{B}_0^*(\Omega \cap \mathbf{B}_{R_0}^e \cap \mathbf{E}_\delta, \mathbf{C}^3),$$

$$(26) \quad (D_0 - i\rho_0 c_S \sqrt{\lambda}) \mathbf{u}_H \in \mathcal{B}_0^*(\Omega \cap \mathbf{B}_{R_0}^e \cap \mathbf{D}_\delta, \mathbf{C}^3).$$

(II) For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(27) \quad \limsup_{R \rightarrow \infty} \frac{1}{R} \left| \int_{\Omega \cap \mathbf{B}_R \cap \mathbf{B}_{R_0}^c \cap \mathbf{D}_\delta} \left(D_0 - i\rho_0 c_R \sqrt{\lambda} \right) \mathbf{u}_V \cdot \bar{\mathbf{u}} dx \right| < \varepsilon,$$

where $\mathbf{B}_{R_0}^c$ is the same as in (A3). Then $\mathbf{u} \equiv 0$.

The conditions in (I) are for the body waves. The one in (II) deals with the Rayleigh wave. Note that (27) is formulated by the quadratic form not by the norm. In fact, one can show that one *cannot* prove (27) by the norm on Ω .

We briefly explain the reason why the radiation conditions are divided into (24)-(27). In [11, Theorem 1], the scattering wave for an incident (P, SV, SH, R) plane wave was shown to be a linear combination of P, SV, SH, R -spherical waves. As is already mentioned, $\mathbf{d}_P, \mathbf{d}_{SV}$ and \mathbf{d}_{SH} are orthonormal bases in \mathbf{R}^3 , and satisfy $\mathbf{d}_{SH} \cdot \mathbf{d}_R^{(\ell)} = 0$ and $\mathbf{d}_j \cdot \mathbf{d}_R^{(\ell)} \neq 0$ ($j = P, SV$ and $\ell = 1, 2$). The scattering waves in \mathbf{E}_δ and \mathbf{D}_δ are almost equal to linear combinations of $P, S = (SV + SH)$ and SH, R -spherical waves, which are also L, T and H, V -components of the scattering wave, respectively (see (2.12)-(2.15) in [12]). As they have different phases and amplitudes, we need to formulate the radiation condition separately for each body wave and surface wave.

We have thus seen that the spherical wave solution to the elastic equation in a perturbed half-space \mathbf{R}_+^3 exists uniquely. This is a basic fact for the forward problem of scattering theory. In the case of the inverse problem for the wave equation, as well as the elastic equation in \mathbf{R}^3 , the Rellich type theorem was the first important step. To extend the Rellich type theorem for the elastic equation in the half-space remains an open problem.

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