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On active sonar-type wave scattering inverse problems for different types of cavities

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Abstract. In this paper, we survey the active sonar-type inverse problem based on the wave equation. We adopt the enclosure method and discuss the inverse problem of detecting different types of cavities. We address the issues caused by 'mixed media' by dividing them into two types: 'separated' and 'non-separated' for mixed media where the sign of the indicator function is not determined. Furthermore, in a new attempt, we assume high smoothness at the boundary of the cavities and perform a higher-order expansion of the indicator function.

1. Introduction

The inverse problem of wave phenomena has attracted the interest of many researchers for a long time. When a plane wave comes from far field and is reflected by an object, the reflected wave gradually becomes a spherical wave when it is far enough away from the object, but the spherical wave we obtain from the reflection is not uniform in all directions. The function that describes this is called the far-field pattern (or scattering amplitude), and the problem of estimating the original object from the far-field pattern is called the 'scattering inverse problem'. Based on this concept, many excellent results have been reported. On the other hand, the problem we would like to address in this paper is different from the inverse scattering problem and is based on the concept of active sonar used in marine exploration and breast cancer screening. More simply, it is a problem of obtaining information about obstacles by emitting waves from a certain location and observing the waves reflected by obstacles at a certain location. In the following, we refer to this as the 'active sonar-type scattering problem', and we will provide an overview of the results obtained thus far in its mathematical analysis and discuss future challenges.

To limit the scope of the discussion, we assume that wave propagation is governed by the wave equation and that the objects D under investigation are only cavities. An incident wave is emitted from B at time 0, and reflected waves are

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observed at the same position B until time $T > 0$. The observation data are given by the solution of the wave equation in $[0, T] \times B$. The objective is to estimate the cavity D from the observation data.

Various methods have been developed to approach inverse problems, but among them, the enclosure method is a mathematical method that is particularly well suited to active sonar-type inverse problems. In [3, 4], M. Ikehata developed the enclosure method for the cavity estimation problem of Laplace's equation. It has also been confirmed to be effective for time-dependent systems such as wave and heat equations (see, e.g., [5, 6, 12]). In the enclosure method, similar to other reconstruction methods for inverse problems, a function called the indicator function is introduced based on the observed data, and information about the object is obtained through its analysis. It is known that the indicator function I_τ in the enclosure method contains a large parameter $\tau > 1$, and if the propagation speed is constant $\sqrt{\gamma_0} > 0$, the shortest distance $\mathrm{dist}(D, B)$ (that is, the minimum length for a wave originating from $y \in \overline{B}$ to reach a point x on the object $\overline{D}$) is obtained from the asymptotic behavior of I_τ when $\tau \to \infty$.

The indicator function gives different signs depending on the type of cavity boundary conditions. For example, $I_\tau < 0$ for cavities with the Dirichlet boundary conditions (negative cavities), but $I_\tau > 0$ for cavities with the Neumann conditions (positive cavities). Even if there are two or more cavities, if each cavity has the same sign for the indicator function, we say that the 'monotonicity condition' is satisfied. When the monotonicity condition is satisfied, even if there are multiple types of cavities, the same argument can be applied as in the case of a single cavity. However, the situation changes significantly when the positive and negative cavities are mixed. When both positive and negative cavities exist in the medium, this is called the 'combined case'. In the combined case, analysis of the indicator function is more difficult because the positive or negative sign of the indicator function is uncertain in asymptotic behavior. In this article, we will explain the combined cases in the following sections. Since the symbols used vary according to previous papers, this paper mainly uses symbols in accordance with [17].

The contents of each section are as follows. Section 2 of this paper provides a formulation of the combined case and gives a brief explanation of the basic idea of cavity detection using the enclosure method. Based on previous research, we explain what kind of information can be obtained using the enclosure method. Section 3 provides an overview of the 'combined but separated' case discussed in [16], limiting the discussion to case where the targets are only cavities. The situation 'combined but separated' is the case where positive and negative cavities coexist but are clearly separated from each other. Under the assumption 'separated', by focusing on the points around the shortest distance and refining the estimates, the same argument as in the case of monotonicity holds. Therefore, assuming 'separated', it is possible to analyze the indicator function without placing strong assumptions on the smoothness of the cavity boundary. In [15], we explain the necessity of classifying combined cases into separated and non-separated cases and present an outline of the proof for separated cases. Furthermore, as explained in Section 3 below, when considering the actual exploration of multiple cavities, it is not desirable to assume 'separated case'. To remove this restriction, [17] discusses the case where 'separated' is not assumed, i.e., the non-separated case. In Section 4, we discuss the results of the 'non-separated' case dealt with in [17]. In [17], an 'asymptotic solution' is used to investigate the behavior of the indicator function without assuming separated. The asymptotic solution allows us to derive more detailed information, but it also has the weakness of requiring higher assumptions about the regularity of the cavity boundary. Section 5 shows that when the cavity boundary and data are sufficiently smooth, an asymptotic expansion of any order can be obtained for the indicator function I_τ. This result is new in this paper. It will serve as a stepping stone for more detailed analysis in the future.

2. Enclosure method and the indicator function

Let $D = D^n \cup D^d$ be a bounded open set satisfying $\overline{D^n} \cap \overline{D^d} = \varnothing$ and have C^1 boundary ∂D. We set $\Omega = \mathbb{R}^3 \setminus \overline{D}$, being an exterior domain in $\mathbb{R}^3$. Let $\nu(x) = {}^t(\nu_1(x), \nu_2(x), \nu_3(x))$ $(x \in \partial D = \partial\Omega)$ be the unit outer normal vector pointing the outside of D. Let $\lambda_j \in L^\infty(\partial D^n)$ $(j = 0, 1)$ be real-valued functions and $\lambda_1(x) \geq 0$ a.e. on ∂D^n.

For a fixed $T > 0$, consider the following initial-boundary-value problem for the wave equation:

$$
(2.1) \qquad \begin{cases} (\partial_t^2 - \gamma_0 \Delta) u(t, x) = 0 & \text{in } (0, T) \times \Omega, \\ (\gamma_0 \nu_x \cdot \nabla_x - \lambda_1(x)\partial_t - \lambda_0(x)) u(t, x) = 0 & \text{on } (0, T) \times \partial D^n, \\ u(t, x) = 0 & \text{on } (0, T) \times \partial D^d, \\ u(0, x) = 0, \ \ \partial_t u(0, x) = f(x) & \text{on } \Omega, \end{cases}
$$

where $\gamma_0 > 0$ is a constant and $\lambda_j \in L^\infty(\partial D^n)$ $(j = 0, 1)$. In (2.1), D^n and D^d correspond to cavities with the Neumann and Dirichlet boundary conditions, respectively. When $D^n = \varnothing$ (or $D^d = \varnothing$), this means that there are no cavities with the Neumann boundary conditions (or the Dirichlet boundary conditions). In this way, there are multiple unknown cavities D^n and D^d in a homogeneous medium Ω, and the objective is to obtain information about these cavities.

As an observation site, we choose a bounded set $B \subset \mathbb{R}^3$ away from the cavities D^n and D^d (i.e., $\overline{B} \cap \overline{(D^n \cup D^d)} = \varnothing$). Incident waves are emitted from B and waves reflected by the cavities D^d and D^n are observed in the same region B. This corresponds to choosing initial data f so that $\operatorname{supp} f \subset \overline{B}$ and observing the solution $u(t,x)$ to (2.1) in $[0,T] \times B$. Here, f is selected as follows so that the incident wave is reliably emitted.

$$(2.2)\quad \begin{cases} f \in L^2(\Omega) \text{ with } \operatorname{supp} f \subset \overline{B} \text{ and there exists a constant } c > 0 \text{ such that} \\ f(x) \geq c \text{ a.e. } x \in \overline{B} \text{ or } -f(x) \geq c \text{ a.e. } x \in \overline{B}. \end{cases}$$

This condition (2.2) is called the emission condition.

In this paper, the solution of (2.1) is considered in the sense of a weak solution. On ∂D^d, the Dirichlet boundary condition is satisfied, so we introduce a function space $H^1_{0,\partial D^d}(\Omega) = \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \partial D^d \text{ in the trace sense}\}$ that is compatible with it. Here, $H^1(\Omega)$ is the usual L^2-Sobolev space. We denote the trace operator on ∂D^n by $\gamma_{\partial D^n} : H^1(\Omega) \to H^{1/2}(\partial D^n)$. We set

$$\langle \psi, \varphi \rangle_\Omega = \langle \psi, \varphi \rangle_{(H^1_{0,\partial D^d}(\Omega))' \times H^1_{0,\partial D^d}(\Omega)}.$$

Definition 2.1. *A function $u \in C([0,T]; H^1_{0,\partial D^d}(\Omega))$ is a weak solution of (2.1) if and only if u satisfies $\partial_t u \in C([0,T]; L^2(\Omega))$, $\partial_t^2 u \in L^2(0,T; (H^1(\Omega))')$, $u(0,x) = 0$, $\partial_t u(0,x) = f(x)$, and $\frac{d}{dt} \int_{\partial D^+} \lambda_1(x) \gamma_{\partial D^n} u(t,x) \varphi(x) dS_x$ is well-defined and*

$$\langle \partial_t^2 u(t,\cdot), \varphi \rangle_\Omega + \int_\Omega \gamma_0 \nabla_x u(t,x) \cdot \nabla_x \varphi(x) dx + \frac{d}{dt} \int_{\partial D^n} \lambda_1(x) \gamma_{\partial D^n} u(t,x) \varphi(x) dS_x$$
$$+ \int_{\partial D^n} \lambda_0(x) \gamma_{\partial D^n} u(t,x) \varphi(x) dS_x = 0 \quad \textit{a.e. on } (0,T)$$

holds for all $\varphi \in H^1_{0,\partial D^n}(\Omega)$.

It is well known that for any $f \in L^2(\Omega)$, there exists a unique weak solution $u(t,x)$ of (2.1). For example, the weak solution can be obtained by [1] Chap.18, Sections 5 and 6. We consider the weak solutions in this class.

The active sonar-type inverse problem considered here is the problem of estimating a cavity from measurement data $u|_{[0,T]\times B}$ corresponding to a single function f satisfying the emission condition (2.2). This corresponds to a single measurement, and note that one measurement yields one piece of data. In other words, this is a practicable framework, and by repeating a finite number of measurements while changing f and B, more information can be obtained.

In the enclosure method, based on these measurement data, the indicator function I_τ is constructed using weak forms commonly used in the theory of differential equations. Examining the asymptotic behavior of I_τ when $\tau \to \infty$, we can find the minimum length (shortest distance) l_0 of the line segment with endpoints at the point on the boundary of the object to be searched and the observation site B. For example, let $B = B(p,\delta)(= \{x \in \mathbb{R}^3 \mid |x-p| < \delta\})$, that is, an open ball with center $p \in \mathbb{R}^3$ and radius $\delta > 0$. In this case, if the shortest length $l_0(p)$ is determined, then $D \subset (B(p, \delta + l_0(p)))^c$. If we move p further and take measurements, we can see that $D \subset \cap_{p,|p|>>1}(B(p, \delta_0 + l_0(p)))^c$, and we can enclose D.

Let us introduce the indicator function I_τ for (2.1). For $f \in L^2(\Omega)$ satisfying (2.2), consider the weak solution $u(t,x)$ of (2.1). We extend f to 0 outside the domain Ω. We use the same notation to express the extended function in this manner. For extended f, consider the weak solution $v(\cdot;\tau) \in H^1(\mathbb{R}^3)$ of

$$(\gamma_0 \triangle - \tau^2)v(x;\tau) + f(x) = 0 \quad \text{in } \mathbb{R}^3, \tag{2.3}$$

and set

$$\mathcal{L}_T u(x;\tau) = \int_0^T e^{-\tau t} u(t,x)dt \quad (x \in \Omega).$$

For these, we introduce the indicator function:

$$I_\tau = \int_B f(x)(\mathcal{L}_T u(x;\tau) - v(x;\tau))dx.$$

Because we know $u(t,x)$ on $[0,T] \times \overline{B}$ as the measurement data, we can compute I_τ ($\tau \geq 1$). As described in the introduction, the enclosure method allows us to obtain the shortest distance $l_0 = \text{dist}(B, D^n \cup D^d)$ as

$$\lim_{\tau\to\infty} \frac{\sqrt{\gamma_0}}{2\tau} \log|I_\tau| = -l_0. \tag{2.4}$$

Depending on the boundary conditions of the cavity that achieves the shortest distance, the sign of I_τ may be positive or negative in the limit of $\tau \to \infty$. From this point of view, Neumann-type boundary conditions must also be treated separately for cavities where $\lambda_1(x) > \sqrt{\gamma_0}$ and for cavities where $\lambda_1(x) < \sqrt{\gamma_0}$. Thus, we assume that D^n can be divided into two types of cavity groups, D^{n+} and D^{n-}, which satisfy the following conditions.

$$\begin{cases} D^n = D^{n+} \cup D^{n-},\ \overline{D^{n+}} \cap \overline{D^{n-}} = \varnothing, \text{ and there exists a constant } \mu_1 > 0 \\ \text{such that } \mp(\lambda_1(x) - \sqrt{\gamma_0}) > \mu_1 \text{ a.e. on } \partial D^{n\pm}. \end{cases}$$

Thus, we assume that D is $D = D^{n+} \cup D^{n-} \cup D^d$ and $D^\alpha \cap D^\beta = \varnothing$ for $\alpha, \beta \in \{n_+, n_-, d\}$ with $\alpha \neq \beta$.

As summarized in the following table, previous studies handle the case that only one type of inclusions or cavities, for example, only the case of the Neumann boundary condition (i.e., $\lambda_1 = \lambda_0 = 0$).

No	boundary condition of cavities	type of objects	sign of I_τ	literature	asymptotics of I_τ
1	Neumann($\partial_\nu u = 0$)	only D^{n+}	$+$	[7]	None
2	Neumann with dissipation as in (2.1)	only D^{n+}	$+$	[8], [11]	[8] None
		only D^{n-}	$-$		[11] Given
3	Dirichlet($u = 0$) (bistatic case)	only D^d	$-$	[9]	Given
4	Robin($(\partial_\nu - \lambda_2)u = 0$)	only D^{n+}	$+$	[10]	Given

As in No.3 in the table, Ikehata in [9] formulates the bistatic case that the place measuring the reflected waves is different from the place generating the incident waves. Some studies have also investigated the asymptotic behavior of I_τ under additional assumptions to achieve the shortest distance l_0. Furthermore, there are also many studies on cases where the object is not a cavity but an inclusion, and results are also available for cases where the background medium (i.e., a medium without cavity and inhomogeneity) is a two-layer structure with a flat transmission boundary (e.g., [13, 14]). However, in these studies, since the object is a single type, the 'monotonicity condition' is satisfied. Because the monotonicity condition also provides a priori information about the target, this situation is special and cannot be considered a natural assumption. Therefore, when considering practical situations, it is important not to assume monotonicity.

3. Separated case

Among the three types of cavities $\{D^{n+}, D^{n-}, D^d\}$ defined in Section 2, when $\tau \to \infty$, the indicator function I_τ becomes positive for D^{n+}. On the other hand, for D^{n-} and D^d, the indicator function I_τ becomes negative. Thus, D^{n+} is a positive cavity, while D^{n-} and D^d are negative cavities. In this section, we consider the combined case where these exist simultaneously, particularly when the cavities are separated*.

The shortest distance for each group represented by

$$l_0^+ = \mathrm{dist}(B, D^{n_+}), \quad l_0^- = \min\Big\{\mathrm{dist}(B, D^{n_-}), \mathrm{dist}(B, D^d)\Big\}.$$

Thus, the shortest distance l_0 for all the cavities in the medium is given by $l_0 = \min\{l_0^+, l_0^-\}$. As mentioned in Section 1, 'separate' means that the negative cavities and positive cavities are clearly separated. Therefore, when $l_0^+ \neq l_0^-$ is known in advance, we call this case the 'separated case'.

In the enclosure method, the indicator function I_τ corresponds to the weak form that appears in the boundary value problems for elliptic operators. For this reason, the method of using elliptic estimates is very effective in the enclosure method, and originated from its use in boundary inverse problems of Laplace's equation ([3, 4]). This method has also been introduced in the reconstruction of wave equations using the enclosure method in the research by Ikehata [7, 8]. Hereafter, this method will be referred to as 'methods of elliptic estimates'. The strength of the 'method of elliptic estimates' is that it can be applied when conditions for the elliptic estimates are satisfied. For example, it is possible to set an assumption of low regularity for the cavity boundary. Therefore, let us apply the method of elliptic estimates to the combined case as well.

In the combined case, because of the existence for both positive and negative cavities, I_τ is approximately expected to have the following form :

(3.1)

$$I_\tau = \big(p_+(\tau)e^{-\frac{2}{\sqrt{\gamma_0}}l_0^+\tau} - p_-(\tau)e^{-\frac{2}{\sqrt{\gamma_0}}l_0^-\tau}\big), \ \ 0 < C_\pm \tau^{\alpha_1^\pm} \le p_\pm(\tau) \le C'_\pm \tau^{\alpha_2^\pm} \ (\tau >> 1),$$

where $C'_\pm > C_\pm > 0$ and $\alpha_2^\pm \ge \alpha_1^\pm$ are constants independent of $\tau >> 1$. However, it is difficult to show an asymptotic form such as (3.1), so instead we show:

$$I_\tau \ge \big(q_+(\tau)e^{-\frac{2}{\sqrt{\gamma_0}}l_0^+\tau} - q_-(\tau)e^{-\frac{2}{\sqrt{\gamma_0}}l_0^-\tau}\big) \quad (\tau >> 1), \tag{3.2}$$

$$-I_\tau \ge \big(\tilde{q}_-(\tau)e^{-\frac{2}{\sqrt{\gamma_0}}l_0^-\tau} - \tilde{q}_+(\tau)e^{-\frac{2}{\sqrt{\gamma_0}}l_0^+\tau}\big) \quad (\tau >> 1), \tag{3.3}$$

where $q_\pm$ and $\tilde{q}_\pm$ are functions satisfying $0 < C_\pm \tau^{\alpha_1^\pm} \le q_\pm(\tau) \le C'_\pm \tau^{\alpha_2^\pm}$ and $0 < C_\pm \tau^{\alpha_1^\pm} \le \tilde{q}_\pm(\tau) \le C'_\pm \tau^{\alpha_2^\pm}$ and for $\tau >> 1$. This is actually possible by the methods of elliptic estimates (cf. (3.8) and (3.12) in [16]).

Here, assuming that the cavities are 'separated', the shortest distance l_0 can

*Reference [16] considers cases where plus and minus inclusions exist simultaneously in addition to cavities; such cases are referred to as mixed cases. Here, we discuss only combined cases.

be obtained from these estimates as follows. If $l_0 = l_0^+ < l_0^-$ is known in advance, from (3.2) and (3.3) it follows that there exists a constant $C > 0$ such that

$$0 < C^{-1}\tau^{\alpha_1^+} \le e^{\frac{2}{\sqrt{\gamma_0}}l_0\tau} I_\tau \le C\tau^{\alpha_2^+} \quad (\tau >> 1).$$

This implies that

$$\frac{C^{-1}\alpha_1^+ \log\tau}{\tau} \le \frac{2}{\sqrt{\gamma_0}} l_0 + \log|I_\tau| \le \frac{C\alpha_2^+ \log\tau}{\tau} \quad (\tau >> 1).$$

Then, we have (2.4). If $l_0 = l_0^- < l_0^+$, we also obtain (2.4) as well.

Thus, under the assumption that the cavities are 'separated', since $l_0^+ \ne l_0^-$, the term with l_0^+ or l_0^- as the exponent, which does not match the shortest distance l_0, can be regarded as a remainder term. From this, if we can obtain estimates (3.2) and (3.3), we can obtain the shortest distance. This suggests that the separated case is similar to a case with a monotonicity condition, even if it is combined.

Let us consider the case where the cavities are not 'separated'. The problem is that $l_0^+ = l_0^- = l_0$ may occur. When $l_0^+ = l_0^- = l_0$, (3.1) becomes

$$I_\tau = \big(p_+(\tau) - p_-(\tau)\big)e^{-\frac{2}{\sqrt{\gamma_0}}l_0\tau}, \quad 0 < C_\pm\tau^{\alpha_1^\pm} \le p_\pm(\tau) \le C'_\pm\tau^{\alpha_2^\pm} \quad (\tau >> 1),$$

so it is necessary to know the more concrete forms of $p_+(\tau)$ and $p_-(\tau)$. Since the method of elliptic estimates is based on the estimation of inequality using exponential decay, further analysis is required to obtain the specific form of the polynomial.

When the monotonicity condition is satisfied, $p_+(\tau) = 0$ or $p_-(\tau) = 0$, but even in this case, it is very meaningful to investigate the asymptotic behavior of p_+ or p_-. Ikehata [9, 10, 11] employs appropriate elliptic estimates for each case and, when the regularity of the cavity boundary is C^3, provides the main terms of the asymptotic expansion (i.e., $p_+(\tau)$ or $p_-(\tau)$) using a skillful technique. However, to the best of the author's knowledge, a monotonicity condition seems to be necessary when estimating the error. Based on the above points, in the non-separated case where the separated assumption does not hold, we decided to adopt an asymptotic solution commonly used in scattering theory to investigate the asymptotic behavior of I_τ more directly. This is discussed in Section 4.

4. Non-separated case -heuristic argument-

In this section, we discuss cases where the cavities are not separated. Before going into detail, we divide I_τ into the main term J_τ and the remainder term. We set $\lambda(x;\tau) = \lambda_1(x)\tau + \lambda_0(x)$ and $\mathcal{B}^n_\tau w = (\nu_x \cdot \gamma_0 \nabla_x - \lambda(x;\tau))w$. Because $\mathcal{L}_T u(x;\tau)$ satisfies

$$\begin{cases} (L_{\gamma_1} - \tau^2)\mathcal{L}_T u(x;\tau) + f(x) = e^{-\tau T}\big(\partial_t u(T,x) + \tau u(T,x)\big) & \text{in } \Omega, \\ \mathcal{B}^n_\tau \mathcal{L}_T u(x;\tau) = \lambda_1(x) e^{-\tau T} u(T,x) & \text{on } \partial D^n, \\ \mathcal{L}_T u(x;\tau) = 0 & \text{on } \partial D^d, \end{cases}$$

$\mathcal{L}_T u(x;\tau)$ is approximated by

$$\|\mathcal{L}_T u(\cdot;\tau) - w(\cdot;\tau)\|_{L^2(\Omega)} \le C\tau^{-1} e^{-\tau T} \quad (\tau >> 1), \tag{4.1}$$

where $w(x;\tau)$ is the solution of

$$\begin{cases} (L_{\gamma_1} - \tau^2) w(x;\tau) + f(x) = 0 & \text{in } \Omega, \\ \mathcal{B}^n_\tau w(x;\tau) = 0 & \text{on } \partial D^n, \\ w(x;\tau) = 0 & \text{on } \partial D^d \end{cases} \tag{4.2}$$

(cf. Appendix C in [12]). We call (4.2) the reduced problem of (2.1), and the weak solution $v(\cdot;\tau)$ of (2.3) is described as

$$v(x;\tau) = \int_\Omega \Phi_\tau(x,y) f(y) dy \ \text{ with } \ \Phi_\tau(x,y) = \frac{1}{4\pi\gamma_0} \frac{e^{-\tau|x-y|/\sqrt{\gamma_0}}}{|x-y|}.$$

For the solution $w(x;\tau)$ of (4.2) and the solution $v \in H^1(\mathbb{R}^3)$ of (2.3), we set

$$J_\tau = \int_B f(x)(w(x;\tau) - v(x;\tau)) dx.$$

Then, the usual elliptic estimates yield the following.

$$I_\tau = J_\tau + O(\tau^{-1} e^{-\tau T}) \quad (\tau \to \infty).$$

Therefore, it is sufficient to deal with only J_τ instead of I_τ. By integration by parts, J_τ is decomposed as follows.

$$J_\tau = J^{n+}_\tau + J^{n-}_\tau + J^d_\tau, \tag{4.3}$$

where

$$J^{n\pm}_\tau = \int_{\partial D^{n\pm}} \mathcal{B}^n_\tau v(x;\tau)\, w(x;\tau) dS_x, \quad J^d_\tau = -\gamma_0 \int_{\partial D^d} \partial_{\nu_x} w(x;\tau)\, v(x;\tau)\, dS_x. \tag{4.4}$$

If we set $w(x;\tau) = v(x;\tau)|_\Omega + \tilde{w}(x;\tau)$, $\tilde{w}(x;\tau)$ corresponds to the reflected wave of the incident wave $v(x;\tau)$. Let $\tilde{w}(x;\tau)$ have a kernel representation:

$$\tilde{w}(x;\tau) = \int_\Omega \tilde{\Psi}_\tau(x,y) f(y)\, dy,$$

then the integral kernel $\tilde{\Psi}_\tau(x,y)$ satisfies

$$\begin{cases} (\gamma_0 \Delta - \tau^2)\tilde{\Psi}_\tau(x,y) = 0 & \text{in } \Omega, \\ \mathcal{B}^n_\tau \tilde{\Psi}_\tau(x,y) = -\mathcal{B}^n_\tau \Phi_\tau(x,y) & \text{on } \partial D^n, \\ \tilde{\Psi}_\tau(x,y) = -\Phi_\tau(x,y) & \text{on } \partial D^d. \end{cases} \tag{4.5}$$

Here, as in Section 4 of [17], we construct a concrete approximate solution for $\tilde{\Psi}_\tau(x,y)$ using asymptotic solutions. First, let us have a formal discussion without considering convergence or anything else. For x near ∂D, we construct the asymptotic solution:

$$\Psi_{\tau,N}(x,y) = e^{-\tau\phi(x,y)} \sum_{j=0}^{N} (-\tau)^{-j} b_j(x,y) \quad (y \in \overline{B}) \tag{4.6}$$

as an approximation of $\tilde{\Psi}_\tau(x,y)$. We construct the phase function ϕ and the amplitude functions b_j for x near ∂D satisfying

$$\begin{cases} \gamma_0 |\nabla_x \phi|^2 = 1 & \text{in } \Omega, \\ \phi(x,y) = |x-y|/\sqrt{\gamma_0} & \text{on } \partial D, \\ \partial_{\nu_x}\phi(x,y) > 0 & \text{on } \partial D, \end{cases} \tag{4.7}$$

and

$$\begin{cases} T_\phi b_j(x,y) + \gamma_0 \Delta b_{j-1}(x,y) = 0 & \text{in } \Omega, \\ \big((\gamma_0 \partial_{\nu_x}\phi) + \lambda_1(x)\big) b_j(x,y) + \mathcal{B}^n_0 b_{j-1}(x,y) = -a_j(x,y) & \text{on } \partial D^n, \\ b_j(x,y) = -\frac{\delta_{0,j}}{4\pi\gamma_0|x-y|} & \text{on } \partial D^d, \end{cases} \tag{4.8}$$

where $T_\phi = 2(\gamma_0 \nabla_x \phi)\cdot\nabla_x + \gamma_0 \Delta\phi$ and $\mathcal{B}^n_0 w = \gamma_0 \partial_{\nu_x} w - \lambda_0(x) w$, $\delta_{0,j} = 1$ for $j = 0$, and $\delta_{0,j} = 0$ for $j \geq 1$ and a_j $(j = 0, 1)$ is defined by

$$a_0 = \frac{1}{4\pi|x-y|}\Big(\frac{\nu_x\cdot(x-y)}{\sqrt{\gamma_0}|x-y|} + \frac{\lambda_1(x)}{\gamma_0}\Big), \quad a_1 = \frac{-1}{4\pi|x-y|}\Big(\frac{\nu_x\cdot(x-y)}{|x-y|^2} + \frac{\lambda_0(x)}{\gamma_0}\Big),$$

and $a_j = 0$ for $j \geq 2$. Then, note that

$$\mathcal{B}_\tau^n \Phi_\tau(x,y) = (-\tau a_0(x,y) + a_1(x,y))e^{-\tau|x-y|/\sqrt{\gamma_0}}.$$

Substituting $\Psi_{\tau,N}(x,y)$ into $\tilde{\Psi}_\tau(x,y)$ in (4.5), we find that $\Psi_{\tau,N}(x,y)$ is an approximate solution to $\tilde{\Psi}_\tau(x,y)$ with remainders of order $\tau^{-N}e^{-\tau\phi(x,y)}$.

Then, if we substitute $\Psi_{\tau,N}(x,y)$ into (4.4) and formally take the limit as $N \to \infty$, we obtain

$$J_\tau^{n_\pm} = \sum_{k=-1}^{\infty} \tau^{-k} K_{\tau,-k}^{n_\pm} \quad \text{and} \quad J_\tau^d = \sum_{k=-1}^{\infty} \tau^{-k} K_{\tau,-k}^d,$$

where $K_{\tau,-k}^\alpha$ ($\alpha \in \{n_+, n_-, d\}$, $k = -1, 0, 1, \ldots$)) are given by

$$K_{\tau,-k}^\alpha = \int_{B\times B} dy d\tilde{y} f(y) f(\tilde{y}) \int_{\partial D^\alpha} e^{-\frac{\tau}{\sqrt{\gamma_0}}(|x-y|+|x-\tilde{y}|)} \kappa_{-k}^\alpha(x,y,\tilde{y}) dS_x. \tag{4.9}$$

The top term of the expansion (when $k = -1$) can be expressed specifically as follows:

$$\kappa_1^{n_\pm}(x,y,\tilde{y}) = -a_0(x,y)\Big\{\frac{1}{4\pi\gamma_0|x-\tilde{y}|} + b_0(x,\tilde{y})\Big\}, \tag{4.10}$$

$$\kappa_1^d(x,y,\tilde{y}) = \frac{1}{(4\pi)^2\gamma_0|x-y|}\Big\{-\frac{1}{\sqrt{\gamma_0}}\frac{\nu_x\cdot(x-\tilde{y})}{|x-\tilde{y}|^2} + \frac{\partial_{\nu_x}\phi(x,\tilde{y})}{|x-\tilde{y}|}\Big\}. \tag{4.11}$$

For $k = 0, 1, \ldots$, we have

(4.12)
$$\kappa_{-k}^\alpha(x,y,\tilde{y}) = (-1)^k\Big\{a_0(x,y)b_{k+1}(x,\tilde{y}) + a_1(x,y)\Big(\frac{\delta_{0,k}}{4\pi\gamma_0|x-\tilde{y}|} + b_k(x,\tilde{y})\Big)\Big\}$$
$$(\alpha \in \{n_+, n_-\}),$$

(4.13)
$$\kappa_{-k}^d(x,y,\tilde{y}) = \frac{1}{4\pi|x-y|}\Big(-\delta_{0,k}\frac{\nu_x\cdot(x-\tilde{y})}{4\pi\gamma_0|x-\tilde{y}|^3} + (-1)^k\partial_{\nu_x}b_k(x,\tilde{y})\Big)$$

(cf. p.12 of [17]).

From this expression, we can determine the coefficients of the highest terms for $p_+(\tau)$ and $p_-(\tau)$ of (3.1) in Section 3 for $\tau \to \infty$. For this purpose, the Laplace method is used:

Lemma 4.1. *Let U be an arbitrary open set of $\mathbb{R}^n$. Assume that $h \in C^{2,\beta_0}(\overline{U})$*

with some $\beta_0 > 0$ *and* $h(x) > h(x_0)$ *for all* $x \in \overline{U} \setminus \{x_0\}$ *at a point* $x_0 \in U$*, and* $det(Hess(h)(x_0)) > 0$*. Then, for a given* $\varphi \in C^{0,\beta_0}(\overline{U})$ *it holds that*

$$\int_U e^{-\tau h(x)}\varphi(x)dx = \frac{e^{-\tau h(x_0)}}{\sqrt{det(Hess(h)(x_0))}}\left(\frac{2\pi}{\tau}\right)^{n/2}\Big(\varphi(x_0) + \|\varphi\|_{C^{0,\beta_0}(\overline{U})}O(\tau^{-\beta_0/2})\Big).$$

Moreover there exists a positive constant C *such that, for all* $\tau \geq 1$

$$\left|\int_U e^{-\tau h(x)}\varphi(x)dx\right| \leq \frac{Ce^{-\tau h(x_0)}}{\tau^{n/2}}\|\varphi\|_{C(\overline{U})}.$$

For a proof, see e.g. Appendix A in [12].

Set $L_0(x,y) = |x-y|$ and $E_0 = \{(x_0,y_0) \in \overline{D} \times \overline{B} \mid L_0(x_0,y_0) = l_0\}$. By comparing (4.9) with the statement of Lemma 4.1, we can see that the dominant part of (4.9) is determined by point $(x_0,y_0,\tilde{y}_0)$ which gives the minimum value of the function $L(x,y,\tilde{y}) = |x-y|+|x-\tilde{y}| = L_0(x,y)+L_0(x,\tilde{y})$. The point $(x_0,y_0,\tilde{y}_0)$ is of the form (x_0,y_0,y_0) for the point $(x_0,y_0) \in E_0$ that gives the minimum value of $L_0(x,y)$.

To use Lemma 4.1, we introduce the following 'non-degenerate condition' (Definition 4.2).

Definition 4.2. *(non-degenerate condition) We say that* B *and* D *satisfy the non-degenerate condition with respect to the function* $L_0(x,y)$ *if and only if for any* $(x_0,y_0) \in E_0$*, there exist constants* $\delta > 0$ *and* $c_0 > 0$ *such that*

$$L_0(x,y) \geq l_0 + c_0(|x-x_0|^2 + |y-y_0|^2) \quad ((x,y) \in (\partial D \cap B_\delta(x_0)) \times (\partial B \cap B_\delta(y_0))),$$

where $B_\delta(x_0) = \{x \in \mathbb{R}^3 \mid |x-x_0| < \delta\}$*.*

From the non-degenerate condition, we can see that E_0 is a discrete set. Therefore, the following expressions will be used from now on:

$$E_0 = \cup_{\alpha \in \{n_+,n_-,d\}} E_0^\alpha, \quad E_0^\alpha = \{(x_j^\alpha, y_j^\alpha) \in \partial D^\alpha \times \partial B \mid j = 1,2,\ldots,M^\alpha\}.$$

Applying the Laplace method, the geometric information of the cavity boundary appears in the coefficient of the top term of the expansion of the indicator function. For simplicity, we take B as a ball with radius $a > 0$, and we denote the principal curvatures of ∂D^α at $x \in \partial D^\alpha$ by $\kappa_{1,\alpha}(x)$ and $\kappa_{2,\alpha}(x)$, where $\kappa_{1,\alpha}(x) \leq \kappa_{2,\alpha}(x)$ for

$\alpha \in \{n_+, n_-, d\}$. We set

$$\mathcal{A}_\alpha(x) = \prod_{j=1}^{2} \Big(\kappa_{j,\alpha}(x) + \frac{1}{l_0 + a} \Big) \quad (\alpha \in \{n_+, n_-, d\}).$$

As in Section 2 of [17], non-degenerate condition yields $\mathcal{A}_\alpha(x_j^\alpha) > 0$ for any $x_j^\alpha \in \partial D^\alpha$ satisfying $(x_j^\alpha, y_j^\alpha) \in E_0^\alpha$ for some $y_j^\alpha \in \partial B$. Take a local coordinate $x = s^\alpha(\sigma)$, $y = b^\alpha(u)$ near (x_j^α, y_j^α) and set $\tilde{L}^\alpha(\sigma, u, \tilde{u}) = L_0(s^\alpha(\sigma), b^\alpha(u)) + L_0(s^\alpha(\sigma), b^\alpha(\tilde{u}))$. Then, we have

$$\det(\mathrm{Hess}(\tilde{L}^\alpha)(0,0,0)) = \frac{4(a + l_0)^4}{(l_0)^4 a^4} \mathcal{A}_\alpha(x_j^\alpha). \tag{4.14}$$

Thus, from Lemma 4.1, we can obtain the asymptotic behavior of I_τ. However, the above asymptotic solution does not converge in general. Thus, if we take the first N-terms as in (4.6), then we need to consider the remainder estimates. Further discussion is required to concretize this intuitive argument. This was performed in [17], which reached the following conclusion.

THEOREM 4.3. *(Theorem 3.3 in* [17]*) Assume that ∂D^n and ∂D^d are of class C^4, B is a ball with radius $a > 0$, and D and B satisfy the non-degenerate condition for $L_0(x, y)$, $\lambda_0 \in L^\infty(\partial D^n)$, $\lambda_1 \in C^2(\partial D^n)$ and $f \in C^1(\overline{B})$. Then, there exists $\delta > 0$ such that*

$$I_\tau = \frac{\pi \gamma_0}{\tau^4} e^{-\frac{\tau}{\sqrt{\gamma_0}} 2 l_0} \Big\{ \mathcal{T}_0 + O(\tau^{-\frac{1}{2}}) \Big\} + O(e^{-\frac{\tau}{\sqrt{\gamma_0}}(2 l_0 + \delta)}) + O(\tau^{-1} e^{-\tau T}) \quad (\tau \to \infty),$$

where $\mathcal{T}_0 = \sum_{\alpha \in \{n_+, n_-, d\}} \sum_{j=1}^{M^\alpha} \frac{a^2}{2(l_0+a)^2} b_\alpha(x_j^\alpha) \frac{(f(y_j^\alpha))^2}{\sqrt{\mathcal{A}_\alpha(x_j^\alpha)}}$, $(x_j^\alpha, y_j^\alpha) \in E_0^\alpha$, $b_\alpha(x) = \frac{\sqrt{\gamma_0} - \lambda_1(x)}{\sqrt{\gamma_0} + \lambda_1(x)}$ *for* $\alpha \in \{n_+, n_-\}$ *and* $b_d(x) = -1$.

From the above result, even if $l_0^+ = l_0^-$, the shortest distance l_0 can be determined if $\mathcal{T}_0 \neq 0$. The detection of l_0 is impossible only in the special case where $\mathcal{T}_0 = 0$.

Taking the case where D^{n_+} and D^d are equidistant from the observation site B, we consider the effects of curvature and energy dissipation. The curvatures of the cavity boundary appear in the denominator of the coefficients of the highest-order term in the asymptotic expansion. Therefore, the curvature of the cavity boundary affects the detectability of the cavity. For example, when there is no effect of energy dissipation on ∂D^{n_+} (i.e., $\lambda_1(x) = 0$ in $x \in \partial D^{n_+}$), between cavities D^{n_+} and D^d, the cavity with the smaller curvature is detected at the arrival point. In addition, the effect of energy dissipation on ∂D^n reduces the detectability of D^n. When energy is dissipated on ∂D^n, the measured reflected waves become weaker, making it difficult to detect D^{n_+}. For example, if $\lambda_1(x) > 0$ holds for $x \in \partial D^{n_+}$, then D^d is detected if the curvatures at the arrival points on the cavity boundary are equal (i.e., if $\kappa_{j,n_+}(x_0^{n_+}) = \kappa_{j,d}(x_0^d)$ $(j = 1, 2)$).

The asymptotic solution requires high regularity assumptions on the cavity boundary ∂D. In fact, the above result uses the asymptotic solution $\Psi_{\tau,0}(x, y)$ of order $N = 0$, but requires condition C^4 for the boundary ∂D and $\lambda_1 \in C^2(\partial D^n)$. To obtain higher-order asymptotic expansions, the corresponding higher regularity assumptions are required.

5. Asymptotic expansion in the smooth case

Here, we show that when ∂D, ∂B, λ_0, λ_1, and f are smooth enough, an asymptotic expansion of any order can be obtained for the indicator function I_τ. In this section, we also assume the non-degenerate condition (Definition 4.2). In [17], the following terminology is introduced for ∂D, λ_0, and λ_1.

Definition 5.1. *We say $(\partial D, \lambda)$ is of m-class if ∂D is of class C^{2m+4}, $\lambda_1 \in C^{2m+2}(\partial D^n)$ and $\lambda_0 \in C^{2m}(\partial D^n)$ for integer $m \geq 1$, and $(\partial D, \lambda)$ is of 0-class(that is, $m = 0$) if ∂D is C^4, $\lambda_1 \in C^2(\partial D^n)$ and $\lambda_0 \in L^\infty(\partial D^n)$.*

Theorem 4.3 is the asymptotic expansion of I_τ when $(\partial D, \lambda)$ is of 0-class. In this section we show that the following holds.

Theorem 5.2. *Assume that $(\partial D, \lambda)$ is of N-class for some $N \in \mathbb{N}$, and ∂B has C^∞ boundary and $f \in C^\infty(\overline{B})$. Then, the indicator function I_τ has the asymptotic expansion of the form:*

(5.1)
$$I_\tau = \frac{(2\pi)^3(\sqrt{\gamma_0})^5}{\tau^4} e^{-\frac{\tau}{\sqrt{\gamma_0}}2l_0} \Big\{ \sum_{l=0}^{N} \mathcal{S}_l \tau^{-l} + O(\tau^{-N-1}) \Big\} + O(\tau^{-1} e^{-\tau T}) \quad (\tau \to \infty)$$

for certain constants $\mathcal{S}_l$. If we assume that B is a ball with radius $a > 0$, then (5.1) is also represented as

$$I_\tau = \frac{\pi\gamma_0}{\tau^4} e^{-\frac{\tau}{\sqrt{\gamma_0}}2l_0} \Big\{ \sum_{l=0}^{N} \mathcal{T}_l \tau^{-l} + O(\tau^{-N-1}) \Big\} + O(\tau^{-1} e^{-\tau T}) \quad (\tau \to \infty), \tag{5.2}$$

where $\mathcal{T}_0$ is given in Theorem 4.3.

To prove Theorem 5.2, let us define the local coordinates $x = s^{\alpha}(\sigma)$ and $y = b^{\alpha}(u)$ for each $(x_j^{\alpha}, y_j^{\alpha})$. Set $\mathcal{U}_k^{\alpha} = \cup_{j=1}^{M^{\alpha}} B_{kr_0}(x_j^{\alpha})$ $(1 \leq k \leq 4,\ j = 1, 2, \ldots, M^{\alpha}$, $\alpha \in \{n_+, n_-, d\})$, $\overline{\mathcal{U}_k^{\alpha}} \subset \mathcal{U}_{k+1}^{\alpha}$ $(k = 1, 2, 3)$. Let $r_0 > 0$ be small enough so that $\overline{B_{4r_0}(x_j^{\alpha})} \cap \overline{B_{4r_0}(y_j^{\alpha})} = \varnothing$ and $(B_{4r_0}(x_j^{\alpha}) \times B_{4r_0}(y_j^{\alpha})) \cap E_0 = \{(x_j^{\alpha}, y_j^{\alpha})\}$. From (2.2) of [17], there exists a constant $c_0 > 0$ such that

$$(5.3) \qquad |x - y| \geq l_0 + c_0 \quad \text{if } (x, y) \in (\partial D \setminus \mathcal{U}_1) \times \overline{B} \ \text{ or } \ (x, y) \in \partial D \times (\overline{B} \setminus \mathcal{V}_1),$$

where $\mathcal{V}_1 = \cup_{\alpha \in \{n_+, n_-, d\}} \cup_{j=1}^{M^{\alpha}} B_{r_0}(y_j^{\alpha})$. From (5.3), we can see that the main part of each $K_{\tau,-k}^{\alpha}$ is given only in the neighborhood of $(x_j^{\alpha}, y_j^{\alpha}, y_j^{\alpha})$.

According to the discussion in Sections 4 and 5 of [17], if we appropriately select an open set $\tilde{B}$ such that $\overline{B} \subset \tilde{B}$, then the solution ϕ and b_j of (4.7) and (4.8) exist in $(\overline{\Omega} \cap \mathcal{U}_4) \times \tilde{B}$, respectively, and we know that $\phi \in \tilde{C}^{2N+4}((\overline{\Omega} \cap \mathcal{U}_4) \times \tilde{B})$ and $b_j \in \tilde{C}^{2(N-j)+2}((\overline{\Omega} \cap \mathcal{U}_4) \times \tilde{B})$, $j = 1, \ldots, N$. Here, $\tilde{C}^k((\overline{\Omega} \cap \mathcal{U}_4) \times \tilde{B}) = \{\phi \in C((\overline{\Omega} \cap \mathcal{U}_4) \times \tilde{B}) \mid \partial_x^{\alpha} \partial_y^{\beta} \phi \in C((\overline{\Omega} \cap \mathcal{U}_4) \times \tilde{B})$ for $|\alpha| \leq k$ and $|\beta| \geq 0\}$. Therefore, the asymptotic solution defined in (4.6) is valid in $\overline{\Omega} \cap \mathcal{U}_4$. Take a cutoff function $\chi \in C_0^{\infty}(\mathbb{R}^3)$ with $\chi(x) = 1$ for $x \in \mathcal{U}_1$, $\mathrm{supp}\chi \subset \mathcal{U}_2$. Then, as in p.11 of [17], the solution $w(x; \tau)$ of (4.2) can be given in the form $w(x; \tau) = w_N(x; \tau) + w_N^r(x; \tau)$, where we can see that

$$w_N(x; \tau) = \int_B \big(\Phi_{\tau}(x, y) + \chi(x) \Psi_{\tau,N}(x, y)\big) f(y)\, dy,$$

and there exist constant $C > 0$ and $\tau_0 > 1$ such that $w_N^r(x; \tau)$ satisfies

$$\|w_N^r\|_{L^2(\partial D^n)} \leq C \tau^{-N-7/2} e^{-\frac{\tau}{\sqrt{\gamma_0}} l_0} \quad (\tau \geq \tau_0),$$

$$\Big| \int_{\partial D^d} \gamma_0 \partial_{\nu_x} w_N^r(x; \tau) g(x) dS_x \Big| \leq C \tau^{-N-3} e^{-\tau l_0/\sqrt{\gamma_0}} \|g\|_{H_{\tau}^{1/2}(\partial D^d)} \qquad (g \in H^{1/2}(\partial D^d), \tau \geq \tau_0)$$

(cf. (6.16) and (6.19) of [17]). From this fact, (4.3), (4.1) and Lemma 6.3 of [17], we obtain

(5.4)

$$I_{\tau} = \sum_{\alpha \in \{n_+, n_-, d\}} \vartheta_{\alpha} \sum_{k=-1}^{N} K_{\tau,-k}^{\alpha} \tau^{-k} + O(\tau^{-N-5} e^{-\frac{2}{\sqrt{\gamma_0}} l_0 \tau}) + O(\tau^{-1} e^{-\tau T}) \quad (\tau \to \infty).$$

Here, $\vartheta_{n_{\pm}} = 1$, $\vartheta_d = -1$, each $K_{\tau,-k}^{\alpha}$ is given by (4.9) and $\kappa_1^{\alpha}(x, y, \tilde{y})$ is given by (4.10) and (4.11). Moreover, for $0 \leq k \leq N$, $\kappa_{-k}^{\alpha}(x, y, \tilde{y})$ is given by (4.12) and

(4.13). Here, we set $b_k = 0$ $(k \geq N+1)$. If we note (5.3), we have

$$K^{\alpha}_{\tau,-k} = \sum_{j=1}^{M^{\alpha}} K^{0,\alpha}_{\tau,-k,j} + O(e^{-\frac{2\tau}{\sqrt{\gamma_0}}(l_0+\delta)}) \quad (\tau \to \infty) \tag{5.5}$$

for some $\delta > 0$, where we set for any $(x^{\alpha}_j, y^{\alpha}_j)$ $(\alpha \in \{n_+, n_-, d\}$, $j = 1, 2, \ldots, M^{\alpha})$

(5.6)

$$K^{0,\alpha}_{\tau,-k,j} = \int_{\partial D^{\alpha} \cap B_{r_0}(x^{\alpha}_j)} \left(\int_{(B \cap B_{r_0}(y^{\alpha}_j))^2} e^{-\frac{\tau}{\sqrt{\gamma_0}} L(x,y,\tilde{y})} \kappa^{\alpha}_{-k}(x,y,\tilde{y}) f(y) f(\tilde{y}) dy d\tilde{y} \right) dS_x.$$

Above, the form of $\kappa^{\alpha}_{-k}(x,y,\tilde{y})$ differs from that in [17], but noting that $\chi(x) = 1$ in $\mathcal{U}_1$, we see that they coincide at least on $\mathcal{U}_1 \times (B \cap B_{r_0}(y^{\alpha}_j))^2$. This fact and (5.3) show that $K^{\alpha}_{\tau,-k}$ can be decomposed as (5.5) and (5.6).

In order to obtain the complete asymptotic expansion of I_τ, in addition to the discussion in Theorem 4.3, an asymptotic expansion for each $K^{0,\alpha}_{\tau,-k,j}$ is required. Here, we use the following Laplace method.

Proposition 5.3. *Let U be an arbitrary open set of $\mathbb{R}^n$. Assume that $h \in C^{2N+2}(\overline{U})$ for some $N \in \mathbb{N}$, and $h(x) > h(x_0)$ for all $x \in \overline{U} \setminus \{x_0\}$ at a point $x_0 \in U$, and $\det(\mathit{Hess}(h)(x_0)) > 0$. Then, for a given $\varphi \in C^{2N}(\overline{U})$ it holds that*

$$\int_U e^{-\tau h(x)} \varphi(x) dx = \frac{e^{-\tau h(x_0)}}{\sqrt{\det(\mathit{Hess}(h)(x_0))}} \left(\frac{2\pi}{\tau}\right)^{n/2} \left(\sum_{j=0}^{N-1} \mathcal{L}_j \varphi(x_0) \tau^{-j} + R_N(\tau) \right) \quad (\tau \to \infty),$$

where $R_N(\tau)$ is estimated by

$$|R_N(\tau)| \leq C_N \tau^{-N} \sum_{|\beta| \leq 2N+2} \sup_{x \in U} |\partial^{\beta}_x h(x)| \sum_{|\beta| \leq 2N} \sup_{x \in U} |\partial^{\beta}_x \varphi(x)| \quad (\tau \geq 1).$$

Each $\mathcal{L}_j$ $(j = 0, 1, \ldots, N-1)$ is linear differential operators of order $2j$ with coefficients consisting of differentials $(\partial^{\alpha}_x h)(x_0)$ $(2 \leq |\alpha| \leq 2(j+1))$, which is given by

$$\mathcal{L}_j \varphi(x_0) = \sum_{\substack{l-k=j \\ 3k \leq 2l}} \frac{1}{l!k!} \frac{(-1)^k}{2^l} \left[((2H(x_0))^{-1} \partial_x, \partial_x)^l (\Psi(x))^k \varphi \right](x_0). \tag{5.7}$$

Here, $H(x_0) = \frac{1}{2}(\text{Hess}(h)(x_0)) > 0$*, and* $\Psi(x)$ *is defined by*

$$\Psi(x) = ((H^1(x) - H(x_0))(x - x_0), (x - x_0))_{\mathbb{R}^n}, \qquad H^1(x) = (H^1_{ij}(x)),$$

where

$$H^1_{ij}(x) = \sum_{k=0}^{2N-1} \frac{1}{(k+1)(k+2)} \sum_{|\alpha|=k} \frac{1}{\alpha!} \partial_x^\alpha h_{x_i x_j}(x_0)(x - x_0)^\alpha.$$

If we suppose that $h \in C^{3N+1}(U)$ and $\varphi \in C^{2N}(U)$, Proposition 5.3 is the same as Theorem 7.7.5 of Hörmander [2]. In Proposition 5.3, we assume that $h \in C^{2N+2}(U)$, we need to give additional arguments to obtain

Remark 5.4. *(1) In the differential operators* $\mathcal{L}_j$ *defined by (5.7), formally, differentials of order greater than* $2j$ *may appear. Hence, the terms* $\left[((2H(x_0))^{-1}\partial_x, \partial_x)^l (\Psi(x))^k \varphi\right](x_0)$ *has to be written with* $\left[((2H(x_0))^{-1}\partial_x, \partial_x)^l (\Psi(x))^k \varphi_1\right](x_0)$*, where*

$$\varphi_1(x) = \sum_{|\alpha| \le 2N} \frac{1}{\alpha!} \partial_x^\alpha \varphi(x_0)(x - x_0)^\alpha.$$

However, since $\Psi(x) = O(|x - x_0|^3)$ $(x \to x_0)$*, for any smooth function* φ*, for* $3l > 2k$*, every coefficient of* $\left[((2H(x_0))^{-1}\partial_x, \partial_x)^l (\Psi(x))^k \varphi\right]$ *at* $x = x_0$ *vanishes. This is the reason why* $\mathcal{L}_j$ *is of order* $2j$*, and we write* $\mathcal{L}_j$ *as (5.7) even though* $\varphi \in C^{2N}(U)$*.*
(2) In $\mathcal{L}_0$*, the sum is taken over for* $k = l = 0$*, which yields* $\mathcal{L}_0\varphi(x_0) = \varphi(x_0)$*. However,* $\mathcal{L}_j$ $(j \ge 1)$ *is complicated even when* $j = 1$ *(see e.g. section 6).*

Proof of Theorem 5.2. Choose $(x_j^\alpha, y_j^\alpha) \in E_0$ and take an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for $T_{x_j^\alpha}(\partial D^\alpha)$. Then, we have $y_j^\alpha = x_j^\alpha + l_0 \nu_{x_j^\alpha}$. As in p.5 of [17], for sufficiently small $r_0 > 0$, we take a local coordinate system of $\partial D^\alpha \cap B_{4r_0}(x_j^\alpha)$ and $\partial B \cap B_{4r_0}(x_j^\alpha)$ as

$$U_{x_j^\alpha} \ni \sigma = (\sigma_1, \sigma_2) \mapsto s^\alpha(\sigma) = x_j^\alpha + \sigma_1 \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 - g^\alpha(\sigma)\nu_{x_j^\alpha} \in \partial D^\alpha \cap B_{4r_0}(x_j^\alpha),$$
$$U_{y_j^\alpha} \ni u = (u_1, u_2) \mapsto b^\alpha(u) = x_j^\alpha + u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + (l_0 + h^\alpha(u))\nu_{x_j^\alpha} \in \partial B \cap B_{4r_0}(y_j^\alpha),$$

where $U_{x_j^\alpha}$ and $U_{y_j^\alpha}$ are open neighborhoods of $(0,0) \in \mathbb{R}^2$, $g^\alpha = g^\alpha_{x_j^\alpha} \in C^{2N+4}(\mathbb{R}^2)$ and $h^\alpha = h^\alpha_{y_j^\alpha} \in C^\infty(\mathbb{R}^2)$ satisfying $g^\alpha(0,0) = h^\alpha(0,0) = 0$, $\nabla g^\alpha(0,0) = \nabla h^\alpha(0,0) = 0$. To describe a local coordinate system of $B \cap B_{r_0}(x_j^\alpha)$, we set $\tilde{U}_{y_j^\alpha} =$

$\{u \in U_{y_j^\alpha} \mid b^\alpha(u) \in \partial B \cap B_{r_0}(x_j^\alpha)\}$, $t_j^\alpha(u) = \sup\{u_3 \in [0, 3r_0] \mid b^\alpha(u) + u_3\nu_{x_j^\alpha} \in B \cap B_{3r_0}(y_j^\alpha)\}$ and $y = y^\alpha(u, u_3) = b^\alpha(u) + u_3\nu_{x_j^\alpha}$ $(u \in \tilde{U}_{y_j^\alpha},\ 0 \le u_3 \le t_j^\alpha(u))$. We also set $\tilde{U}_{x_j^\alpha} = \{\sigma \in U_{x_j^\alpha} \mid s^\alpha(\sigma) \in \partial D \cap B_{r_0}(x_j^\alpha)\}$. Note that $|g^\alpha(\sigma)| \le r_0$ $(\sigma \in \tilde{U}_{x_j^\alpha})$ and $|h^\alpha(u)| \le r_0$ and $r_0 \le t_j^\alpha(u) \le 3r_0$ $(u \in \tilde{U}_{y_j^\alpha})$.

Using these local coordinate systems, $K^{0,\alpha}_{\tau,-k,j}$ is expressed as

$$K^{0,\alpha}_{\tau,-k,j} = \int_{\tilde{U}_{x_j^\alpha}} d\sigma \int_{\tilde{U}_{y_j^\alpha} \times \tilde{U}_{y_j^\alpha}} du d\tilde{u} J_{\partial D}(\sigma) J_{\partial B}(u) J_{\partial B}(\tilde{u})$$

$$\int_0^{t_j^\alpha(\tilde{u})} \int_0^{t_j^\alpha(u)} e^{-\frac{\tau}{\sqrt{\gamma_0}} L(s^\alpha(\sigma), y^\alpha(u,u_3), y^\alpha(\tilde{u},\tilde{u}_3))} \theta^\alpha_{-k,j}(\sigma, u, \tilde{u}, u_3, \tilde{u}_3) du_3 d\tilde{u}_3, \tag{5.8}$$

where $J_{\partial D}(\sigma) = \sqrt{1 + (g^\alpha_{\sigma_1})^2 + (g^\alpha_{\sigma_2})^2}$, $J_{\partial D}(\sigma) = \sqrt{1 + (h^\alpha_{u_1})^2 + (h^\alpha_{u_2})^2}$ and

$$\theta^\alpha_{-k,j}(\sigma, u, \tilde{u}, u_3, \tilde{u}_3) = \kappa^\alpha_{-k}(s^\alpha(\sigma), y^\alpha(u, u_3), y^\alpha(\tilde{u}, \tilde{u}_3)) f(y^\alpha(u, u_3)) f(y^\alpha(\tilde{u}, \tilde{u}_3))$$
$$(\sigma \in \tilde{U}_{x_j^\alpha}, u, \tilde{u} \in \tilde{U}_{y_j^\alpha}, 0 \le u_3 \le t_j^\alpha(u) \text{ and } 0 \le \tilde{u_3} \le t_j^\alpha(\tilde{u})).$$

We set

$$A^\alpha(\sigma, u, u_3) = \frac{2(l_0 + g^\alpha(\sigma) + h^\alpha(u))u_3 + u_3^2}{L_0(s^\alpha(\sigma), y^\alpha(u, u_3)) + L_0(s^\alpha(\sigma), b^\alpha(u))}.$$

Then, since we have $L_0(s^\alpha(\sigma), y^\alpha(u, u_3)) = L_0(s^\alpha(\sigma), b^\alpha(u)) + A^\alpha(\sigma, u, u_3)$, the integral of u_3 and $\tilde{u}_3$ of (5.8) can be expressed by

$$\int_0^{t_j^\alpha(\tilde{u})} \int_0^{t_j^\alpha(u)} e^{-\frac{\tau}{\sqrt{\gamma_0}} A^\alpha(\sigma,u,u_3)} e^{-\frac{\tau}{\sqrt{\gamma_0}} A^\alpha(\sigma,\tilde{u},\tilde{u}_3)} \theta^\alpha_{-k,j}(\sigma, u, \tilde{u}, u_3, \tilde{u}_3) du_3 d\tilde{u}_3.$$

Because $(\partial_{u_3} A^\alpha)(\sigma, u, 0) = A^\alpha_{u_3}(\sigma, u, 0) = \frac{l_0 + g^\alpha(\sigma) + h^\alpha(u)}{L_0(s^\alpha(\sigma), b^\alpha(u))} \ge \frac{l_0 - 2r_0}{l_0 + 2r_0}$ $((\sigma, u) \in \tilde{U}_{x_j^\alpha} \times \tilde{U}_{y_j^\alpha})$, we can assume that there exists a constant $c_0 > 0$ such that $A^\alpha_{u_3}(\sigma, u, u_3) \ge c_0 > 0$ for $(\sigma, u) \in \tilde{U}_{x_j^\alpha} \times \tilde{U}_{y_j^\alpha}$ and $0 \le u_3 \le t_j^\alpha(u)$ if we take $r_0 > 0$ sufficiently small again if necessary.

We set $Q = Q(\sigma, u, u_3, \partial_{u_3}) = \frac{1}{A^\alpha_{u_3}(\sigma,u,u_3)} \partial_{u_3}$. Then, it follows that

$$Q(\sigma, u, u_3, \partial_{u_3}) e^{-\frac{\tau}{\sqrt{\gamma_0}} A^\alpha(\sigma,u,u_3)} = -\frac{\tau}{\sqrt{\gamma_0}} e^{-\frac{\tau}{\sqrt{\gamma_0}} A^\alpha(\sigma,u,u_3)}.$$

Hence, integration by parts yields

$$\begin{aligned}
&\int_0^{t_j^\alpha(u)} e^{-\frac{\tau}{\sqrt{\gamma_0}}A^\alpha(\sigma,u,u_3)}\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},u_3,\tilde{u}_3)du_3\\
&=\frac{\sqrt{\gamma_0}}{\tau}\Big(\frac{\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},0,\tilde{u}_3)}{A^\alpha_{u_3}(\sigma,u,0)}-e^{-\frac{\tau}{\sqrt{\gamma_0}}A^\alpha(\sigma,u,t_j^\alpha(u))}\frac{\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},t_j^\alpha(u),\tilde{u}_3)}{A^\alpha_{u_3}(\sigma,u,t_j^\alpha(u))}\Big)\\
&\quad+\frac{\sqrt{\gamma_0}}{\tau}\int_0^{t_j^\alpha(u)} e^{-\frac{\tau}{\sqrt{\gamma_0}}A^\alpha(\sigma,u,u_3)}\ {}^tQ(\sigma,u,u_3,\partial_{u_3})(\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},u_3,\tilde{u}_3))du_3,
\end{aligned}$$

where ${}^tQ(\sigma,u,u_3,\partial_{u_3})=\frac{1}{A^\alpha_{u_3}(\sigma,u,u_3)}\partial_{u_3}-\frac{A^\alpha_{u_3u_3}(\sigma,u,u_3)}{(A^\alpha_{u_3}(\sigma,u,u_3))^2}$ is the formal transpose operator of $Q(\sigma,u,u_3,\partial_{u_3})$. Because $s^\alpha(\sigma)\in B_{r_0}(x_j^\alpha)$ ($\sigma\in\tilde{U}_{x_j^\alpha}$), and $b^\alpha(u)\in B_{r_0}(y_j^\alpha)$ and $y^\alpha(u,u_3)\in B_{3r_0}(y_j^\alpha)$ ($u\in\tilde{U}_{y_j^\alpha}$), it follows that $A^\alpha(\sigma,u,t_j^\alpha(u))\geq\frac{l_0-2r_0}{l_0+3r_0}t_j^\alpha(u)\geq\frac{l_0-2r_0}{l_0+3r_0}r_0>0$ ($u\in\tilde{U}^\alpha_{y_j}$). From this estimate, we obtain

$$\left|e^{-\frac{\tau}{\sqrt{\gamma_0}}A^\alpha(\sigma,u,t_j^\alpha(u))}\frac{\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},t_j^\alpha(u),\tilde{u}_3)}{A^\alpha_{u_3}(\sigma,u,t_j^\alpha(u))}\right|\leq Ce^{-c_1\tau}\quad(\tau>1)$$

for some constants $C>0$ and $c_1>0$. Since $\theta^\alpha_{-k,j}$ and $A^\alpha(\sigma,u,u_3)$ are of C^∞ class for (u,u_3), this operation can be repeated as many times as necessary. Then, for any $m\in\mathbb{N}\cup\{0\}$ we obtain

$$\begin{aligned}
&\int_0^{t_j^\alpha(u)} e^{-\frac{\tau}{\sqrt{\gamma_0}}A^\alpha(\sigma,u,u_3)}\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},u_3,\tilde{u}_3)du_3\\
&=\sum_{p=0}^m\Big(\frac{\sqrt{\gamma_0}}{\tau}\Big)^{1+p}\frac{({}^tQ(\sigma,u,u_3,\partial_{u_3}))^p(\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},u_3,\tilde{u}_3))|_{u_3=0}}{A^\alpha_{u_3}(\sigma,u,0)}\\
&+\Big(\frac{\sqrt{\gamma_0}}{\tau}\Big)^{1+m}\int_0^{t_j^\alpha(u)} e^{-\frac{\tau}{\sqrt{\gamma_0}}A^\alpha(\sigma,u,u_3)}\ ({}^tQ(\sigma,u,u_3,\partial_{u_3}))^{1+m}(\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},u_3,\tilde{u}_3))du_3\\
&+O(e^{-c_1\tau})\\
&=\sum_{p=0}^m\Big(\frac{\sqrt{\gamma_0}}{\tau}\Big)^{1+p}\frac{({}^tQ(\sigma,u,u_3,\partial_{u_3}))^p(\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},u_3,\tilde{u}_3))|_{u_3=0}}{A^\alpha_{u_3}(\sigma,u,0)}+O(\tau^{-1-m})\\
&\hspace{25em}(\tau\to\infty).
\end{aligned}$$

Using $\tilde{Q}(\sigma,\tilde{u},\tilde{u}_3,\partial_{\tilde{u}_3})=\frac{1}{A^\alpha_{\tilde{u}_3}(\sigma,\tilde{u},\tilde{u}_3)}\partial_{\tilde{u}_3}$, the same operation yields the following.

$$\int_0^{t_j^\alpha(\tilde{u})}\int_0^{t_j^\alpha(u)} e^{-\frac{\tau}{\sqrt{\gamma_0}}A^\alpha(\sigma,u,u_3)}e^{-\frac{\tau}{\sqrt{\gamma_0}}A^\alpha(\sigma,\tilde{u},\tilde{u}_3)}\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},u_3,\tilde{u}_3)du_3d\tilde{u}_3$$
$$=\sum_{p=0}^{m+1}\Big(\frac{\sqrt{\gamma_0}}{\tau}\Big)^{1+p}\int_0^{t_j^\alpha(\tilde{u})} e^{-\frac{\tau}{\sqrt{\gamma_0}}A^\alpha(\sigma,\tilde{u},\tilde{u}_3)}\frac{({}^tQ(\sigma,u,u_3,\partial_{u_3}))^p(\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},u_3,\tilde{u}_3))|_{u_3=0}}{A^\alpha_{u_3}(\sigma,u,0)}d\tilde{u}_3$$
$$+O(\tau^{-2-m})$$
$$=\sum_{i=0}^{m}\Big(\frac{\sqrt{\gamma_0}}{\tau}\Big)^{2+i}\Theta^{\alpha,j,0}_{-k,i}(\sigma,u,\tilde{u})+O(\tau^{-2-m})\quad(\tau\to\infty),$$

where

$$\Theta^{\alpha,j,0}_{-k,i}(\sigma,u,\tilde{u})$$
$$=\sum_{p=0}^{i}\frac{(\tilde{Q}(\sigma,\tilde{u},\tilde{u}_3,\partial_{\tilde{u}_3}))^{i-p}({}^tQ(\sigma,u,u_3,\partial_{u_3}))^p(\theta^\alpha_{-k,j}(\sigma,u,\tilde{u},u_3,\tilde{u}_3))|_{u_3=0,\tilde{u}_3=0}}{A^\alpha_{u_3}(\sigma,u,0)A^\alpha_{u_3}(\sigma,\tilde{u},0)}.$$

We set $L^\alpha_j(\sigma,u,\tilde{u}) = L_0(s^\alpha(\sigma),b^\alpha(u)) + L_0(s^\alpha(\sigma),b^\alpha(\tilde{u}))$ for $(\sigma,u,\tilde{u}) \in \tilde{U}_{x^\alpha_j}\times\tilde{U}_{y^\alpha_j}\times\tilde{U}_{y^\alpha_j}$. This expansion and (5.8) yield

$$K^{0,\alpha}_{\tau,-k,j}=\sum_{i=0}^{m}\Big(\frac{\sqrt{\gamma_0}}{\tau}\Big)^{2+i}\int_{\tilde{U}_{x^\alpha_j}\times\tilde{U}_{y^\alpha_j}\times\tilde{U}_{y^\alpha_j}} e^{-\frac{\tau}{\sqrt{\gamma_0}}L^\alpha_j(\sigma,u,\tilde{u})}\Theta^{\alpha,j}_{-k,i}(\sigma,u,\tilde{u})\,d\sigma dud\tilde{u}$$
$$+\int_{\tilde{U}_{x^\alpha_j}\times\tilde{U}_{y^\alpha_j}\times\tilde{U}_{y^\alpha_j}} e^{-\frac{\tau}{\sqrt{\gamma_0}}L^\alpha_j(\sigma,u,\tilde{u})}r^{\alpha,j}_{\tau,m}(\sigma,u,\tilde{u})\,d\sigma dud\tilde{u},\tag{5.9}$$

where $\Theta^{\alpha,j}_{-k,i}(\sigma,u,\tilde{u}) = \Theta^{\alpha,j,0}_{-k,i}(\sigma,u,\tilde{u})J_{\partial D}(\sigma)J_{\partial B}(u)J_{\partial B}(\tilde{u})$ and $r^{\alpha,j}_{\tau,m}(\sigma,u,\tilde{u})$ is a function satisfying

$$\sup_{(\sigma,u,\tilde{u})\in\tilde{U}_{x^\alpha_j}\times\tilde{U}_{y^\alpha_j}\times\tilde{U}_{y^\alpha_j}}|r^{\alpha,j}_{\tau,m}(\sigma,u,\tilde{u})|\le C_m\tau^{-2-m}\quad(\tau\ge\tau_1)$$

for some constants $C_m>0$ and $\tau_1\ge 1$. Since the second integral of (5.9) can be estimated by

$$\Big|\int_{\tilde{U}_{x^\alpha_j}\times\tilde{U}_{y^\alpha_j}\times\tilde{U}_{y^\alpha_j}} e^{-\frac{\tau}{\sqrt{\gamma_0}}L^\alpha_j(\sigma,u,\tilde{u})}r^{\alpha,j}_{\tau,m}(\sigma,u,\tilde{u})\,d\sigma dud\tilde{u}\Big|$$
$$\le C_m\tau^{-2-m}\int_{\tilde{U}_{x^\alpha_j}\times\tilde{U}_{y^\alpha_j}\times\tilde{U}_{y^\alpha_j}} e^{-\frac{\tau}{\sqrt{\gamma_0}}L^\alpha_j(\sigma,u,\tilde{u})}\,d\sigma dud\tilde{u},$$

from Lemma 4.1 we obtain

$$\left|\int_{\tilde{U}_{x_j^\alpha}\times\tilde{U}_{y_j^\alpha}\times\tilde{U}_{y_j^\alpha}} e^{-\frac{\tau}{\sqrt{\gamma_0}}L_j^\alpha(\sigma,u,\tilde{u})} r_{\tau,m}^{\alpha,j}(\sigma,u,\tilde{u})\,d\sigma du d\tilde{u}\right| \le C_m\tau^{-5-m}e^{-2\frac{\tau}{\sqrt{\gamma_0}}l_0}. \tag{5.10}$$

From (5.9) and (5.10), it follows that

(5.11)

$$K_{\tau,-k,j}^{0,\alpha} = \sum_{i=0}^{m}\left(\frac{\sqrt{\gamma_0}}{\tau}\right)^{2+i}\int_{\tilde{U}_{x_j^\alpha}\times\tilde{U}_{y_j^\alpha}\times\tilde{U}_{y_j^\alpha}} e^{-\frac{\tau}{\sqrt{\gamma_0}}L_j^\alpha(\sigma,u,\tilde{u})}\Theta_{-k,i}^{\alpha,j}(\sigma,u,\tilde{u})\,d\sigma du d\tilde{u}$$
$$+ O(\tau^{-5-m}e^{-2\frac{\tau}{\sqrt{\gamma_0}}l_0}) \quad (m=0,1,\ldots).$$

Remark 5.5. *Since $\Theta_{-k,i}^{\alpha,j}(\sigma,u,\tilde{u})$ is bounded in $\tilde{U}_{x_j^\alpha}\times\tilde{U}_{y_j^\alpha}\times\tilde{U}_{y_j^\alpha}$, similar to obtaining (5.10), we also have*

$$\left|\int_{\tilde{U}_{x_j^\alpha}\times\tilde{U}_{y_j^\alpha}\times\tilde{U}_{y_j^\alpha}} e^{-\frac{\tau}{\sqrt{\gamma_0}}L_j^\alpha(\sigma,u,\tilde{u})}\Theta_{-k,i}^{\alpha,j}(\sigma,u,\tilde{u})\,d\sigma du d\tilde{u}\right| \le C\tau^{-3}e^{-\frac{2\tau}{\sqrt{\gamma_0}}l_0}.$$

From this estimate and (5.11), we obtain

$$K_{\tau,-k,j}^{0,\alpha} = O(\tau^{-5}e^{-\frac{2\tau}{\sqrt{\gamma_0}}l_0}) \quad (\tau\to\infty). \tag{5.12}$$

With the above preparations, let us find the asymptotic expansion of I_τ. First, from (5.4), (5.5) and (5.12), we obtain the following:

(5.13)

$$I_\tau = \sum_{\alpha\in\{n_+,n_-,d\}}\vartheta_\alpha\sum_{k=-1}^{N-1}\sum_{j=1}^{M^\alpha}K_{\tau,-k,j}^{0,\alpha}\tau^{-k} + O(\tau^{-N-5}e^{-\frac{2\tau}{\sqrt{\gamma_0}}l_0}) + O(\tau^{-1}e^{-\tau T}) \quad (\tau\to\infty).$$

From (5.11), it suffices to consider the following Laplace integrals:

$$\int_{\tilde{U}_{x_j^\alpha}\times\tilde{U}_{y_j^\alpha}\times\tilde{U}_{y_j^\alpha}} e^{-\frac{\tau}{\sqrt{\gamma_0}}L_j^\alpha(\sigma,u,\tilde{u})}\Theta_{-k,i}^{\alpha,j}(\sigma,u,\tilde{u})\,d\sigma du d\tilde{u}. \tag{5.14}$$

For these integrals, we set $h(\sigma,u,\tilde{u}) = L_j^\alpha(\sigma,u,\tilde{u})$ and $\varphi(\sigma,u,\tilde{u}) = \Theta_{-k,i}^{\alpha,j}(\sigma,u,\tilde{u})$, and apply Proposition 5.3. Since $(\partial D,\lambda)$ is of N-class, $h\in C^{2N+4}(\tilde{U}_{x_j^\alpha}\times\tilde{U}_{y_j^\alpha}\times\tilde{U}_{y_j^\alpha})$, and $\phi\in\tilde{C}^{2N+4}((\overline{\Omega}\cap\mathcal{U}_4)\times\tilde{B})$, $b_j\in\tilde{C}^{2(N-j)+2}((\overline{\Omega}\cap\mathcal{U}_4)\times\tilde{B})$. From this fact, (4.10), (4.11), (4.12) and (4.13), $\kappa_{-k}^\alpha\in\tilde{C}^{2(N-k)}((\overline{\Omega}\cap\mathcal{U}_4)\times\tilde{B})$, then $\varphi\in C^{2(N-k)}(\tilde{U}_{x_j^\alpha}\times$

$\tilde{U}_{y_j^\alpha} \times \tilde{U}_{y_j^\alpha})$. Therefore, Proposition 5.3 implies that (5.14) can be expanded as

$$\frac{e^{-\frac{2\tau}{\sqrt{\gamma_0}}l_0}}{\sqrt{(\mathrm{Hess}\, L_j^\alpha)(0,0,0)}}\Big(\frac{2\pi\sqrt{\gamma_0}}{\tau}\Big)^3\Big(\sum_{p=0}^{N-k-1}\mathcal{L}_p^{\alpha,j}\Theta_{-k,i}^{\alpha,j}(0,0,0)\Big(\frac{\sqrt{\gamma_0}}{\tau}\Big)^p + O(\tau^{-(N-k)})\Big).$$

Here, $\mathcal{L}_p^{\alpha,j}$ is the differential operator determined using L_j^α as in Proposition 5.3. From (5.11) with $m = N-1-k$, (5.13) and the above asymptotics for (5.14), it follows that

$$\begin{aligned}
I_\tau = &\sum_{\alpha\in\{n_+,n_-,d\}}\vartheta_\alpha\sum_{j=1}^{M^\alpha}\sum_{k=-1}^{N-1}\Bigg\{\sum_{i=0}^{N-1-k}\Big(\frac{\sqrt{\gamma_0}}{\tau}\Big)^{2+i}\frac{e^{-\frac{2\tau}{\sqrt{\gamma_0}}l_0}}{\sqrt{(\mathrm{Hess}\, L_j^\alpha)(0,0,0)}}\Big(\frac{2\pi\sqrt{\gamma_0}}{\tau}\Big)^3\\
&\quad\Big(\sum_{p=0}^{N-k-1}\mathcal{L}_p^{\alpha,j}\Theta_{-k,i}^{\alpha,j}(0,0,0)\Big(\frac{\sqrt{\gamma_0}}{\tau}\Big)^p + O(\tau^{-(N-k)})\Big) + O(\tau^{-5-N+k}e^{-2\frac{\tau}{\sqrt{\gamma_0}}l_0})\Bigg\}\tau^{-k}\\
&\qquad + O(\tau^{-N-5}e^{-\frac{2\tau}{\sqrt{\gamma_0}}l_0}) + O(\tau^{-1}e^{-\tau T})\\
= &\, e^{-\frac{2\tau}{\sqrt{\gamma_0}}l_0}\frac{(2\pi)^3(\sqrt{\gamma_0})^5}{\tau^5}\sum_{\alpha\in\{n_+,n_-,d\}}\vartheta_\alpha\sum_{j=1}^{M^\alpha}\frac{1}{\sqrt{(\mathrm{Hess}\, L_j^\alpha)(0,0,0)}}\\
&\quad\sum_{k=0}^{N}\sum_{i=0}^{N-k}\tau^{1-k}\Big(\frac{\sqrt{\gamma_0}}{\tau}\Big)^i\sum_{p=0}^{N-k}\Big\{\Big(\frac{\sqrt{\gamma_0}}{\tau}\Big)^p\mathcal{L}_p^{\alpha,j}\Theta_{1-k,i}^{\alpha,j}(0,0,0) + O(\tau^{-(N+1-k)})\Big\}\\
&\qquad + O(\tau^{-N-5}e^{-\frac{2\tau}{\sqrt{\gamma_0}}l_0}) + O(\tau^{-1}e^{-\tau T})\\
= &\, e^{-\frac{2\tau}{\sqrt{\gamma_0}}l_0}\frac{(2\pi)^3(\sqrt{\gamma_0})^5}{\tau^4}\sum_{\alpha\in\{n_+,n_-,d\}}\vartheta_\alpha\sum_{j=1}^{M^\alpha}\frac{1}{\sqrt{(\mathrm{Hess}\, L_j^\alpha)(0,0,0)}}\\
&\quad\sum_{k=0}^{N}\sum_{i=0}^{N-k}\sum_{p=0}^{N-k}\tau^{-(k+i+p)}(\sqrt{\gamma_0})^{i+p}\mathcal{L}_p^{\alpha,j}\Theta_{1-k,i}^{\alpha,j}(0,0,0)\\
&\qquad + O(\tau^{-N-5}e^{-\frac{2\tau}{\sqrt{\gamma_0}}l_0}) + O(\tau^{-1}e^{-\tau T}).
\end{aligned}$$

Taking into account $0 \le k+i+p \le 2N-k$, $0 \le k \le N$, we have

$$\sum_{k=0}^{N}\sum_{i=0}^{N-k}\sum_{p=0}^{N-k}\tau^{-(k+i+p)}(\sqrt{\gamma_0})^{i+p}\mathcal{L}_p^{\alpha,j}\Theta_{1-k,i}^{\alpha,j}(0,0,0)$$
$$=\sum_{l=0}^{N}\tau^{-l}\sum_{k+i+p=l}(\sqrt{\gamma_0})^{i+p}\mathcal{L}_p^{\alpha,j}\Theta_{1-k,i}^{\alpha,j}(0,0,0)+O(\tau^{-N-1}),$$

we obtain (5.1), where

$$\mathcal{S}_l=\sum_{\alpha\in\{n_+,n_-,d\}}\vartheta_\alpha\sum_{j=1}^{M^\alpha}\frac{1}{\sqrt{(\mathrm{Hess}\,L_j^\alpha)(0,0,0)}}\sum_{k+i+p=l}(\sqrt{\gamma_0})^{i+p}\mathcal{L}_p^{\alpha,j}\Theta_{1-k,i}^{\alpha,j}(0,0,0).$$

If B is a ball, then (4.14) implies (5.2) if we set

$$\mathcal{T}_l=8\pi^2(\sqrt{\gamma_0})^3\sum_{\alpha\in\{n_+,n_-,d\}}\vartheta_\alpha\sum_{j=1}^{M^\alpha}\frac{(l_0)^2a^2}{2(a+l_0)^2\sqrt{\mathcal{A}_\alpha(x_j^\alpha)}}\sum_{k+i+p=l}(\sqrt{\gamma_0})^{i+p}\mathcal{L}_p^{\alpha,j}\Theta_{1-k,i}^{\alpha,j}(0,0,0).$$

Since $\mathcal{L}_0^{\alpha,j}g=g$ for any function g, and $A_{u_3}(0,0,0)=1$, it follows that

$$\mathcal{L}_0^{\alpha,j}\Theta_{1,0}^{\alpha,j}(0,0,0)=(f(y_0^\alpha))^2\kappa_1^\alpha(x_j^\alpha,y_j^\alpha,y_j^\alpha).$$

As in p.14 of [16], we obtain

$$\kappa_1^\alpha(x_j^\alpha,y_j^\alpha,y_j^\alpha)=\begin{cases}\frac{1}{8\pi^2(\sqrt{\gamma_0})^3(l_0)^2}\frac{\sqrt{\gamma_0}-\lambda_1(x_j^\alpha)}{\sqrt{\gamma_0}+\lambda_1(x_j^\alpha)} & (\alpha\in\{n_+,n_-\}),\\ \frac{1}{8\pi^2(\sqrt{\gamma_0})^3(l_0)^2} & (\alpha=d).\end{cases}$$

Thus, the above $\mathcal{T}_l$ as $l=0$ coincides with $\mathcal{T}_0$ given in Theorem 4.3. This completes the proof of Theorem 5.2. □

6. Asymptotics of Laplace integrals

In this section, we show Proposition 5.3. By Taylor's theorem for h at $x=x_0$, it follows that

$$h(x)=h(x_0)+\big(H(x)(x-x_0),x-x_0\big)_{\mathbb{R}^n},$$

where

$$H(x) = (H_{ij}(x)), \qquad H_{ij}(x) = \int_0^1 (1-\theta) h_{x_i x_j}(x_0 + \theta(x - x_0)) d\theta. \tag{6.1}$$

Since $h \in C^{2N+2}(\overline{U}) \subset C^3(\overline{U})$, there exists a constant $C > 0$ such that

$$|H_{ij}(x) - H_{ij}(x_0)| \le C|x - x_0| \qquad (x \in \overline{U}, i, j = 1, 2, \ldots, n).$$

Let $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n > 0$ be the eigenvalues of $H(x_0)(= (\mathrm{Hess}(f)(x_0)) > 0)$. We take an orthogonal matrix P satisfying ${}^tPH(x_0)P = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_n)$ (diagonal matrix), and set $y = {}^tP(x - x_0)$ and $B(y) = {}^tP(H(x) - H(x_0))P$, $\tilde{\varphi}(y) = \varphi(x)$. We also set $\Phi_0(y) = \sum_{j=1}^{n} \mu_j y_j^2$ and $\Phi_1(y) = (B(y)y, y)_{\mathbb{R}^n}$.

Take any $\delta_1 > 0$ sufficiently small. Then, there exists a constant $c_1 > 0$ such that $h(x) \ge h(x_0) + c_1$ $(x \in \overline{U}, |x - x_0| \ge \delta_1)$, which yields

$$\left| \int_{\{x \in U \mid |x - x_0| \ge \delta_1\}} e^{-\tau h(x)} \varphi(x) dx \right| \le C e^{-\tau h(x_0)} e^{-c_1 \tau} \max_{x \in U} |\varphi(x)|.$$

Hence, if we take a cut-off function $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(y) = 1$ $(|y| \le \delta_1/3)$ and $\psi(y) = 0$ $(|y| \ge 2\delta_1/3)$, we obtain

$$\int_U e^{-\tau h(x)} \varphi(x) dx = e^{-\tau h(x_0)} \left\{ I(\tau) + O(e^{-c_1 \tau}) \|\varphi\|_{C(\overline{U})} \right\} \ (\tau \to \infty),$$

where

$$I(\tau) = \int_{\mathbb{R}^n} e^{-\tau(\Phi_0(y) + \Phi_1(y))} \tilde{\varphi}(y) \psi(y) dy.$$

Because $h_{x_i x_j} \in C^{2N}(\overline{U})$, using Taylor's theorem for (6.1), we can decompose $H(x)$ into $H(x) = H^1(x) + \tilde{H}^1(x)$, where $H^1(x) = (H^1_{ij}(x))$, $\tilde{H}^1(x) = (\tilde{H}^1_{ij}(x))$,

$$H^1_{ij}(x) = \sum_{k=0}^{2N-1} \frac{1}{(k+1)(k+2)} \sum_{|\alpha| = k} \frac{1}{\alpha!} \partial_x^\alpha h_{x_i x_j}(x_0)(x - x_0)^\alpha,$$

$$\tilde{H}^1_{ij}(x) = 2N \sum_{|\alpha| = 2N} \frac{(x - x_0)^\alpha}{\alpha!} \int_0^1 \theta^{|\alpha|}(1-\theta) \int_0^1 (1-s)^{2N-1} \partial_x^\alpha h_{x_i x_j}(x_0 + s\theta(x - x_0)) ds d\theta.$$

We set $B^1(y) = {}^tP(H^1(x) - H(x_0))P$, $\tilde{B}^1(y) = {}^tP\tilde{H}^1(x)P$, $\Phi_1^1(y) = (B^1(y)y, y)$ and $\tilde{\Phi}_1^1(y) = (\tilde{B}^1(y)y, y)$. Then, we have $\Phi_1(y) = \Phi_1^1(y) + \tilde{\Phi}_1^1(y)$. Because $N \geq 1$, it follows that

$$|H^1(x) - H(x_0)| \leq C|x - x_0|, \quad \tilde{H}^1(x) = O(|x - x_0|^{2N}),$$

which yields $B^1(y) = O(|y|)$, $\tilde{B}^1(y) = O(|y|^{2N})$ $(y \to 0)$. Hence, taking $\delta_1 > 0$ small enough, we obtain

$$|\Phi_1^1(y)| + |\tilde{\Phi}_1^1(y)| \leq C|y|^3 < \frac{1}{2}\Phi_0(y) \qquad (|y| \leq \delta_1) \tag{6.2}$$

and

$$|\tilde{\Phi}_1^1(y)| \leq C_N|y|^{2N+2} \leq C_N(\Phi_0(y))^{N+1} \quad (|y| \leq \delta_1). \tag{6.3}$$

To decompose $\tilde{\varphi}$ into $\tilde{\varphi}(y) = \tilde{\varphi}_1(y) + \tilde{\varphi}_2(y)$, we define $\varphi_1(x)$ and $\varphi_2(x)$ by

$$\varphi_1(x) = \sum_{|\alpha| \leq 2N} \frac{1}{\alpha!}\partial_x^\alpha \varphi(x_0)(x - x_0)^\alpha, \qquad \varphi_2(x) = \varphi(x) - \varphi_1(x)$$

and set $\tilde{\varphi}_1(y) = \varphi_1(x)$, $\tilde{\varphi}_2(y) = \varphi_2(x)$, where $x = x_0 + Py$. Since $\varphi \in C^{2N}(\overline{U})$, it follows that $\varphi_1 \in C^{2N}(\mathbb{R}^n)$, $\varphi_2 \in C^0(\overline{U})$ and $|\varphi_2(x)| \leq C|x - x_0|^{2N}$ $(|x - x_0| \leq \delta_1)$, which yields

$$|\tilde{\varphi}_2(y)| \leq C_N(\Phi_0(y))^N \qquad (|y| \leq \delta_1). \tag{6.4}$$

We set

$$I_1(\tau) = \int_{\mathbb{R}^n} e^{-\tau(\Phi_0(y) + \Phi_1^1(y))}\tilde{\varphi}_1(y)\psi(y)dy. \tag{6.5}$$

Then we have

$$\begin{aligned} I(\tau) - I_1(\tau) = \int_{\mathbb{R}^n} & e^{-\tau(\Phi_0(y) + \Phi_1^1(y))}\big(e^{-\tau\tilde{\Phi}_1^1(y)} - 1\big)\tilde{\varphi}(y)\psi(y)\,dy \\ & + \int_{\mathbb{R}^n} e^{-\tau(\Phi_0(y) + \Phi_1^1(y))}\tilde{\varphi}_2(y)\psi(y)dy. \end{aligned} \tag{6.6}$$

From $|e^X - 1| = |X\int_0^1 e^{\theta X}d\theta| \leq |X|e^{|X|}$ $(X \in \mathbb{R})$, (6.2), (6.3) and (6.4), it follows that

$$|e^{-\tau(\Phi_0(y)+\Phi_1^1(y))}\big(e^{-\tau\tilde{\Phi}_1^1(y)}-1\big)| \le e^{-\tau\Phi_0(y)}e^{\tau|\Phi_1^1(y)|}|\tau||\tilde{\Phi}_1^1(y)|e^{\tau|\tilde{\Phi}_1^1(y)|}$$
$$\le C_N\tau e^{-\tau\Phi_0(y)/2}(\Phi_0(y))^{N+1} \le C_N\tau^{-N}e^{-\tau\Phi_0(y)/4} \quad (|y|\le\delta_1)$$

and

$$|e^{-\tau(\Phi_0(y)+\Phi_1^1(y))}\tilde{\varphi}_2(y)| \le C_N(\Phi_0(y))^N e^{-\tau\Phi_0(y)}e^{\tau|\Phi_1^1(y))|} \le C_N(\Phi_0(y))^N e^{-\tau\Phi_0(y)/2}$$
$$\le C_N\tau^{-N}e^{-\tau\Phi_0(y)/4} \qquad (|y|\le\delta_1).$$

From these estimates, (6.6) and Lemma 4.1, we obtain

$$|I(\tau)-I_1(\tau)| \le C_N\tau^{-N}\int_{\mathbb{R}^n} e^{-\tau\Phi_0(y)/4}\psi(y)dy \le C_N\tau^{-N-n/2}.$$

Thus, the problem can be reduced to performing the asymptotic expansion of the integral given in (6.5).

In (6.5), the functions Φ_0, Φ_1^1 and $\tilde{\varphi}_1$ are smooth functions in $\mathbb{R}^n$, and satisfy $\Phi_0(y)+\Phi_1^1(y) > 0(=\Phi_0(0)+\Phi_1^1(0))$ in $\mathrm{supp}\,(\tilde{\varphi}_1\psi)$. Hence, we can use Theorem 7.7.5 of Hörmander [2]. This completes the proof of Proposition 5.3. □

Finally, we calculate $\mathcal{L}_1\varphi(x_0)$. We need $N\ge 2$ to handle $\mathcal{L}_1\varphi(x_0)$, that is, $h\in C^6(\overline{U})$ and $\varphi\in C^4(\overline{U})$ at least. For $j=1$, the possible cases of (k,l) are $(0,1)$, $(1,2)$ and $(2,3)$. We set

$$\Psi_0(x) = \frac{1}{6}\sum_{i,j,k=1}^{n} h_{x_ix_jx_k}(x_0)(x_i-(x_0)_i)(x_j-(x_0)_j)(x_k-(x_0)_k),$$
$$\Psi_1(x) = \frac{1}{24}\sum_{i,j,k,l=1}^{n} h_{x_ix_jx_kx_l}(x_0)(x_i-(x_0)_i)(x_j-(x_0)_j)(x_k-(x_0)_k)(x_l-(x_0)_l).$$

Since $\Psi(x)=\Psi_0(x)+\Psi_1(x)+O(|x-x_0|^5)$, $\Psi_0(x)=O(|x-x_0|^3)$, $\Psi_1(x)=O(|x-x_0|^4)$ $(x\to x_0)$, it follows that $\big[((H(x_0))^{-1}\partial_x,\partial_x)^l(\Psi(x))^k\varphi\big](x_0)$ are of the form:

$$((H(x_0))^{-1}\partial_x,\partial_x)\varphi(x_0) \qquad \text{for } (k,l)=(0,1),$$
$$((H(x_0))^{-1}\partial_x,\partial_x)^2(\Psi_0(x)\varphi)(x_0)+((H(x_0))^{-1}\partial_x,\partial_x)^2(\Psi_1(x))(x_0)\varphi(x_0) \qquad \text{for } (k,l)=(1,2),$$
$$((H(x_0))^{-1}\partial_x,\partial_x)^3(\Psi_0(x))^2(x_0)\varphi(x_0) \qquad \text{for } (k,l)=(2,3).$$

Note that for $k \geq 1$, only the term $((H(x_0))^{-1}\partial_x, \partial_x)^2(\Psi_0(x)\varphi)(x_0)$ includes the derivative of φ. Hence, we obtain

$$\mathcal{L}_1\varphi(x_0) = \frac{1}{4}((H(x_0))^{-1}\partial_x, \partial_x)\varphi(x_0) - \frac{1}{32}((H(x_0))^{-1}\partial_x, \partial_x)^2(\Psi_0(x)\varphi)(x_0)$$
$$+ \frac{1}{32}\Big(-((H(x_0))^{-1}\partial_x, \partial_x)^2(\Psi_1(x))(x_0) + \frac{1}{24}((H(x_0))^{-1}\partial_x, \partial_x)^3(\Psi_0(x))^2)(x_0)\Big)\varphi(x_0),$$

where it contains only the first-order derivative of φ at $x = x_0$ since $\Psi_0(x) = O(|x - x_0|^3)$. Therefore, this representation does not depend on N.

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