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Note on monotonicity tests for the magnetic Schrödinger operator

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Abstract. We give a method to detect unknown obstacles in the magnetic Schrödinger equation using monotonicity tests. In [6], we gave monotonicity tests to detect unknown obstacles under a monotonicity assumption on the scalar potential q. In this paper, we show that unknown obstacles can also be detected under a monotonicity assumption on the vector potential A.

1. Introduction and main results

We refer to [2] for monotonicity tests using localized potentials. The technique of localized potentials was first proposed in [1]. The monotonicity based method is due to the fundamental work of B. Harrach, V. Pohjola and M. Salo [4], which extended the previous result in [3]. In [5], they improved the bounds for the dimension of a space in [4]. In [6], we assumed a monotonicity relation for a potential q in a neighborhood of $\Sigma \subset \partial\Omega$, we proved local uniqueness by the monotonicity based method for the magnetic Schrödinger equation. In addition, we proved monotonicity tests to detect an unknown obstacle for the magnetic Schrödinger equation. In this paper, we prove monotonicity tests under a monotonicity assumption on the vector potential A from partial boundary data. Before we define the Neumann to Dirichlet map (we say ND map for short) for the magnetic Schrödinger equation, we assume the following admissibility condition.

Let $\Omega \subset \mathbb{R}^n (n = 2, 3)$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Here, for $A = (A_1, A_2, \dots, A_n)$, let

$$D_A^2 u := \sum_{j=1}^{n} D_{A,j}(D_{A,j}u), \quad D_{A,j} := \frac{1}{i}\partial_j + A_j.$$

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Throughout this paper, we assume that $A \in C^1(\overline{\Omega}; \mathbb{R}^n), q \in L^\infty(\Omega; \mathbb{R})$.

Definition 1.1 (Admissibility). We say that $A(x)$ and $q(x)$ are admissible if 0 is not an eigenvalue of the problem:

$$\begin{cases} -D_A^2 u + qu = 0 \quad \text{in } \Omega, \\ \nu \cdot (\nabla + iA)u = 0 \text{ on } \partial\Omega, \end{cases}$$

where ν is the outward unit normal vector to Ω.

To define the ND map, we give the definition of a weak solution.

Definition 1.2 (Weak solution). Suppose $A(x)$ and $q(x)$ are admissible. Let $\Sigma \subset \partial\Omega$ be an open subset of $\partial\Omega$. For a given $g \in L^2(\Sigma)$, there exists a unique weak solution $u \in H^1(\Omega)$ to the following boundary value problem:

$$-D_A^2 u + qu = 0 \quad \text{in } \Omega, \quad \nu \cdot (\nabla + iA)u = \begin{cases} g \text{ on } \Sigma, \\ 0 \text{ else.} \end{cases} \tag{1}$$

Namely, u satisfies

$$\int_\Omega (D_A u) \cdot \overline{D_A \varphi} - qu\overline{\varphi}\, dx = \int_\Sigma g\overline{\varphi}\, dS \tag{2}$$

for any $\varphi \in H_A^1(\Omega)$. Here, $\overline{\varphi}$ is the complex conjugate of φ.

We may use the solution by $u_{A,q}^{(g)}$ instead of u to clarify the dependence of g and A, q. Here, we define the inner product of $H_A^1(\Omega)$ and $L^2(\Omega)$ as follows:

$$\langle u, v\rangle_{H_A^1(\Omega)} := \int_\Omega (D_A u) \cdot \overline{D_A v} + u\overline{v}\, dx, \quad \langle u, v\rangle_{L^2(\Omega)} := \int_\Omega u\overline{v}\, dx.$$

We use the bounded linear operators as in [6].

$$K_A := j_A^* j_A, \quad \textit{and} \quad K_{A,q} := j_A^* M_q j_A,$$

which are compact self-adjoint linear operators from $H_A^1(\Omega)$ to $H_A^1(\Omega)$. Here, I_A denote the identity operator, j_A is the compact embedding from $H_A^1(\Omega)$ to $L^2(\Omega)$, and M_q is the multiplication operator by q. $\gamma_{A,\Sigma}$ denote the compact trace operator from $H_A^1(\Omega)$ to $L^2(\Sigma)$. We can write (2) as

$$\langle (I_A - K_A - K_{A,q})u, \varphi\rangle_{H_A^1} = \int_\Sigma g\overline{(\gamma_{A,\Sigma}\varphi)}\, dS, \quad \varphi \in H_A^1(\Omega).$$

Since A and q are admissible, $I_A - K_A - K_{A,q}$ is continuously invertible. Then, we can define the Neumann to Dirichlet operator.

$$\Lambda_{A,q} : g \in L^2(\Sigma) \to u|_\Sigma \in L^2(\Sigma),$$

where $u \in H^1_A(\Omega)$ solves (2). So, the Neumann to Dirichlet operator can be written by

$$\Lambda_{A,q} = \gamma_{A,\Sigma}(I_A - K_A - K_{A,q})^{-1}\gamma^*_{A,\Sigma}.$$

We note that $\Lambda_{A,q}$ is a compact self-adjoint linear operator. Now, we give the formula of monotonicity tests to detect an unknown obstacle D for the magnetic Schrödinger equation, which also generalizes the results in [4]. The unknown obstacle is assumed $D \subset \Omega$ with $\overline{D} \subset \Omega$ and connected complement $\Omega \setminus \overline{D}$. We assume that u_1 is constant in an open ball $B \subset \Omega$ and zero outside. In the next section, we show that a given function u_1 can be approximated by functions in a subspace of solutions of the magnetic Schrödinger equation in Ω. We assume that $A = A_1 = A_2$ in $\Omega \setminus D$ (a.e.) and that q is fixed in Ω. Here, we define the operator T_B for an open ball $B \subset \Omega$:

$$T_B : L^2(\Sigma) \to L^2(\Sigma), \quad \int_\Sigma g\overline{T_B h}\, dS := \int_B u_1^{(g)}\overline{u_1^{(h)}}\, dx,$$

where $u_1^{(g)}, u_1^{(h)} \in H^1_A(\Omega)$ solve (2). It is easy to see that T_B is a compact self-adjoint operator on $L^2(\Sigma)$.

Theorem 1.3. *Assume $A \in C^1(\overline{\Omega})$ and*

$$|A_1| \geq |A_2| \quad \textit{for all } x \in D\,(a.e.).$$

1. *If a ball $B \subset D$, then*

$$\alpha T_B \leq_{d(A,q_{max})} \Lambda_{A_2,q} - \Lambda_{A_1,q} \quad \textit{for all } \alpha \leq |A_1| - |A_2|.$$

   Here, for the meaning of this inequality and the definition of $d(A,q)$, see Definition 3.1 and Proposition 3.2.

2. *If a ball $B \not\subseteq D$, then $\Lambda_{A_2,q} - \Lambda_{A_1,q} - \alpha T_B$ has infinitely many negative eigenvalues for all $\alpha > 0$.*

Throughout this paper we count the number of eigenvalues for a compact self-adjoint operator with multiplication according to the dimension of the associated eigenspaces.

REMARK 1.4. In [6], we proved monotonicity tests under the condition to $q(x) = 1$ in $\Omega \setminus D$ (a.e.) and A is fixed in Ω. In addition, we assumed

$$1 < q_{\min} \leq q(x) \leq q_{\max} \quad \text{for all } x \in D \, (a.e.),$$

or

$$q_{\min} \leq q(x) \leq q_{\max} < 1 \quad \text{for all } x \in D \, (a.e.).$$

2. Localized potentials and Runge approximation

In this section, we show that we can approximate a given function $\varphi \in L^2(O)$, defined on a measurable subset $O \subseteq \Omega$, by functions in a subspace of solutions of the magnetic Schrödinger equation in Ω. Local uniqueness result is controlled by using localized potentials. We can construct localized potentials for a magnetic Schrödinger equation in a similar way to [4].

PROPOSITION 2.1. *Let $B, D \subset \overline{\Omega}$ be measurable sets. Assume $A \in C^1(\overline{\Omega})$ and $q \in L^\infty(\Omega) \setminus \{0\}$ are admissible, D and $B \setminus \overline{D}$ possess positive measures and $\overline{\Omega} \setminus \overline{D}$ is connected and $(\overline{\Omega} \setminus \overline{D}) \cap \Sigma \neq \varnothing$. Then, there exists a sequence $\{g_j\}_{j \in \mathbb{N}} \subset V^\perp$ such that*

$$\int_B |u_{A,q}^{(g_j)}|^2 \, dx \to \infty \quad \text{and} \quad \int_D |u_{A,q}^{(g_j)}|^2 \, dx \to 0$$

for any subspace $V \subset L^2(\Sigma)$ with $\dim V < \infty$, where $u_{A,q}^{(g_j)} \in H_A^1(\Omega)$ solves the magnetic Schrödinger equation (2) with Neumann boundary data g_j.

Proposition 2.1 can be proved using arguments as in [4]. For the proof of Proposition 2.1, see [6]. Now, we give the Theorem that we can take a given function $\varphi \in L^2(O)$ on a measurable subset $O \subseteq \Omega$.

THEOREM 2.2. *Let $D \subseteq \Omega$ be a measurable set and $C \subset \Omega$ be a closed ball such that $C \cap \overline{D} = \varnothing$ and $\overline{\Omega} \setminus (C \cup \overline{D})$ is connected to Σ. Then, for any function $\varphi \in L^2(C \cup \overline{D})$, there exists a sequence of solutions $u \in H^1(\Omega)$ that approximates φ in the $L^2(C \cup \overline{D})$-norm. u is solutions of*

$$-D_A^2 u + qu = 0 \quad \text{in} \, \Omega, \quad \nu \cdot (\nabla + iA)u = \begin{cases} g \text{ on } \Sigma, \\ 0 \text{ else}, \end{cases}$$

for any subspace $V \subset L^2(\Sigma)$ *with* $\dim V < \infty$ *and a function* φ *satisfies*

$$\varphi|_{\overline{D}} \equiv 0 \quad and \quad \varphi|_{\overline{B}} \not\equiv 0$$

for all subset $B \subseteq C$ *with positive measure.*

Here, we show Theorem 2.2 implies Proposition 2.1.

COROLLARY 2.3. *Let* $B, D \subset \overline{\Omega}$ *be measurable sets. Assume* $A \in C^1(\overline{\Omega})$ *and* $q \in L^\infty(\Omega) \setminus \{0\}$ *are admissible,* D *and* $B \setminus \overline{D}$ *possess positive measures and* $\overline{\Omega} \setminus \overline{D}$ *is connected and* $(\overline{\Omega} \setminus \overline{D}) \cap \Sigma \neq \varnothing$. *Then, there exists a sequence* $\{g_j\}_{j\in\mathbb{N}} \subset V^\perp$ *such that*

$$\int_B |u_{A,q}^{(g_j)}|^2 \, dx \to \infty \quad and \quad \int_D |u_{A,q}^{(g_j)}|^2 \, dx \to 0$$

for any subspace $V \subset L^2(\Sigma)$ *with* $\dim V < \infty$, *where* $u_{A,q}^{(g_j)} \in H_A^1(\Omega)$ *solves the magnetic Schrödinger equation* (2) *with Neumann boundary data* g_j.

PROOF. From [4, Lemma 4.3], we can take a closed ball $C \subset \Omega$ such that $B \cap C$ has positive measure, $C \cap \overline{D} = \varnothing$, and $\overline{\Omega} \setminus (\overline{D} \cup C)$ is connected to Σ. Using Theorem 2.2, there exists a function $\varphi \in L^2(C \cup \overline{D})$ and $\{\tilde{u}^{(j)}\}_{j\in\mathbb{N}} \subset H^1(\Omega)$ of

$$-D_A^2 \tilde{u}^{(j)} + q\tilde{u}^{(j)} = 0 \quad \text{in}\, \Omega, \quad \nu \cdot (\nabla + iA)\tilde{u}^{(j)} = \begin{cases} g_j \text{ on}\, \Sigma, \\ 0 \ \text{ else}, \end{cases}$$

$$\|\tilde{u}^{(j)}|_{B\cap C}\|_{L^2(B\cap C)} \to \|\varphi\|_{L^2(B\cap C)} > 0 \quad \text{and} \quad \|\tilde{u}^{(j)}|_{\overline{D}}\|_{L^2(\overline{D})} \to 0.$$

That is, we can take g_j as follow

$$g_j := \frac{\nu \cdot (\nabla + iA)\tilde{u}^{(j)}}{\sqrt{\|\tilde{u}^{(j)}|_{\overline{D}}\|_{L^2(\overline{D})}}} \in V^\perp.$$

□

Although we can show Theorem 2.2 by using the arguments as in [4], we give the proof for the sake of completeness. We write $v_{\chi_0} \in L^2(\Omega)$ for the zero extension of a function $v \in L^2(O)$.

LEMMA 2.4. *Let* $O \subset \Omega$ *be measurable set and let* $H \subseteq H_A^1(\Omega)$ *be a subspace of solutions to* $-D_A^2 u + qu = 0$ *in* Ω. *Then, a function* $\varphi \in L^2(O)$ *can be approximated* *by solutions $u \in H$, meaning that*

$$\inf_{u\in H} \|\varphi - u\|_{L^2(O)} = 0,$$

where φ satisfies $\int_O \varphi\overline{v}\,dx = 0$ for any $v \in L^2(O)$ such that the solution $w \in H^1(\Omega)$ of

$$-D_A^2 w + qw = v_{\chi_0}\,\mathrm{in}\,\Omega, \quad \nu\cdot(\nabla + iA)w = 0\,\mathrm{on}\,\partial\Omega \tag{3}$$

also satisfies $\int_{\partial\Omega} \nu\cdot(\nabla + iA)u|_{\partial\Omega}\overline{w|_{\partial\Omega}}\,ds = 0$ for any $u \in H$.

Proof. We assume that

$$\mathcal{R} := \{u|_O : u \in H\} \subseteq L^2(O).$$

Let $v \in L^2(O)$ and let $w \in L^2(\Omega)$ be the solution of (3). Then, for $v \in \mathcal{R}^\perp$, we have

$$\begin{aligned} 0 = \int_O u\overline{v}\,dx &= \int_\Omega u\overline{(-D_A^2 w + qw)}\,dx \\ &= \int_\Omega \overline{w}(-D_A^2 u + qu)\,dx + \int_{\partial\Omega} (\nu\cdot(\nabla + iA)u)\overline{w}\,ds \\ &= \int_{\partial\Omega} (\nu\cdot(\nabla + iA)u)\overline{w}\,ds = 0 \end{aligned}$$

for any $u \in H$. Since $\overline{\mathcal{R}} = (\mathcal{R}^\perp)^\perp$, the desired result follows. □

We characterize the functions w as follows.

Lemma 2.5. *Let V be a finite-dimensional subspace of $L^2(\Sigma)$, and let $O \subset \Omega$ be a closed set such that its complement $\overline{\Omega}\setminus O$ is connected to Σ. Define the spaces*

$$W := \left\{ w \in H^1(\Omega) : \begin{array}{c} \exists v \in L^2(O) \text{ s.t. } -D_A^2 w + qw = v_{\chi_0}, \\ \nu\cdot(\nabla + iA)w|_{\partial\Omega} = 0,\ w|_\Sigma \in V \end{array} \right\},$$

$$W_0 := \left\{ w \in H^1(\Omega) : \begin{array}{c} \exists v \in L^2(O) \text{ s.t. } -D_A^2 w + qw = v_{\chi_0}, \\ \nu\cdot(\nabla + iA)w|_{\partial\Omega} = 0,\ w|_\Sigma = 0 \end{array} \right\}.$$

Then, the codimension $d := \dim(W/W_0)$ of W_0 in W satisfies $d \le \dim(V)$.

Although we consider the magnetic Schrödinger equation, we can prove in the same way as Lemma 4.11 in [4].

Proof of Theorem 2.2. Let $D \subseteq \Omega$ be a measurable set and $C \subset \Omega$ be a closed ball for which $C \cap \overline{D} = \varnothing$ and $\overline{\Omega} \setminus (C \cup \overline{D})$ is connected to Σ. Let V be a finite-dimensional subspace of $L^2(\Sigma)$. From Lemma 2.4, we assume $O := C \cup \overline{D}$ and

$$H := \{u \in H^1(\Omega) : -D_A^2 u + qu = 0 \text{ in } \Omega, \nu\cdot(\nabla + iA)u|_{\partial\Omega\setminus\Sigma} = 0, \nu\cdot(\nabla+iA)u|_\Sigma = g\}.$$

A function $w \in H^1$ satisfies (3) and

$$\int_{\partial\Omega} \nu \cdot (\nabla + iA)u|_{\partial\Omega}\overline{w|_{\partial\Omega}}\, ds = 0,$$

for any $u \in H$ if and only if $w \in W$. Therefore, by Lemma 2.4 a function $\varphi \in L^2(O)$ can be approximated by solution $u \in H$ if and only if

$$\int_O \varphi\overline{(-D_A^2 w + qw)}\, dx = 0 \tag{4}$$

for any $w \in W$. Next, we show that there exists a function $\varphi \in L^2(O)$ satisfying (4) which vanishes on D, but does not vanish on any subset of C with positive measure. By unique continuation, Neumann data that are linearly independent give rise to solutions whose restrictions to the open ball C^0 remain linearly independent. Therefore, there exist infinitely many linearly independent solutions.

$$\varphi_j \in H^1(C^0) \quad \text{with} \quad -D_A^2\varphi_j + q\varphi_j = 0 \quad \text{in} \quad C^0, \quad j \in \mathbb{N}. \tag{5}$$

We extend φ_j by zero on $\overline{D} \cup \partial C$ to $\varphi_j \in L^2(O)$. Every $w_0 \in W_0$ has vanishing Cauchy data on $\partial\Omega$. That is, $w_0|_{\partial\Omega} = 0$ and $\nu \cdot (\nabla + iA)w|_{\partial\Omega} = 0$ by unique continuation. Consequently, for all $w_0 \in W_0$ and $j \in \mathbb{N}$, we obtain

$$\begin{aligned}\int_O \varphi_j\overline{(-D_A^2 w_0 + qw_0)}\, dx &= \int_C \varphi_j\overline{(-D_A^2 w_0 + qw_0)}\, dx \\ = \int_{\partial C}(\varphi_j|_{\partial C}&\overline{\nu\cdot(\nabla + iA)w_0|_{\partial C}} + \nu\cdot(\nabla + iA)\varphi_j|_{\partial C}\overline{w_0|_{\partial C}})\, ds \\ &\qquad + \int_C \overline{w}(-D_A^2\varphi_j + q\varphi_j)\, dx = 0.\end{aligned}$$

By a dimensional argument, one can find a nontrivial finite linear combination φ of the infinitely many linearly independent functions φ_j such that

$$\int_O \varphi\overline{(-D_A^2 w_k + qw_k)}\, dx = 0$$

for each of the finitely many functions $w_1, \cdots, w_d \in W$ given in Lemma 2.5. Then we obtain a function $\varphi \in L^2(O)$ satisfying $\varphi|_{\overline{D}} \equiv 0, \varphi|_{C^0} \not\equiv 0$, and

$$\int_O \varphi\overline{(-D_A^2 w_k + q w_k)}\, dx = 0 \quad \text{for all} \quad w \in W = W_0 + \operatorname{span}\{w_1, \cdots, w_d\}.$$

φ is solution of (5). By [6, Theorem 3.4] we have $\varphi|_B \not\equiv 0$ for all $B \subseteq C^0$ with positive measure. Since ∂C is a null set, the latter also holds all $B \subseteq C$ with positive measure. The desired result then follows from Lemma 2.4. □

3. A key compact self-adjoint operator A

First, we recall the following definition in [4].

DEFINITION 3.1 ([4]). Let $A, B : X \to X$ be two self-adjoint compact linear operators on a Hilbert space X. For a number $d \in \mathbb{N}_0$, we write

$$A \leq_d B \quad \text{or} \quad \langle Ax, x\rangle_X \leq_d \langle Bx, x\rangle_X$$

if $B - A$ has at most d negative eigenvalues. We also write $A \leq_{\text{fin}} B$ if $A \leq_d B$ holds for some $d \in \mathbb{N}_0$, and we write $A \leq B$ if $A \leq_d B$ holds for $d = 0$.

We present the monotonicity relation as follows.

PROPOSITION 3.2. *Let $A_1, A_2 \in C^1(\overline{\Omega})$ and $q_1, q_2 \in L^\infty(\Omega)\backslash\{0\}$. Assume that A_1, q_1 and A_2, q_2 are admissible and let $d(A_2, q_2) \in N_0$ be the number of eigenvalues of $K_{A_2} + K_{A_2,q_2}$ that are larger than 1. Then, there exists a subspace $V \subset L^2(\Sigma)$ with $\dim(V) \leq d(A_2, q_2)$ such that*

$$\int_\Sigma g\overline{(\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1})g}\, dS \geq \int_\Omega |D_{A_1} u_1^{(g)}|^2 - |D_{A_2} u_1^{(g)}|^2\, dx + \int_\Omega (q_2 - q_1)|u_1^{(g)}|^2\, dx,$$

for all $g \in V^\perp$.

REMARK 3.3. When $q = q_1 = q_2$, we can also obtain

$$\langle g, (\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1})g\rangle_{L^2(\Sigma)} \leq \int_\Omega |D_{A_1} u_2^{(g)}|^2 - |D_{A_2} u_2^{(g)}|^2\, dx$$

for all $g \in V^\perp$ with $\dim V \leq d(A_1, q_1)$.

For the proof of Proposition 3.2, see [6]. From the monotonicity inequality in

Proposition 3.2, we can obtain

$$\langle g, (\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1})g\rangle_{L^2(\Sigma)} \geq \langle g, \mathcal{A}g\rangle_{L^2(\Sigma)} \tag{6}$$

for any $g \in V^{\perp}$ with a finite dimensional subspace $V \subset L^2(\Sigma)$, where $\mathcal{A} : L^2(\Sigma) \to L^2(\Sigma)$ is defined as follows.

$$\begin{aligned}\mathcal{A} = &\sum_{k=1}^{n}(\frac{1}{i}S_1^* j_{A_1}^* M_{(A_1-A_2)_k} D_k S_1 - \frac{1}{i}S_1^* D_k^* M_{(A_1-A_2)_k} j_{A_1} S_1) \\ &+ S_1^* j_{A_1}^* M_{(|A_1|^2-|A_2|^2)} j_{A_1} S_1 + S_1^* j_{A_1}^* M_{(q_2-q_1)} j_{A_1} S_1\end{aligned} \tag{7}$$

where $S_1 : g \to u_1^{(g)}$ is the solution operator and $D_j : u \to \frac{\partial u}{\partial x_j}$ is the differential operator, which is bounded from $H^1_{A_1}(\Omega)$ to $L^2(\Omega)$. We note that $\mathcal{A}$ is a compact self-adjoint linear operator on $L^2(\Sigma)$.

4. Proof of Theorem 1.3

In this section, we show a monotonicity tests to detect an unknown obstacle. To prove Theorem 1.3, we need the following Lemma.

LEMMA 4.1. *Assume $A_1, A_2 \in C^1(\overline{\Omega})$ and $q_1, q_2 \in L^{\infty}(\Omega)\setminus\{0\}$ are admissible. If $A_1(x) = A_2(x), q_1(x) = q_2(x)$ for all x a.e. outside a measurable set $\overline{D} \subset \Omega$, then there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \int_D |u_1^{(g)}|^2\, dx \leq \int_D |u_2^{(g)}|^2\, dx \leq C_2 \int_D |u_1^{(g)}|^2\, dx$$

for all $g \in L^2(\Sigma)$, where $u_1^{(g)}, u_2^{(g)} \in H^1_A(\Omega)$ are solutions of (2) with Neumann boundary data g and $A = A_1, A_2, q = q_1, q_2$, respectively.

Although we consider the magnetic Schrödinger equation, we can prove in the same way as Theorem 4.2 in [4]. We write $u_1^{(g)} = u_{A_1,q}^{(g)}$ and $u_2^{(g)} = u_{A_2,q}^{(g)}$ for simplicity.

Proof of Theorem 1.3. 1. From assumption and $q = q_1 = q_2$, u_1 is constant in an open ball $B \subset \Omega$ and zero outside. There exists a subspace $V \subset L^2(\Sigma)$ with $\dim V \leq d(A_2, q)$, where

$$\begin{aligned}\int_\Sigma g\overline{(\Lambda_{A_2,q}-\Lambda_{A_1,q})g}\,dS &\geq \langle g, \mathcal{A}g\rangle_{L^2(\Sigma)}\\ &= \int_\Omega |D_{A_1}u_1^{(g)}|^2 - |D_{A_2}u_1^{(g)}|^2 + (q_2-q_1)|u_1^{(g)}|^2\\ &\geq \int_B (|A_1|^2-|A_2|^2)|u_1^{(g)}|^2\,dx\end{aligned}$$

for all $g \in V^\perp$. If $B \subset D$ and $\alpha \leq |A_1| - |A_2|$, we have

$$\int_B (|A_1|^2-|A_2|^2)|u_1^{(g)}|^2\,dx \geq \int_B \alpha|u_1^{(g)}|^2\,dx = \alpha\int_\Sigma g\overline{T_B g}\,dS,$$

for all $g \in L^2(\Sigma)$. Therefore,

$$\int_\Sigma g\overline{(\Lambda_{A_2,q}-\Lambda_{A_1,q})g}\,dS \geq \alpha\int_\Sigma g\overline{T_B g}\,dS$$

for all $g \in V^\perp$.

2. We prove the claim by contradiction. Let $B \not\subseteq D, \alpha > 0$ and we assume that

$$\Lambda_{A_2,q} - \Lambda_{A_1,q} \geq_{fin} \alpha T_B.$$

From Remark 3.3 and Lemma 4.1, there exists a finite dimensional subspace $V \subset L^2(\Sigma)$ with $\dim V \leq d(A_1, q)$ and a constant $C > 0$ such that

$$\begin{aligned}\int_\Sigma g\overline{(\Lambda_{A_2,q}-\Lambda_{A_1,q})g}\,dS &\leq \int_D |D_{A_1}u_2^{(g)}|^2 - |D_{A_2}u_2^{(g)}|^2\,dx\\ &\leq C\int_D (|A_1|^2-|A_2|^2)|u_1^{(g)}|^2\,dx\end{aligned}$$

for all $g \in V^\perp$. By the assumption and the transitivity property, there exists a finite dimensional subspace $\tilde{V} \subset L^2(\Sigma)$ with

$$\alpha\int_\Sigma g\overline{T_B g}\,dS = \int_B \alpha|u_1^{(g)}|^2\,dx \leq C\int_D (|A_1|^2-|A_2|^2)|u_1^{(g)}|^2\,dx$$

for all $g \in \tilde{V}^\perp$. However, this is contradicted by the localized potentials.

That is, by Proposition 2.1, there exists a sequence $(g_j)_{j\in\mathbb{N}} \subset \tilde{V}^\perp$ with

$$\int_B |u_{A,q}^{(g_j)}|^2\,dx \to \infty \quad \text{and} \quad \int_D |u_{A,q}^{(g_j)}|^2\,dx \to 0,$$

Therefore, $\Lambda_{A_2,q} - \Lambda_{A_1,q} - \alpha T_B$ has infinitely many negative eigenvalues. □

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