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Scattering theory for one-dimensional wave equations with thin bond layers

Hisashi MORIOKA and Steeve GRÉAUX

Abstract. In this note, we consider a mathematical justification of a reconstruction of wave velocities via in situ ultrasonic interferometry from a measurement of the travel time of the acoustic pulse. In order to do this, the impulse response for the one-dimensional wave equation on the stratified media is studied. Moreover, due to the situation of an experiment, we try to justify the bond correction method which is used to remove the perturbation of the measurement by the thin bound layers. On the other hand, for the sake of simplicity of our model, we restrict our study to wave propagations without mode changes.

1. Introduction

In this note, we report a recent collaborative study between applied mathematics and experimental geophysics in a nondestructive testing of minerals. Our main aim is to justify mathematically a reconstruction of wave velocities via in situ ultrasonic interferometry from a measurement of the travel time of the acoustic pulse. The measurement is based on the back scattering of elastic waves. The wave velocity of the sample is calculated from the sample length and the travel time through the sample (see Figures 1 and 2).

Note that we do not use the scattering matrix in our argument. In mathematical studies of inverse scattering problems for wave equations and Schrödinger equations, one often considers the scattering matrix as a measurement and developed some techniques of the reconstruction of coefficients or potentials. For this topic, see, e.g., [3], [7] and the references therein. However, in the experimental situation which we consider in this note, we cannot get the data of the scattering

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matrix. Then we study an auxiliary method of the reconstruction without the knowledge of the scattering matrix.

Another perspective is the bond correction method ([9],[16],[15]). In the experimental situation, there are some thin bond layers between the buffer rod or the pressure maker and the sample layer. Thus, the reflected wave is perturbed by the bond layers. We have to remove the effect of the bond layer from the measurement.

Mathematical study of stationary scattering theory for elastic wave equations is one of basic issues in partial differential equations. There are some developments in the asymptotic behavior of scattered waves and characterizations of the generalized eigenfunctions on $\mathbf{R}^3$ or on the half space $\mathbf{R}^3_+$ ([6], [4], [17, 18], [11], [8]). Since we adopt a restricted situation like a real experiment, we have to consider the elastic equation in a cylindrical domain with the free boundary condition. In this case, the general theory of scattering is too complicated in order to apply the argument to the bond correction method. Then we restrict the incident wave to the case where it is perpendicular to the interface and reduce the problem to the impulse response problem for the one-dimensional wave equation. This topic is going to be

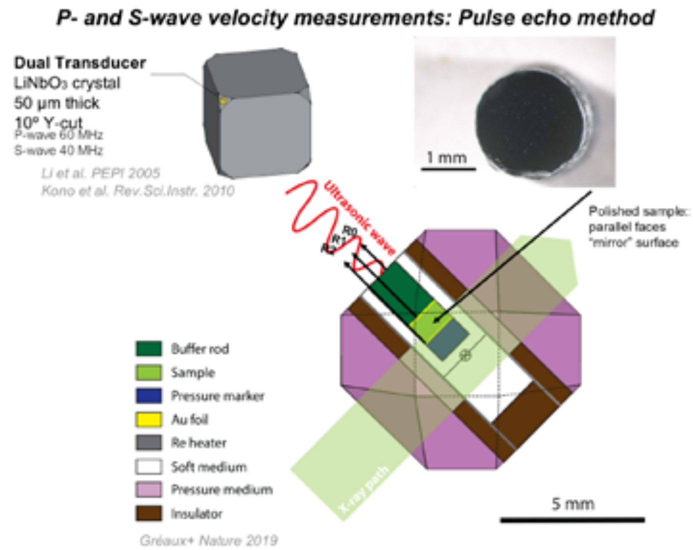

Figure 1. The situation of experiments ([14], [13], [5]). The length of every layer can be observed by the X-ray tomography. The travel time of elastic waves are observed in view of pulse echoes by the oscilloscope.

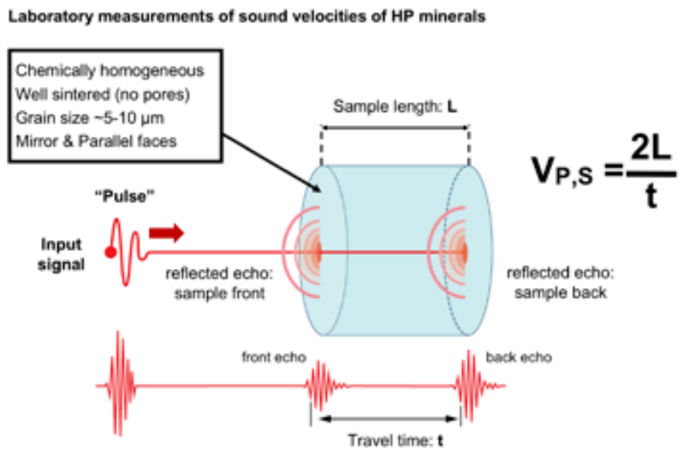

Figure 2. The situation of ultrasonic experiments. We input a pulse and observe pulse echoes in view of the back scattering. There are perturbations by the thin bond layers between the buffer rod and the sample layer.

discussed in §2 and §3.1.

In §3, we derive an ideal model of pulse echoes by using an impulse response of the wave equation. The solution of the wave equation in view of the impulse response is represented by the propagation of Dirac's measure with suitable intensities. This kind of propagation problems is studied in the research area of inverse problems on quantum graphs (e.g., [2], [1]) in view of the boundary control method (e.g., [10]). We adopt their construction of solutions as the impulse response for wave equations on the stratified media. However, in our setting, the coefficient which appears in the wave equation is piecewise constant. Then we can use a combinatorial approach instead of Duhamel's principle in order to derive the propagation of singularities. Actually, our combinatorial method has been used for the scattering theory of discrete time quantum walks ([12]).

As an application of these arguments, we derive a reconstruction formula of the wave speed of the sample from a measurement of the travel time of singularities, removing the effect of the thin bond layers. The main result is Theorem 3.5.

2. Preliminaries for elastic wave equations

2.1. Elastic wave equations in stratified media

Let us consider the scattering theory for the elastic wave equation in stratified media. In this paper, we only consider the scattered wave associated with the incoming plane wave which is perpendicular to the interface. Then we omit the general theory.

Let $\mathbf{R}^3 = \Omega_{a,+} \cup \Omega_{a,-} \cup S_a$ for fixed $a \in \mathbf{R}$ where

$$\Omega_{a,\pm} = \{x \in \mathbf{R}^3 \ ; \ \pm(x_1 - a) > 0\}, \quad S_a = \{x \in \mathbf{R}^3 \ ; \ x_1 = a\}.$$

We assume that every domain $\Omega_{a,\pm}$ is an isotropic elastic continuum with the interface S_a. Let $\rho, \lambda, \mu \in L^\infty(\mathbf{R}^3)$ be the density and the Lamé coefficients. Suppose that

(A-1) ρ, λ, and μ are piecewise positive constant functions, i.e.,

$$\alpha(x) = \alpha_\pm > 0, \quad x \in \Omega_{a,\pm}, \quad \alpha = \rho, \lambda, \mu. \tag{1}$$

For the displacement $\mathbf{w}(t,x) = [w_1(t,x), w_2(t,x), w_3(t,x)]^\mathsf{T}$ at $(t,x) \in \mathbf{R} \times \mathbf{R}^3$, the stress tensor $[\sigma_{j,k}\mathbf{w}]_{j,k=1}^3$ is given by

$$\sigma_{j,k}\mathbf{w} = \lambda(\cdot)\delta_{j,k} \mathrm{div}\, \mathbf{w} + 2\mu(\cdot)\epsilon_{j,k}\mathbf{w},$$

where

$$\epsilon_{j,k}\mathbf{w} = \frac{1}{2}\left(\partial_{x_k} w_j + \partial_{x_j} w_k\right).$$

Letting $\mathbf{w}(t,x) = \phi(t)\mathbf{u}(x)$, the wave equation

$$\partial_t^2 \mathbf{w} = \left[-\frac{1}{\rho(\cdot)} \sum_{k=1}^3 \partial_{x_k} \sigma_{j,k} \mathbf{w} \right]_{j=1,2,3}$$

is reduced to $H_0 \mathbf{u} = k^2 \mathbf{u}$ in $\mathbf{R}^3$ for a constant $k > 0$, where the operator H_0 is defined by

$$H_0 \mathbf{v} = \left[-\frac{1}{\rho(\cdot)} \sum_{k=1}^3 \partial_{x_k} \sigma_{j,k} \mathbf{v} \right]_{j=1,2,3}, \quad \mathbf{v} \in C_0^\infty(\mathbf{R}^3; \mathbf{C}^3).$$

We use the Hilbert space $\mathcal{H}_0 = L^2(\mathbf{R}^3; \mathbf{C}^3, \rho dx)$ equipped with the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{H}_0} = \sum_{j=1}^{3} \int_{\mathbf{R}^3} u_j(x) \overline{v_j(x)} \rho(x) dx,$$

and the associated norm $\|\mathbf{u}\|_{\mathcal{H}_0} = (\langle \mathbf{u}, \mathbf{u} \rangle_{\mathcal{H}})^{1/2}$. For a nonnegative integer m, the Sobolev space $H^m(\mathbf{R}^3; \mathbf{C}^3)$ is defined by the norm

$$\|\mathbf{u}\|^2_{H^m(\mathbf{R}^3;\mathbf{C}^3)} = \sum_{|\alpha| \leq m} \|\partial_x^\alpha \mathbf{u}\|^2_{L^2(\mathbf{R}^3;\mathbf{C}^3, dx)},$$

for multi-indices α. It is well-known that H_0 is self-adjoint on $\mathcal{H}_0$ as follows (see [17]).

LEMMA 2.1. *The operator H_0 has the self-adjoint extension with the domain*

$$\mathsf{D}(H_0) = \{\mathbf{u} \in H^1(\mathbf{R}^3; \mathbf{C}^3) \ ; \ H_0 \mathbf{u} \in \mathcal{H}_0\}, \tag{2}$$

in the sense of

$$\langle H_0 \mathbf{u}, \mathbf{v} \rangle_{\mathcal{H}_0} = \int_{\mathbf{R}^3} \left(\lambda \operatorname{div} \mathbf{u} \, \overline{\operatorname{div} \mathbf{v}} + 2\mu \sum_{j,k=1}^{3} \epsilon_{j,k} \mathbf{u} \, \overline{\epsilon_{j,k} \mathbf{v}} \right) dx$$

for any $\mathbf{v} \in H^1(\mathbf{R}^3; \mathbf{C}^3)$. The operator H_0 is nonnegative.

Let us note the continuity of $\mathbf{u} \in \mathsf{D}(H_0)$ across S_a.

LEMMA 2.2. *If $\mathbf{u} \in \mathsf{D}(H_0)$, we have*

$$\mathbf{u}\big|_{x_1 = a+0} = \mathbf{u}\big|_{x_1 = a-0}, \quad \sigma_{j,1} \mathbf{u}\big|_{x_1 = a+0} = \sigma_{j,1} \mathbf{u}\big|_{x_1 = a-0}, \tag{3}$$

for $j = 1, 2, 3$.

PROOF. Since there exists the trace operator from $H^1(\mathbf{R}^3; \mathbf{C}^3)$ to $L^2(S_a; \mathbf{C}^3)$, we have $\mathbf{u}\big|_{x_1 = a} \in L^2(S_a; \mathbf{C}^3)$. This implies the continuity of $\mathbf{u}$ across the interface S_a. In view of Lemma 2.1, we have $\langle H_0 \mathbf{u}, \mathbf{v} \rangle_{\mathcal{H}_0} = \langle \mathbf{u}, H_0 \mathbf{v} \rangle_{\mathcal{H}_0}$ for any $\mathbf{u}, \mathbf{v} \in \mathsf{D}(H_0)$.

Then, by the integration by parts, we have from $\langle H_0\mathbf{u},\mathbf{v}\rangle_{\mathcal{H}_0} = \langle \mathbf{u}, H_0\mathbf{v}\rangle_{\mathcal{H}_0}$

$$0 = \sum_{j=1}^{3}\left(\int_{\mathbf{R}^2} [\mathbf{u}]^a_{j,1}\cdot \overline{v_j}\big|_{x_1=a} dx' - \int_{\mathbf{R}^2} u_j\big|_{x_1=a}\cdot \overline{[\mathbf{v}]^a_{j,1}} dx'\right) + \sum_{j,k=1}^{3}\int_{\mathbf{R}^3}\left(\sigma_{j,k}\mathbf{u}\cdot\overline{\partial_{x_k} v_j} - \partial_{x_k} u_j\cdot\overline{\sigma_{j,k}\mathbf{v}}\right)dx,$$

where $[\mathbf{u}]^a_{j,k}(x') = \sigma_{j,k}\mathbf{u}(a+0,x') - \sigma_{j,k}\mathbf{u}(a-0,x')$. For the second term on the right-hand side, we note that

$$\begin{aligned}&\sum_{j,k=1}^{3}\int_{\mathbf{R}^3}\sigma_{j,k}\mathbf{u}\cdot\overline{\partial_{x_k} v_j}dx\\ &= \sum_{j=1}^{3}\int_{\mathbf{R}^3}\lambda\partial_{x_j} u_j\cdot\overline{\partial_{x_j} v_j}dx + \sum_{j,k=1}^{3}\int_{\mathbf{R}^3}\mu\left(\partial_{x_k} u_j + \partial_{x_j} u_k\right)\overline{\partial_{x_k} v_j}dx\\ &= \sum_{j,k=1}^{3}\int_{\mathbf{R}^3}\partial_{x_k} u_j\cdot\overline{\sigma_{j,k}\mathbf{v}}dx.\end{aligned}$$

Then we have

$$\sum_{j=1}^{3}\int_{\mathbf{R}^2} [\mathbf{u}]^a_{j,1}\cdot \overline{v_j}\big|_{x_1=a} dx' = \sum_{j=1}^{3}\int_{\mathbf{R}^2} u_j\big|_{x_1=a}\cdot \overline{[\mathbf{v}]^a_{j,1}} dx'. \tag{4}$$

Taking an arbitrary $v_1 \in C_0^\infty(\mathbf{R}^3)$ and letting $\mathbf{v} = [v_1, 0, 0]^{\mathsf{T}}$, we obtain from (4)

$$\sigma_{1,1}\mathbf{u}\big|_{x_1=a-0} = \sigma_{1,1}\mathbf{u}\big|_{x_1=a+0}.$$

The proofs of $\sigma_{j,1}\mathbf{u}\big|_{x_1=a-0} = \sigma_{j,1}\mathbf{u}\big|_{x_1=a+0}$ for $j = 2, 3$ are similar. □

Let us turn to the generalized eigenfunction with the incident wave perpendicular to the interface S_a. There are P-waves and S-waves. Due to our aim, we ignore surface waves, namely, Rayleigh waves or evanescent waves (see, e.g., [8]). We put

$$c_p = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}, \quad c_s = \left(\frac{\mu}{\rho}\right)^{1/2},$$

as the wave speed of a P-wave and a S-wave, respectively. Letting $\omega \in S^2$, a P-wave is of the form $\beta(k)e^{ikx\cdot\omega/c_p}\omega$ for a wave number $k > 0$ and an complex amplitude

$\beta(k)$. On the other hand, a S-wave is of the form $\gamma(k)e^{ikx\cdot\theta/c_s}\eta$ where $\gamma(k)$ is a complex amplitude, and $\theta, \eta \in S^2$ with $\theta \cdot \eta = 0$.

Now we consider the equation $H_0\mathbf{u} = k^2\mathbf{u}$ for a fixed wave number $k > 0$ with the boundary condition (3). We seek the solution in $H^1_{loc}(\mathbf{R}^3; \mathbf{C}^3)$. By using the notation (1), we put

$$c_{p,\pm} = \left(\frac{\lambda_\pm + 2\mu_\pm}{\rho_\pm}\right)^{1/2}, \quad c_{s,\pm} = \left(\frac{\mu_\pm}{\rho_\pm}\right)^{1/2}.$$

We seek the solution to $H_0\mathbf{u} = k^2\mathbf{u}$ of the form

$$\mathbf{u}(x) = -e^{-ikx_1/c_{p,+}}\begin{bmatrix}1\\0\\0\end{bmatrix} + R_p(k)e^{ikx\cdot\omega_+/c_{p,+}}\omega_+ + R_s(k)e^{ikx\cdot\theta_+/c_{s,+}}\eta_+, \tag{5}$$

for $x \in \Omega_{a,+}$ where $\omega_+, \theta_+, \eta_+ \in S^2$ with $\theta_+ \cdot \eta_+ = 0$, $R_p(k)$ and $R_s(k)$ are the complex reflection coefficients. In addition, letting $\omega_+ = [\omega_{+,1}, \omega_{+,2}, \omega_{+,3}]^\mathsf{T}$ and $\theta_+ = [\theta_{+,1}, \theta_{+,2}, \theta_{+,3}]^\mathsf{T}$, we can assume that $\omega_{+,1}, \theta_{+,1} > 0$. Note that the first term on the right-hand side is the incident P-wave. In the similar way, for $x \in \Omega_{a,-}$, we put

$$\mathbf{u}(x) = T_p(k)e^{ikx\cdot\omega_-/c_{p,-}}\omega_- + T_s(k)e^{ikx\cdot\theta_-/c_{s,-}}\eta_-, \tag{6}$$

where $\omega_-, \theta_-, \eta_- \in S^2$ with $\theta_- \cdot \eta_- = 0$, $T_p(k)$ and $T_s(k)$ are the complex transmission coefficients, and we assume that $\omega_{-,1}, \theta_{-,1} < 0$.

Proposition 2.3. *Let $k > 0$. We have for (5) and (6)*

$$R_p(k) = e^{-2ika/c_{p,+}}\frac{\rho_- c_{p,-} - \rho_+ c_{p,+}}{\rho_- c_{p,-} + \rho_+ c_{p,+}}, \tag{7}$$

$$T_p(k) = e^{-ika/c_{p,+}}e^{ika/c_{p,-}}\frac{2\rho_+ c_{p,+}}{\rho_- c_{p,-} + \rho_+ c_{p,+}}, \tag{8}$$

and $R_s(k) = T_s(k) = 0$.

PROOF. In view of the boundary condition $\mathbf{u}|_{x_1=a+0} = \mathbf{u}|_{x_1=a-0}$ in (3), we obtain

$$
\begin{aligned}
&- e^{-ika/c_{p,+}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \left(R_p(k) e^{ikx\cdot\omega_+/c_{p,+}} \omega_+ + R_s(k) e^{ikx\cdot\theta_+/c_{s,+}} \eta_+ \right)\Big|_{x_1=a} \\
&= \left(T_p(k) e^{ikx\cdot\omega_-/c_{p,-}} \omega_- + T_s(k) e^{ikx\cdot\theta_-/c_{s,-}} \eta_- \right)\Big|_{x_1=a},
\end{aligned} \tag{9}
$$

for any $(a, x') \in S_a$ with $x' \in \mathbf{R}^2$. Since $e^{ikx\cdot\omega_\pm/c_{p,\pm}}|_{x_1=a}$ and $e^{ikx\cdot\theta_\pm/c_{s,\pm}}|_{x_1=a}$ are analytic in $x' \in \mathbf{R}^2$, it follows from (9) that

$$
\omega_\pm = \theta_\pm = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \eta_\pm = \begin{bmatrix} 0 \\ \eta_{\pm,2} \\ \eta_{\pm,3} \end{bmatrix}. \tag{10}
$$

Now we obtain

$$
R_p(k) e^{ika/c_{p,+}} + T_p(k) e^{-ika/c_{p,+}} = e^{-ika/c_{p,+}}, \tag{11}
$$

$$
R_s(k) e^{ika/c_{s,+}} \begin{bmatrix} \eta_{+,2} \\ \eta_{+,3} \end{bmatrix} = T_s(k) e^{-ika/c_{s,+}} \begin{bmatrix} \eta_{-,2} \\ \eta_{-,3} \end{bmatrix}. \tag{12}
$$

Due to (5), (6) and (10), we have

$$
\begin{aligned}
\sigma_{j,1}\mathbf{u}\Big|_{x_1=a+0} =& \delta_{j,1} ik\rho_+ c_{p,+} \left(e^{-ika/c_{p,+}} + R_p(k) e^{ika/c_{p,+}} \right) \\
&+ (1-\delta_{j,1}) ik\rho_+ c_{s,+} R_s(k) e^{ika/c_{s,+}} \eta_{+,j},
\end{aligned}
$$

and

$$
\sigma_{j,1}\mathbf{u}\Big|_{x_1=a-0} = \delta_{j,1} ik\rho_- c_{p,-} T_p(k) e^{-ika/c_{p,-}} + (1-\delta_{j,1}) ik\rho_- c_{s,-} T_s(k) e^{-ika/c_{s,-}} \eta_{-,j},
$$

for $j = 1, 2, 3$. The boundary condition (3) implies that

$$
\rho_+ c_{p,+} \left(e^{-ika/c_{p,+}} + R_p(k) e^{ika/c_{p,+}} \right) = \rho_- c_{p,-} T_p(k) e^{-ika/c_{s,-}}, \tag{13}
$$

$$
\rho_+ c_{p,+} R_s(k) e^{ika/c_{s,+}} \begin{bmatrix} \eta_{+,2} \\ \eta_{+,3} \end{bmatrix} + \rho_- c_{s,-} T_s(k) e^{-ika/c_{s,-}} \begin{bmatrix} \eta_{-,2} \\ \eta_{-,3} \end{bmatrix} = 0. \tag{14}
$$

From (11) and (13), we have

$$
\begin{bmatrix} e^{ika/c_{p,+}} & e^{-ika/c_{p,-}} \\ \rho_+ c_{p,+} e^{ika/c_{p,+}} & -\rho_- c_{p,-} e^{-ika/c_{p,-}} \end{bmatrix} \begin{bmatrix} R_p(k) \\ T_p(k) \end{bmatrix} = \begin{bmatrix} e^{-ika/c_{p,+}} \\ e^{ika/c_{p,+}} \end{bmatrix}.
$$

Solving this equation, we obtain (7) and (8). From (12) and (14), we have

$$e^{-ika/c_{s,-}} T_s(k) \left(1 + \frac{\rho_- c_{s,-}}{\rho_+ c_{s,+}}\right) \begin{bmatrix} \eta_{-,2} \\ \eta_{-,3} \end{bmatrix} = 0.$$

Recalling $[\eta_{-,2}, \eta_{-,3}]^\mathsf{T} \in S^1$, we obtain $T_s(k) = 0$. Plugging $T_s(k) = 0$ to (12), we see $R_s(k) = 0$. □

For the case where we take an incident S-wave, we seek the solution to $H_0 \mathbf{v} = k^2 \mathbf{v}$ of the form

$$\mathbf{v}(x) = e^{-ikx_1/c_{s,+}} \eta_0 + R_p(k) e^{ikx\cdot\omega_+/c_{p,+}} \omega_+ + R_s(k) e^{ikx\cdot\theta_+/c_{s,+}} \eta_+, \tag{15}$$

for $x \in \Omega_{a,+}$ where $\eta_0 = [0, \eta_{0,2}, \eta_{0,3}]^\mathsf{T} \in S^2$, and

$$\mathbf{v}(x) = T_p(k) e^{ikx\cdot\omega_-/c_{p,-}} \omega_- + T_s(k) e^{ikx\cdot\theta_-/c_{s,-}} \eta_-, \tag{16}$$

for $x \in \Omega_{a,-}$.

Proposition 2.4. *Let $k > 0$. We have for (15) and (16)*

$$R_s(k) \begin{bmatrix} \eta_{+,2} \\ \eta_{+,3} \end{bmatrix} = e^{-2ika/c_{p,+}} \frac{\rho_+ c_{p,+} - \rho_- c_{p,-}}{\rho_+ c_{p,+} + \rho_- c_{p,-}} \begin{bmatrix} \eta_{0,2} \\ \eta_{0,3} \end{bmatrix}, \tag{17}$$

$$T_s(k) \begin{bmatrix} \eta_{+,2} \\ \eta_{+,3} \end{bmatrix} = e^{-ika/c_{p,+}} e^{ika/c_{p,-}} \frac{2\rho_+ c_{p,+}}{\rho_+ c_{p,+} + \rho_- c_{p,-}} \begin{bmatrix} \eta_{0,2} \\ \eta_{0,3} \end{bmatrix}, \tag{18}$$

and $R_p(k) = T_p(k) = 0$.

Proof. The proof is similar to Proposition 2.3. We derive a sketch of the proof. In view of the boundary conditions $\mathbf{v}|_{x_1=a+0} = \mathbf{v}|_{x_1=a-0}$, we have

$$\begin{aligned} &- e^{-ika/c_{s,+}} \eta_0 + \left(R_p(k) e^{ikx\cdot\omega_+/c_{p,+}} \omega_+ + R_s(k) e^{ikx\cdot\theta_+/c_{s,+}} \eta_+ \right)\Big|_{x_1=a} \\ &= \left(T_p(k) e^{ikx\cdot\omega_-/c_{p,-}} \omega_- + T_s(k) e^{ikx\cdot\theta_-/c_{s,-}} \eta_- \right)\Big|_{x_1=a}, \end{aligned}$$

for any $(a, x') \in S_a$ with $x' \in \mathbf{R}^2$. Since $e^{ikx\cdot\omega_\pm/c_{p,\pm}}|_{x_1=a}$ and $e^{ikx\cdot\theta_\pm/c_{s,\pm}}|_{x_1=a}$ are analytic with respect to x', we see (10). We obtain

$$R_p(k) e^{ika/c_{p,+}} + T_p(k) e^{-ika/c_{p,+}} = 0, \tag{19}$$

$$R_s(k) e^{ika/c_{s,+}} \begin{bmatrix} \eta_{+,2} \\ \eta_{+,3} \end{bmatrix} - T_s(k) e^{-ika/c_{s,+}} \begin{bmatrix} \eta_{-,2} \\ \eta_{-,3} \end{bmatrix} = -e^{-ika/c_{s,+}} \begin{bmatrix} \eta_{0,2} \\ \eta_{0,3} \end{bmatrix}. \tag{20}$$

The boundary condition $\sigma_{j,1}\mathbf{v}|_{x_1=a+0} = \sigma_{j,1}\mathbf{v}|_{x_1=a-0}$ implies

$$\rho_+ c_{p,+} R_p(k) e^{ika/c_{p,+}} = \rho_- c_{p,-} T_p(k) e^{-ika/c_{p,+}}, \tag{21}$$

$$\begin{aligned} &\rho_+ c_{p,+} R_s(k) e^{ika/c_{s,+}} \begin{bmatrix} \eta_{+,2} \\ \eta_{+,3} \end{bmatrix} + \rho_- c_{-,s} T_s(k) e^{-ika/c_{s,+}} \begin{bmatrix} \eta_{-,2} \\ \eta_{-,3} \end{bmatrix} \\ &= \rho_+ c_{s,+} e^{-ika/c_{s,+}} \begin{bmatrix} \eta_{0,2} \\ \eta_{0,3} \end{bmatrix}. \end{aligned} \tag{22}$$

The equalities (19) and (21) imply $R_p(k) = T_p(k) = 0$. Plugging (20) with (22), we obtain (17) and (18). □

Proposition 2.3 and (10) imply the following fact.

Corollary 2.5. *The solution* $\mathbf{u}$ *given by (5) and (6) depends only on* x_1*, is of the form*

$$u_1(x_1) = \begin{cases} -\, e^{-ikx_1/c_{p,+}} + R_p(k) e^{ikx_1/c_{p,+}}, & x_1 > a, \\ -\, T_p(k) e^{-ikx_1/c_{p,-}}, & x_1 < a, \end{cases}$$

and satisfies the ODE

$$-c_p^2 u_1''(x_1) = k^2 u_1(x_1) \quad in \quad \mathbf{R} \setminus \{a\},$$

with the boundary condition

$$u_1\big|_{x_1=a+0} = u_1\big|_{x_1=a-0}, \quad (\lambda_+ + 2\mu_+) u_1'\big|_{x_1=a+0} = (\lambda_- + 2\mu_-) u_1'\big|_{x_1=a-0}.$$

Proof. In view of Lemma 2.2, we have $\sigma_{1,1}\mathbf{u}|_{x_1=a+0} = \sigma_{1,1}\mathbf{u}|_{x_1=a-0}$. We also have

$$\begin{aligned} \sigma_{1,1}\mathbf{u}|_{x_1=a+0} &= ik\rho_+ c_{p,+} (e^{-ika/c_{p,+}} + R_p(k) e^{ika/c_{p,+}}), \\ \sigma_{1,1}\mathbf{u}|_{x_1=a-0} &= ik\rho_- c_{p,-} T_p(k) e^{-ika/c_{p,-}}. \end{aligned}$$

Then the boundary condition follows from (7) and (8). □

Similarly, Proposition 2.4 and its proof imply the following one dimensional problem.

Corollary 2.6. *The solution* $\mathbf{v}$ *given by (15) and (16) depends only on* x_1

and satisfies the ODE

$$-c_s^2 v_j''(x_1) = k^2 v_j(x_1) \quad in \quad \mathbf{R} \setminus \{a\}, \quad j = 2, 3,$$

with the boundary condition

$$v_j\big|_{x_1=a+0} = v_j\big|_{x_1=a-0}, \quad \mu_+ v_j'\big|_{x_1=a+0} = \mu_- v_j'\big|_{x_1=a-0}.$$

3. Scattering theory for stratified media with thin bond layers

3.1. Wave equations in stratified cylinders

Due to the arguments in the previous section, we reduce the problem to the one-dimensional wave equation in stratified media with thin bond layers. In the setting of an experiment for elastic wave velocity measurements, the medium is a cylinder. Thus, we give a remark for the elastic wave equation on the cylindrical domain (Figure 3).

Suppose that $l \gg \delta_b > 0$. Let $I = I_0 \cup I_1 \cup I_2 \cup I_3 \cup I_4$ where

$$I_0 = (-\infty, -\delta_b), \quad I_1 = (-\delta_b, 0), \quad I_2 = (0, l), \quad I_3 = (l, l + \delta_b), \quad I_4 = (l + \delta_b, \infty).$$

The cylindrical domain $\Omega := I \times B^2$ with $B^2 = \{x' \in \mathbf{R}^2 \ ; \ |x'| < 1\}$ is decomposed into five subdomains and four interfaces as follows. Let

$$\Omega_j = I_j \times B^2, \quad j = 0, 1, 2, 3, 4. \tag{23}$$

Then there are four interfaces

$$S_{-\delta_b} = \{-\delta_b\} \times B^2, \quad S_0 = \{0\} \times B^2, \quad S_l = \{l\} \times B^2, \quad S_{l+\delta_b} = \{l + \delta_b\} \times B^2.$$

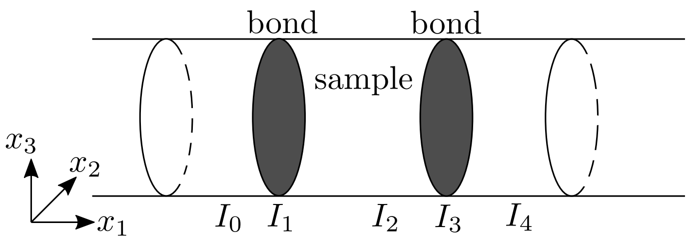

Figure 3. The cylindrical domain Ω with thin bond layers.

In the context of the experiment, Ω_0 is the pressure maker, Ω_1 and Ω_3 are thin bond layers of a same material, Ω_2 is the sample, and Ω_4 is the buffer rod. Here we assume that the pressure maker and the buffer rod are sufficiently longer than the sample.

We assume that every part is an isotropic elastic continuum. Let $\rho, \lambda, \mu \in L^\infty(\Omega)$ be the density and the Lamé coefficients on Ω. Suppose that

(A-2) ρ, λ and μ are piecewise positive constant functions, i.e.,

$$\alpha(x) = \alpha_j \quad \text{for} \quad x \in \Omega_j, \quad \alpha = \rho, \lambda, \mu,$$

where α_j, $j = 0, 1, 2, 3, 4$, are positive constants. We assume that $\alpha_1 = \alpha_3$.

The elastic wave equation in Ω is given by

$$\partial_t^2 \mathbf{w} = \left[-\frac{1}{\rho(\cdot)} \sum_{k=1}^{3} \partial_{x_k} \sigma_{j,k} \mathbf{w} \right]_{j=1,2,3} \quad \text{on} \quad \mathbf{R} \times \Omega, \tag{24}$$

for the displacement $\mathbf{w}(t,x) = [w_1(t,x), w_2(t,x), w_3(t,x)]^\mathsf{T}$. Thus, letting $\mathbf{w}(t,x) = \phi(t)\mathbf{u}(x)$, we have $H\mathbf{u} = k^2 \mathbf{u}$ for a constant $k > 0$ where H is defined by

$$H\mathbf{u} = \left[-\frac{1}{\rho(\cdot)} \sum_{k=1}^{3} \partial_{x_k} \sigma_{j,k} \mathbf{u} \right]_{j=1,2,3}.$$

The Hilbert space $\mathcal{H} = L^2(\Omega; \mathbf{C}^3, \rho dx)$ and the Sobolev space $H^m(\Omega; \mathbf{C}^3)$ is defined in the similar way for Ω_a in §2.1. Then the operator H is self-adjoint on $\mathcal{H}$ with the free boundary condition on the boundary $\partial\Omega := \{x \in \Omega \ ; \ |x'| = 1\}$. For the proof, see [17] and [4].

Lemma 3.1. *Let $\nu(x)$ be the exterior unit normal at $x \in \partial\Omega$. The operator H has the self-adjoint extension with the domain*

$$\mathsf{D}(H) = \{\mathbf{u} \in H^1(\Omega; \mathbf{C}^3) \ ; \ H\mathbf{u} \in \mathcal{H}, \ \mathbf{u} \text{ satisfies } (26)\}, \tag{25}$$

$$\left[\sum_{k=1}^{3} \sigma_{j,k} \mathbf{u}\big|_{x \in \partial\Omega} \nu_k \right]_{j=1,2,3} = 0, \tag{26}$$

in the sense of

$$\langle H\mathbf{u}, \mathbf{v}\rangle_{\mathcal{H}} = \int_{\Omega} \left(\lambda \operatorname{div} \mathbf{u} \, \overline{\operatorname{div} \mathbf{v}} + 2\mu \sum_{j,k=1}^{3} \epsilon_{j,k}\mathbf{u} \, \overline{\epsilon_{j,k}\mathbf{v}} \right) dx$$

for any $\mathbf{v} \in H^1(\Omega; \mathbf{C}^3)$. *The operator* H *is nonnegative.*

The continuity of $\mathbf{u} \in \mathsf{D}(H)$ across each interface can be seen in the similar way for Lemma 2.2.

LEMMA 3.2. *Let* $a \in \{-\delta_b, 0, l, l + \delta_b\}$. *If* $\mathbf{u} \in \mathsf{D}(H)$, *we have*

$$\mathbf{u}\big|_{x_1=a+0} = \mathbf{u}\big|_{x_1=a-0}, \quad \sigma_{j,1}\mathbf{u}\big|_{x_1=a+0} = \sigma_{j,1}\mathbf{u}\big|_{x_1=a-0},$$

for $j = 1, 2, 3$.

The generalized eigenfunction satisfying $H\mathbf{u} = k^2\mathbf{u}$ for $k > 0$ is complicated in general due to so many reflections on the boundary $\partial\Omega$. However, we restrict the incident P- and S-wave to the case where they are perpendicular to the interfaces. As has been seen in Propositions 2.5 and 2.6, it is natural to reduce the equation $H\mathbf{u} = k^2\mathbf{u}$ in Ω to the ODE

$$-c_\gamma^2 v'' = k^2 v \quad \text{on} \quad \mathbf{R} \setminus \{-\delta_b, 0, l, l + \delta_b\}, \tag{27}$$

with the boundary condition at every interface

$$v\big|_{x_1=a-0} = v\big|_{x_1=a+0}, \quad \rho c_\gamma^2 \partial_x v\big|_{x_1=a-0} = \rho c_\gamma^2 \partial_x v\big|_{x_1=a+0}, \tag{28}$$

for $a \in \{-\delta_b, 0, l, l + \delta_b\}$ where $\gamma \in \{p, s\}$, $c_p^2 = (\lambda + 2\mu)/\rho$, and $c_s^2 = \mu/\rho$. As a consequence, we reduce the wave equation (24) to the one-dimensional wave equation

$$\begin{aligned} &\partial_t^2 w(t,x) = c_\gamma^2 \partial_x^2 w(t,x) \quad \text{for} \quad (t,x) \in \mathbf{R} \times (\mathbf{R} \setminus \{-\delta_b, 0, l, l + \delta_b\}), \\ &w(0,x) = f(x), \quad \partial_t w(0,x) = g(x), \quad x \in \mathbf{R}, \\ &w(t, a+0) = w(t, a-0), \quad (m_\gamma \partial_x w(t,\cdot))\big|_{x=a+0} = (m_\gamma \partial_x w(t,\cdot))\big|_{x=a-0}, \end{aligned} \tag{29}$$

for $a \in \{-\delta_b, 0, l, l + \delta_b\}$ and a suitable pair f, g of the initial conditions where $m_p = \lambda + 2\mu$, and $m_s = \mu$.

3.2. Impulse responses of the wave equation

In the remaining part of this paper, we restrict the argument to the case of the P-wave. For the S-wave, the argument is parallel. We consider a scattering theory for the wave equation (29). Recall that our aim is to apply our arguments to a justification of a reconstruction of elastic wave velocity of the sample from the measurement of the travel time of elastic waves. In order to construct a simple model of this problem, we consider the impulse response problem associated with (29). Namely, we study

$$\begin{aligned} &\partial_t^2 w(t,x) = c_p^2 \partial_x^2 w(t,x) \quad \text{for} \quad (t,x) \in \mathbf{R} \times (\mathbf{R} \setminus \{-\delta_b, 0, l, l+\delta_b\}), \\ &w(0,x) = c_p \delta(x-\alpha), \quad \partial_t w(0,x) = c_p^2 \delta'(x-\alpha), \quad \alpha > l + \delta_b, \\ &w(t,a+0) = w(t,a-0), \quad (m_p \partial_x w(t,\cdot))\big|_{x=a+0} = (m_p \partial_x w(t,\cdot))\big|_{x=a-0}, \end{aligned} \tag{30}$$

for $a \in \{-\delta_b, 0, l, l+\delta_b\}$ where $\delta(\cdot-\alpha)$ is Dirac's measure. Since c_p is piecewise constant, the (weak) solution $w(t,x)$ is represented by the translations in the medium and the reflection and the transmission at every interface of Dirac's measure with a suitable intensity (see [2] and [1]).

Let us consider the transmission coefficient and the reflection coefficient for w at every interface. In order to do this, we consider the auxiliary problem

$$\begin{aligned} &\partial_t^2 v(t,x) = c_p^2 \partial_x^2 v(t,x) \quad \text{for} \quad (t,x) \in \mathbf{R} \times (\mathbf{R} \setminus \{a\}), \\ &v(0,x) = c_p \delta(x-\beta), \quad \partial_t v(0,x) = c_p^2 \delta'(x-\beta), \quad \beta > a, \\ &v(t,a+0) = v(t,a-0), \quad (m_p \partial_x v(t,\cdot))\big|_{x=a+0} = (m_p \partial_x v(t,\cdot))\big|_{x=a-0}, \end{aligned} \tag{31}$$

where $x = a$ is an interface and

$$c_p(x) = \begin{cases} c_+, & x > a, \\ c_-, & x < a, \end{cases} \qquad m_p(x) = \begin{cases} m_+ = \lambda_+ + 2\mu_+, & x > a, \\ m_- = \lambda_- + 2\mu_-, & x < a, \end{cases}$$

Then we seek the solution v of the form

$$v(t,x) = \begin{cases} \delta(c_+^{-1}(x-\beta)+t) - R\delta(c_+^{-1}(x-a)-(t-t_1)), & x > a, \\ T\delta(c_+^{-1}(x-a)+(t-t_1)), & x < a, \end{cases} \qquad t > t_1,$$

with $t_1 = c_+^{-1}(\beta - a)$. Due to the boundary condition $v(t,a+0) = v(t,a-0)$, we have

$$\delta(t-t_1) - R\delta(-(t-t_1)) = T\delta(t-t_1),$$

where we have used $c_+^{-1}(a-\beta) = -t_1$. We apply $\delta(-\tau) = \delta(\tau)$ for $\tau \in \mathbf{R}$ to the above equality, we obtain

$$1 - R = T. \tag{32}$$

The boundary condition $m_p \partial_x v(t,\cdot)|_{x=a+0} = m_p \partial_x v(t,\cdot)|_{x=a-0}$ implies that

$$m_+ c_+^{-1} \left(\delta'(t-t_1) - R\delta'(-(t-t_1))\right) = m_- c_-^{-1} T \delta'(t-t_1).$$

Now we use $\delta'(-\tau) = -\delta'(\tau)$ for $\tau \in \mathbf{R}$. Thus, we obtain

$$\rho_+ c_+ (1+R) = \rho_- c_- T. \tag{33}$$

Solving the equations (32)-(33), we can see

$$R = \frac{\rho_- c_- - \rho_+ c_+}{\rho_+ c_+ + \rho_- c_-}, \quad T = \frac{2\rho_+ c_+}{\rho_+ c_+ + \rho_- c_-}. \tag{34}$$

Applying the above arguments to the reflection and the transmission at every interface, we obtain the following lemma.

Lemma 3.3. *Let $R_{j,k}$ and $T_{j,k}$ be the coefficient of reflection and the coefficient of transmission for the case where the singularity hits the interface between I_j and I_k from I_j, respectively. Then we have*

$$R_{j,k} = \frac{Z_k - Z_j}{Z_k + Z_j}, \quad T_{j,k} = \frac{2Z_j}{Z_k + Z_j},$$

for the acoustic impedance $Z_j = \rho_j c_{p,j}$ where $\rho_j = \rho|_{I_j}$ and $c_{p,j} = c_p|_{I_j}$. $R_{j,k}$ and $T_{j,k}$ satisfy the following properties.

1. $R_{j,k} + T_{j,k} = 1$.
2. $|R_{j,k}| < 1$.
3. $R_{j,k} = -R_{k,j}$.
4. $T_{j,k}T_{k,j} - R_{j,k}R_{k,j} = 1$.

3.3. Model of pulse echoes via back scattering measurements

Let us return to the problem (30). Here we construct a model of the pulse echoes in view of the measurement at $x = \alpha$, $t > 0$. Namely, we take $w(0,x) = c_p \delta(x-\alpha)$

and $\partial_t w(0,x) = c_p^2 \delta'(x-\alpha)$ as the input, and we consider the measurement of reflected singularities $w(t,\alpha)$ for $t \in [0,T]$ with suitable $T>0$.

Since the bond layers are sufficiently thin, there are so many reflections and transmissions of singularities. In this paper, we adopt a combinatorial approach via a path counting method for the solution $w(t,x)$. Actually, $w(t,x)$ is represented by a sum of moving Dirac's measures since the coefficient c_p is piecewise constant, i.e., $c_p|_{I_j} = c_{p,j}$ for $j = 0,1,2,3,4$. Recall that $c_{p,1} = c_{p,3}$ due to the assumption (A-2). Thus, the process of scattering can be identified with a set of paths. See Figure 4. Let us split the pulse echoes into some clusters. In particular, we focus the first cluster $\mathcal{W}_1$ and the second cluster $\mathcal{W}_2$ of echoes. Namely, $\mathcal{W}_1$ is the set of signals at $x=\alpha$ which occur by the reflections at the interfaces $x = l+\delta_b$ or $x = l$ without penetrations into the sample layer I_2. On the other hand, $\mathcal{W}_2$ is the set of signals at $x=\alpha$ which occur by the reflections at the interfaces $x=0$ or $x=-\delta_b$

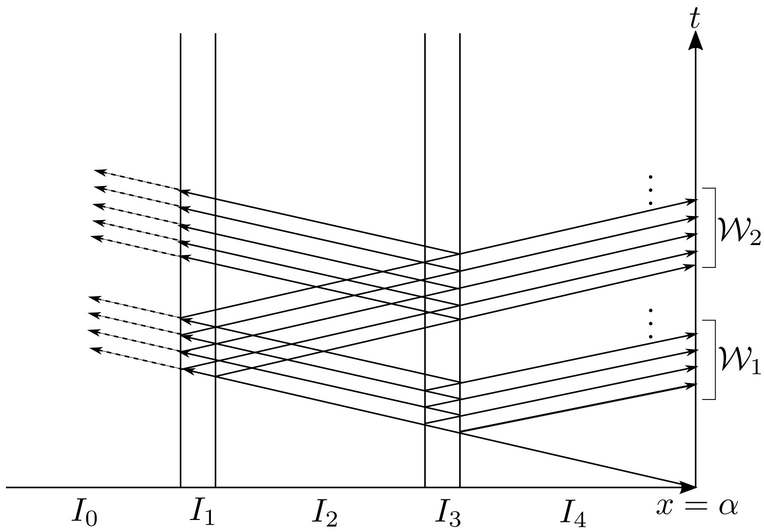

Figure 4. This is a diagram of the propagation of the singularities. We split the observation at $x=\alpha$ into two clusters. The first cluster $\mathcal{W}_1$ of pulse echoes is the set of reflected singularities which occur by the reflections on the I_3-I_4 interface or the I_2-I_3 interface. The second cluster $\mathcal{W}_2$ of pulse echoes is the set of reflected singularities which occur by the reflections on the I_1-I_2 interface or the I_0-I_1 interface

through the sample layer I_2.

LEMMA 3.4. *Let $W_n^{(j)}$ for $n = 0, 1, 2, \ldots,$ and $j = 1, 2$, be the signed intensity of the n-th signal in $\mathcal{W}_j$. $W_n^{(1)}$ is given by*

$$W_n^{(1)} = \begin{cases} R_{4,3}, & n = 0, \\ T_{3,4}T_{4,3}R_{3,4}^{n-1}R_{3,2}^{n}, & n \geq 1. \end{cases}$$

For the case $R_{1,0} \neq R_{3,4}$, $W_n^{(2)}$ is of the form

$$W_0^{(2)} = T_{3,4}T_{4,3}T_{2,3}T_{3,2}R_{2,1},$$

and

$$\begin{aligned} W_n^{(2)} =& T_{3,4}T_{4,3}T_{2,3}T_{3,2} \\ &\cdot \left(T_{1,2}T_{2,1}R_{1,0}R_{3,4}^{n-1}R_{4,3}^{n}\frac{1 - R_{1,0}^{n}R_{3,4}^{n-1}}{1 - R_{1,0}R_{3,4}^{-1}} + (n+1)R_{2,1}R_{3,2}^{n}R_{3,4}^{n} \right). \end{aligned}$$

with $n \geq 1$. For the case $R_{1,0} = R_{3,4}$, $W_n^{(2)}$ is of the form

$$W_n^{(2)} = \begin{cases} T_{3,4}T_{4,3}T_{2,3}T_{3,2}R_{2,1}, & n = 0, \\ T_{3,4}T_{4,3}T_{2,3}T_{3,2}\left(nT_{1,2}T_{2,1}R_{1,0}^{n}R_{1,2}^{n-1} + (n+1)R_{2,1}R_{3,2}^{n}R_{1,0}^{n}\right), & n \geq 1. \end{cases}$$

Consequently, $|W_n^{(1)}|$ and $|W_n^{(2)}|$ are exponentially decreasing as $n \to \infty$.

PROOF. The 0-th signal of the first cluster $\mathcal{W}_1$ comes from the interface $x = l + \delta_b$. Due to Lemma 3.3, we have $W_0^{(1)} = R_{4,3}$. For $n \geq 1$, the n-th signal of the first cluster $\mathcal{W}_1$ is the singularity which reflects n times between the interfaces $x = l$ and $x = l + \delta_b$. In view of Lemma 3.3, we have

$$W_n^{(1)} = T_{3,4}T_{4,3}R_{3,4}^{n-1}R_{4,3}^{n}, \quad n \geq 1.$$

In the similar way, we can see the formula of $W_0^{(2)}$.

Suppose $R_{1,0} \neq R_{3,4}$. For the second cluster, we note that the length of the bond layer I_1 coincides with that of another bond layer I_3. Thus, the n-th signal with $n \geq 1$ of the second cluster consists of the sum of some singularities. Actually, every singularity for the n-th signal has its path with the length $2(l+\delta_b+(\alpha-l-\delta_b))$. Such a propagation consists of (i) a singularity which passes through I_3 $2m$ times with $0 \leq m \leq n-1$, passes through I_1 $n-m$ times, and passes through the sample layer I_2 two times, (ii) a singularity which passes through I_3 $2n$ times and passes through the sample layer I_2 two times. Note that there exists $n+1$ paths in the latter case. Then we have

$$\begin{aligned} W_n^{(2)} =& T_{3,4}T_{4,3}T_{2,3}T_{3,2} \\ & \cdot \left(\sum_{m=0}^{n-1} T_{1,2}T_{2,1}R_{1,0}^m R_{1,2}^m R_{3,4}^{n-m-1} R_{3,2}^{n-m-1} + (n+1)R_{2,1}R_{3,2}^n R_{3,4}^n \right) \end{aligned}$$

for $n \geq 1$. It follows from $R_{1,2} = R_{3,2}$ that

$$\sum_{m=0}^{n-1} R_{1,0}^m R_{1,2}^m R_{3,4}^{-m} R_{3,2}^{-m} = \frac{1 - R_{1,0}^n R_{3,4}^{-n}}{1 - R_{1,0} R_{3,4}^{-1}}.$$

We obtain the formula of $W_n^{(2)}$ for $n \geq 1$ with $R_{1,0} \neq R_{3,4}$.

For the case $R_{1,0} = R_{3,4}$, the situation of the propagation is same as above. Then we have

$$W_n^{(2)} = T_{3,4}T_{4,3}T_{2,3}T_{3,2} \left(\sum_{m=0}^{n-1} T_{1,2}T_{2,1}R_{3,4}^{n-1} R_{3,2}^{n-1} + (n+1)R_{2,1}R_{3,2}^n R_{3,4}^n \right).$$

Thus we obtain the formula of $W_n^{(2)}$ for $n \geq 1$ with $R_{1,0} = R_{3,4}$.

In view of the assertion 2 in Lemma 3.3, $|W_n^{(1)}|$ and $|W_n^{(2)}|$ are exponentially decreasing as $n \to \infty$. □

Finally, we derive a reconstruction formula of $c_{p,2}$ from a measurement of pulse echoes in view of the travel time of singularities. Actually, we can see that the n-th signal in $\mathcal{W}_1$ is observed at $t = t_n^{(1)}$ where

$$t_0^{(1)} = \frac{2(\alpha - l - \delta_b)}{c_{p,4}}, \quad t_n^{(1)} = t_0^{(1)} + \frac{2\delta_b n}{c_{p,1}}, \quad n \geq 1, \tag{35}$$

and n-th signal in $\mathcal{W}_2$ is observed at $t = t_n^{(2)}$ where

$$t_0^{(2)} = t_0^{(1)} + \frac{2\delta_b}{c_{p,1}} + \frac{2l}{c_{p,3}}, \quad t_n^{(2)} = t_0^{(2)} + \frac{2\delta_b n}{c_{p,1}}, \quad n \geq 1. \tag{36}$$

Here we choose $t_0^{(2)} - t_0^{(1)}$ as the measurement of the travel time. Then we can see an application to an inverse back scattering problem via bond correction method as follows.

Theorem 3.5. *Suppose that l, δ_b and $c_{p,j}$ with $j = 0, 1, 3, 4$ are known. If the difference of the travel time $t_0^{(1)} - t_0^{(2)}$ is given, we obtain the reconstruction formula for $c_{p,2}$ as*

$$c_{p,2} = 2l\left(t_0^{(2)} - t_0^{(1)} - \frac{2\delta_b}{c_{p,1}}\right)^{-1}. \tag{37}$$

Proof. In view of (35)-(36), we have

$$t_0^{(2)} - t_0^{(1)} = \frac{2\delta_b}{c_{p,1}} + \frac{2l}{c_{p,2}}. \tag{38}$$

The reconstruction formula (37) follows from (38) immediately. □

3.4. Example and concluding remarks

Finally, let us see a typical example of an experimental setting. We consider the following situation: the backing material (I_0) is NaCl, the buffer rod (I_4) is Alumina, the bond (I_1, I_3) is Au, and the sample (I_2) is MgO. In this case, the wave velocities and the acoustic impedances introduced in Lemma 3.3 are given by Table 1. By using this data, the coefficients of reflections are computed as in Table 2.

For the case where the length of the bond layer is $\delta_b = 2.5$ (µm), the time shift is

$$\frac{2\delta_b}{c_{p,2}} = \frac{2 \cdot 2.5 \cdot 10^{-6}}{3.288 \cdot 10^3} \sim 1.52 \cdot 10^{-9}\,(\text{s}) = 1.52\,(\text{ns}).$$

Medium	Backing material	Buffer rod	Bond	Sample
Material	NaCl	Alumina	Au	MgO
Velocity (s)	4.536×10^3	10.88×10^3	3.288×10^3	9.698×10^3
Z ($\text{Pa} \cdot \text{s/m}^3$)	0.98×10^7	4.33×10^7	6.34×10^7	3.47×10^7

Table 1. An example of a real experiment by P-wave.

	$R_{1,0}$	$R_{1,2} = R_{3,2}$	$R_{3,4}$
P-wave	0.73	0.29	0.19

Table 2. $R_{j,k}$ of a real experiment by P-wave.

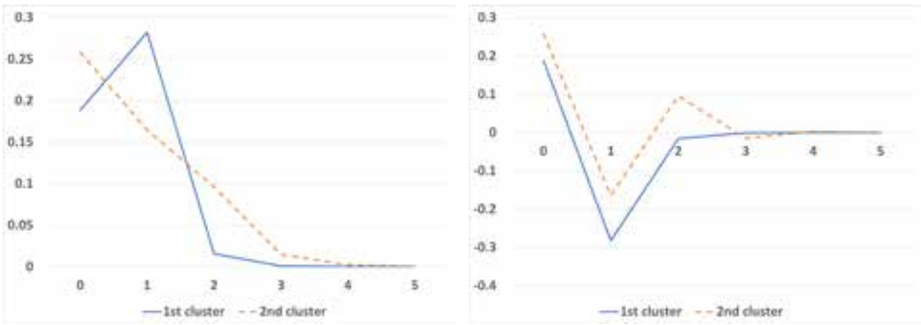

Figure 5. Left: The numerical results of the intensities $|W_n^{(1)}|$ and $|W_n^{(2)}|$ for $n = 0, 1, 2, \ldots,$ for the example Table 1. Right: The numerical results of the signed intensities of signals $W_n^{(1)}$ and $W_n^{(2)}$ for $n = 0, 1, 2, \ldots,$ for the example Table 1.

Then the reconstruction formula in the bond correction method is given by

$$c_{p,2} = 2l \left(t_0^{(2)} - t_0^{(1)} - 1.52 \right)^{-1} \quad (\text{µm/ns}).$$

As a conclusion of this note, we reproduced a model of the bond correction method by using the impulse response for the one-dimensional wave equation. Our construction is based on the propagation of singularities. In order to determine $c_{p,2}$ which is the wave speed of the sample, we have to pick up appropriate echoes. Theoretically, the perturbations by the thin bond layers are exponentially decreasing in time (see Figure 5). One of the measurements is to pick up the time between the 0-th signal in $\mathcal{W}_1$ and the 0-th signal in $\mathcal{W}_2$. Moreover, we derived a formula of the reconstruction of $c_{p,2}$ from the data of the travel time and the length of the sample layer.

We have established a framework for modeling the bond related time delay of waves propagated in a medium. Such models are important for correcting the arrival time of experimentally measured P- and S-wave echoes, which are subsequently used to derive bulk elastic wave velocities and interpret seismic observations. Direct comparison of modeling and experimental measurements however require treatment of noise, which will be addressed in future works.

Acknowledgements. Usually authors are sorted in alphabetical order in arti-

cles of mathematics. On the other hand, in research areas of science and engineering including geophysics, the order of authors is often determined in view of their contributions. In this note, the authors adopt the precedent in geophysics. The arguments in this note were progressed mainly by HM, since they are, for the most part, mathematical fundamentals of our study. SG supported HM in knowledge of geophysics and its experiments. Considering this situation, the authors decided that HM is the first author and SG is the second author of this note.

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Hisashi Morioka
Graduate School of Science and Engineering, Ehime University
Bunkyo-cho 3, Matsuyama, Ehime, 790-8577, Japan
E-mail: morioka.hisashi.ya@ehime-u.ac.jp

Steeve Gréaux
Geodynamics Research Center, Ehime University
Bunkyo-cho 2-5, Matsuyama, Ehime, 790-8577, Japan
E-mail: greaux.steeve_georgi.me@ehime-u.ac.jp