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SUBMANIFOLDS WITH DEGENERATE GAUSS MAPPINGS IN SPHERES

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1. Introduction

This is a survey note of our joint work [19].

An ℓ-dimensional immersed submanifold M in S^n is called tangentially degenerate, if its Gauss mapping

$$\gamma : M \to Gr(\ell+1, \mathbb{R}^{n+1})$$

has rank $r < \ell$, where $r = \max_{p\in M} \operatorname{rank}_p \gamma$. The subset of M on which $\operatorname{rank}_p \gamma$ is constant less than ℓ is foliated by totally geodesic subspheres, along which the tangent space is parallel. The notion of tangential degeneracy is invariant under the projective transformations. Thus, we can consider the problem in the projective space $\mathbb{RP}^n$, but here we concentrate mainly to submanifolds of S^n. To supply examples, we use Riemannian geometry, where this notion is equivalent to the existence of a common null direction of the shape operators.

When M is complete, there exists a number $F(\ell)$ (Ferus number, [11], [12]) such that, if $r < F(\ell)$ then $r = 0$, therefore $M = S^\ell$ and $f(M)$ is a great ℓ-sphere in S^n. The Ferus number $F(\ell)$ is defined by

$$F(\ell) := \min\{k \mid A(k) + k \geq \ell\},$$

where $A(k)$ is the Adams number: the maximal number of linearly independent vector fields over the sphere S^{k-1}, given by $A((2k+1)2^{c+4d}) = 2^c + 8d - 1$, $0 \leq c \leq 3$, $0 \leq d$. The problem we are concerning with is the following:

Problem: Is the inequality $r < F(\ell)$ best possible for the implication $r = 0$? Do there exist tangentially degenerate immersions $M^\ell \to S^n$ with $r = F(\ell)$, M being compact? Moreover can we classify tangentially degenerate immersions $M^\ell \to S^n$ with $r = F(\ell)$ and M compact?

In contrast to the tangentially degenerate submanifolds in $\mathbb{R}^n$ such as cylinders, cones or tangent developable of space curves, all of which have singularities when considered in $\mathbb{RP}^n$, we construct many non-singular compact tangentially degenerate submanifolds in the sphere, some of which even satisfy the Ferus equality. As a byproduct, we obtain many special Lagrangian submanifolds in $\mathbb{C}^{n+1}$, because some of our examples are "austere submanifolds" in the sense of Harvey and Lawson [15].

2. Examples related to isoparametric hypersurfaces

The first author [18] classified the homogeneous tangentially degenerate hypersurfaces in $\mathbb{RP}^n$ or S^n, showing that they are Cartan hypersurfaces, i.e. the projective images of certain tubes of the Veronese surfaces of the four kinds. Suggested by this, we observe examples related to isoparametric hypersurfaces. See [28] for general facts on isoparametric hypersurfaces.

Hypersurfaces in the sphere are tangentially degenerate if they have zero principal curvature. In the simplest case where the principal curvatures are constant, i.e. in the case of isoparametric hypersurfaces, the principal curvatures are given by

$$\lambda_i = \cot\left(\theta_0 + \frac{\pi(i-1)}{g}\right), \quad 0 < \theta_0 < \frac{\pi}{2g}, \quad i = 1, \cdots, g$$

where $g = 1, 2, 3, 4, 6$. Then the tangentially degenerate isoparametric hypersurfaces are

(i) $g = 1$ and M is a great hypersphere
(ii) $g = 3$ and M is the Cartan hypersurfaces ([9]) .

Isoparametric hypersurfaces have two focal submanifolds $M_\pm$. It is well known that all the shape operators S_N of $M_\pm$ have constant eigenvalues given by

(i) 0 for $g = 2$
(ii) $\pm\sqrt{3}$ for $g = 3$
(iii) $\pm 1, 0$ for $g = 4$
(iv) $\pm\sqrt{3}, \pm 1/\sqrt{3}, 0$ for $g = 6$

Remark 2.1 : From these, we know that minimal isoparametric hypersurfaces and their focal submanifolds are austere (§4).

When $g = 2$, $M_\pm$ are totally geodesic subspheres hence tangentially degenerate. Other possibilities are when $g = 4$ or 6. If the kernel of the shape operators have a common non-trivial vector, they are tangentially degenerate. When $g = 6$ and M is homogeneous, both focal submanifolds are tangentially degenerate [24],[26]. Note that they are given by singular orbits of the linear isotropy representation of the rank two symmetric spaces $G_2/SO(4)$ and $G_2 \times G_2/G_2$. Moreover, these satisfy the Ferus equality for $(\ell, r) = (5, 4), (10, 8)$.

When $g = 4$ and M is homogeneous, take the principal curvatures $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ so that m_1 and m_2 are the multiplicities of λ_{odd} and λ_{even} where $m_1 \leq m_2$. Then we obtain

Proposition 2.2. *(3.2 of [19]) Let M be a homogeneous isoparametric hypersurface with $g = 4$ and $(m_1, m_2) = (1, k-2)$, $k \geq 3$, $(2, 2k-3), (4, 4k-5)$, $k \geq 2$. When $(m_1, m_2) = (1, k-2)$, the focal submanifolds M_+ is tangentially degenerate with $(\ell, r) = (2k-3, 2k-4)$; while M_- is not. When $(m_1, m_2) = (2, 2k-3), (4, 4k-5)$,*

M_- is tangentially degenerate with $(\ell, r) = (2k+1, 2k), (4k+3, 4k)$, respectively; while M_+ is not. In particular, there exist infinitely many tangentially degenerate homogeneous submanifolds in the sphere, some of which satisfy the Ferus equality.

On the last assertion, we can easily show that for $p \geq 1$ and $q \geq 2$, $F(2^p+1) = 2^p$ and $F(2^q+3) = 2^q$ hold, hence examples are given by M_- of isoparametric hypersurfaces with $(m_1, m_2) = (2, 2^p - 3)$, $p \geq 2$, and $(4, 2^q - 5)$, $q \geq 3$.

Proposition 2.3. *(3.3 of [19]) Let M be a homogeneous isoparametric hypersurface with $g = 4$. When $(m_1, m_2) = (2, 2)$, the focal submanifolds M_+ is tangentially degenerate with $(\ell, r) = (6, 4)$, satisfying the Ferus equality; while M_- is not. When $(m_1, m_2) = (4, 5)$, M_- is tangentially degenerate with $(\ell, r) = (13, 12)$; while M_+ is not.*

Remark 2.4 : We will discuss the remaining homogeneous case with $(m_1, m_2) = (9, 6)$ in another occasion, as well as all other inhomogeneous examples of Clifford type in [27], [13]. The tangential degeneracy of M_+ for $(m_1, m_2) = (1, k-2)$ and of M_- for $(2, 2k-3), (4, 5)$ follows from Lemma 2.5 below, since the odd dimensional focal submanifolds given as singular orbits of the linear isotropy representation of a Hermitian symmetric space of rank two is tangentially degenerate. See [16], [29].

Lemma 2.5. *(2.2 of [19]) Let $\Sigma^k \subset \mathbb{CP}^n$ be a complex submanifold of complex dimension k. Consider the Hopf fibration $\pi : S^{2n+1}(\subset \mathbb{C}^{n+1}) \to \mathbb{CP}^n$, and set $M^{2k+1} := \pi^{-1}(\Sigma) \subset S^{2n+1}$. Then M is a submanifold with degenerate Gauss mapping of S^{2n+1}. If Σ is compact and not a complex projective subspace, then the rank of Gauss mapping is equal to $2k$.*

This is a consequence of the following important fact on the tangential degeneracy in complex version:

Theorem 2.6. [2][14] *Let Σ^k be a k-dimensional compact complex submanifold in $\mathbb{CP}^n$ and let $\gamma_{\mathbb{C}} : \Sigma \to G_{k+1}(\mathbb{C}^{n+1})$ be the complex Gauss mapping of Σ in $\mathbb{CP}^n$. If the rank of $\gamma_{\mathbb{C}}$ is less than* $\dim_{\mathbb{C}} \Sigma$, *then Σ is necessarily a complex projective subspace $\mathbb{CP}^k$ in $\mathbb{CP}^n$.*

3. Stiefel and Grassmannian manifolds

Here we introduce an outline of the second author's construction of minimal immersions from a circle bundle over surfaces [22], and show how to get tangentially degenerate submanifolds.

Let W be a real vector space with Euclidean inner product $\langle\ ,\ \rangle$. By an *2-frame* in W we mean an ordered set of 2 orthonormal vectors in W. Let $V_2(W)$ be the space of 2-frames in W, i.e.,

$$V_2(W) = \{(\mathrm{f}_1, \mathrm{f}_2) \in W \times W \mid \langle \mathrm{f}_\alpha, \mathrm{f}_\beta \rangle = \delta_{\alpha\beta}\ (\alpha, \beta = 1, 2)\}. \tag{1}$$

Then $V_2(W)$ is a Stiefel manifold with $\dim_{\mathbb{R}} V_2(W) = 2\dim_{\mathbb{R}} W - 3$. Let $\widetilde{G}_2(W)$ be the space of oriented 2-planes in W. Then $V_2(W)$ is a principal fiber bundle over $\widetilde{G}_2(W)$ with structure group S^1 and projection map $\pi : V_2(W) \to \widetilde{G}_2(W)$ defined by

$$\pi((\mathbf{f}_1, \mathbf{f}_2)) = \operatorname{span}\{\mathbf{f}_1, \mathbf{f}_2\}.$$

Let $\widetilde{Q}(\mathbb{C}^{m+1})$ be a submanifold of $S^{2m+1}(\sqrt{2})$ defined by

$$\widetilde{Q}(\mathbb{C}^{m+1}) = \{\mathbf{z} \in S^{2m+1}(\sqrt{2}) \mid {}^t\mathbf{z}\mathbf{z} = 0\}. \tag{2}$$

There is an identification between $\widetilde{Q}(\mathbb{C}^{m+1})$ and $V_2(\mathbb{R}^{m+1})$ as:

$$\widetilde{Q}(\mathbb{C}^{m+1}) \ni \mathbf{z} \mapsto (\operatorname{Re}\mathbf{z}, \operatorname{Im}\mathbf{z}) \in V_2(\mathbb{R}^{m+1}).$$

Then $\widetilde{G}_2(\mathbb{R}^{m+1})$ is identified with the complex quadric

$$Q^{m-1} = \{\pi(\mathbf{z}) \in \mathbb{C}P^m \mid \mathbf{z} \in \widetilde{Q}(\mathbb{C}^{m+1})\}, \tag{3}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{Q}(\mathbb{C}^{m+1}) & \xrightarrow{\sim} & V_2(\mathbb{R}^{m+1}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Q^{m-1} & \xrightarrow{\sim} & \widetilde{G}_2(\mathbb{R}^{m+1}). \end{array}$$

Let $\varphi : \Sigma \to Q^{n-1} \cong \widetilde{G}_2(\mathbb{R}^{n+1})$ be a mapping from a differentiable manifold Σ with $\dim_{\mathbb{R}} \Sigma = \ell$, and let $\pi_\varphi : \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$ be the pullback bundle of the circle bundle $\pi : V_2(\mathbb{R}^{n+1}) \to \widetilde{G}_2(\mathbb{R}^{n+1})$ with respect to φ:

$$\begin{array}{ccc} \varphi^* V_2(\mathbb{R}^{n+1}) & \xrightarrow{\psi} & V_2(\mathbb{R}^{n+1}) \\ {\scriptstyle\pi_\varphi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \Sigma & \xrightarrow{\varphi} & \widetilde{G}_2(\mathbb{R}^{n+1}). \end{array} \tag{4}$$

Let $\Phi : \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n(1)$ be the mapping defined by

$$\Phi = \operatorname{pr}_1 \circ \psi, \tag{5}$$

where $\psi : \varphi^* V_2(\mathbb{R}^{n+1}) \to V_2(\mathbb{R}^{n+1})$ is the bundle mapping in (4) and $\operatorname{pr}_1 : V_2(\mathbb{R}^{n+1}) \to S^n(1)$ is the projection given by

$$\operatorname{pr}_1(\mathbf{f}_1, \mathbf{f}_2) = \mathbf{f}_1.$$

Then we have

$$\Phi(\varphi^* V_2(\mathbb{R}^{n+1})) = \bigcup_{p \in \Sigma} \{\cos\theta \mathbf{f}_1 + \sin\theta \mathbf{f}_2 \mid \pi(\mathbf{f}_1, \mathbf{f}_2) = \varphi(p), \theta \in S^1\}.$$

Hence $\Phi(\varphi^* V_2(\mathbb{R}^{n+1}))$ is a union of (real) ℓ-parameter family of great circles in $S^n(1)$. Taking a local section η of $\pi : V_2(\mathbb{R}^{n+1}) \to \widetilde{G}_2(\mathbb{R}^{n+1})$, denote

$$(\eta \circ \varphi)(q) = (\mathbf{f}_1(q), \mathbf{f}_2(q)) \quad \text{for} \quad q \in U \subset \Sigma. \tag{6}$$

Here $\mathbf{f}_\alpha$ is an $\mathbb{R}^{n+1}$-valued function on U with $\langle \mathbf{f}_\alpha, \mathbf{f}_\beta \rangle = \delta_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$. Write differential maps of $\mathbf{f}_\alpha : U \to \mathbb{R}^{n+1}$ $(\alpha = 1, 2)$ as

$$d\mathbf{f}_1(X) = \lambda(X)\mathbf{f}_2 + \mathbf{p}(X), \ d\mathbf{f}_2(X) = -\lambda(X)\mathbf{f}_1 + \mathbf{q}(X) \quad \text{for} \quad X \in T_q(\Sigma), \tag{7}$$

where λ is a 1-form on U, and $\mathbf{p}$, $\mathbf{q}$ are $\mathbb{R}^{n+1}$-valued 1-forms on U such that $\mathbf{p}(X), \mathbf{q}(X) \perp \operatorname{span}\{\mathbf{f}_1, \mathbf{f}_2\}$. Then the differentials of $\eta \circ \varphi$ and φ are given by

$$\begin{aligned} d(\eta \circ \varphi)(X) &= (d\mathbf{f}_1(X), d\mathbf{f}_2(X)) \\ &= (\lambda(X)\mathbf{f}_2 + \mathbf{p}(X), -\lambda(X)\mathbf{f}_1 + \mathbf{q}(X)), \\ d\varphi(X) &= (d\pi \circ d(\eta \circ \varphi))(X) \\ &= d\pi(\mathbf{p}(X), \mathbf{q}(X)). \end{aligned} \tag{8}$$

Let $\mathbf{e}_1, \cdots, \mathbf{e}_\ell$ be an orthonormal basis of the tangent space $T_q(\Sigma)$ at $q \in U \subset \Sigma$. With respect to $\mathbb{R}^{n+1}$-valued 1-forms $\mathbf{p}$, $\mathbf{q}$ on $U \subset \Sigma$ defined by (8), denote $\mathbf{p}(\mathbf{e}_j) = \mathbf{p}_j$ and $\mathbf{q}(\mathbf{e}_j) = \mathbf{q}_j$ for $j = 1, \cdots, \ell$, and put

$$\Psi_j = \cos\theta \mathbf{p}_j + \sin\theta \mathbf{q}_j \in T_{\Phi_U(\theta,q)}(S^n) \quad \text{for} \quad j = 1, \cdots, \ell. \tag{9}$$

In particular when (Σ^m, J) is a Kähler manifold with $\dim_{\mathbb{C}} \Sigma = m$ and $\varphi : \Sigma^m \to Q^{n-1}$ $(m < n-1)$ is a holomorphic isometric immersion, let $\{\mathbf{e}_{2k-1}, \mathbf{e}_{2k} = J\mathbf{e}_{2k-1} \mid k = 1, \cdots, m\}$ be an orthonormal basis of the tangent space $T_q(\Sigma^m)$ at $q \in U \subset \Sigma^m$. Then we obtain

$$\begin{aligned} \Psi_{2k-1} \wedge \Psi_{2k} &= (\cos\theta \mathbf{p}_{2k-1} + \sin\theta \mathbf{q}_{2k-1}) \wedge (\cos\theta \mathbf{p}_{2k} + \sin\theta \mathbf{q}_{2k}) \\ &= (\cos\theta \mathbf{p}_{2k-1} - \sin\theta \mathbf{p}_{2k}) \wedge (\cos\theta \mathbf{p}_{2k} + \sin\theta \mathbf{p}_{2k-1}) \\ &= \mathbf{p}_{2k-1} \wedge \mathbf{p}_{2k}, \quad (k = 1, \cdots, m) \end{aligned}$$

and $\Psi_1 \wedge \cdots \wedge \Psi_{2m} = \mathbf{p}_1 \wedge \cdots \wedge \mathbf{p}_{2m}$. We have:

Proposition 3.1. *(6.2 of* [19]*) Let (Σ^m, J) be a Kähler manifold of $\dim_{\mathbb{C}} \Sigma = m$ and let $\varphi : \Sigma^m \to Q^{n-1}$ $(m < n-1)$ be a holomorphic immersion. Then the mapping $\Phi : \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n(1)$ defined by* (5) *is non-singular at each point in $\pi_\varphi^{-1}(q)$ if and only if at $q \in \Sigma^m$, φ satisfies*

$$\mathbf{p}_1 \wedge \cdots \wedge \mathbf{p}_{2m} \neq 0. \tag{10}$$

Suppose that a holomorphic immersion $\varphi : \Sigma^m \to Q^{n-1}$ satisfies (10) at each point of Σ^m. Let $V = d\phi_U^{-1}(\partial/\partial\theta, 0)$ be a tangent vector of the fiber $\pi_\varphi^{-1}(q)$ of the submersion $\pi_\varphi : \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$ at $q \in \Sigma$. Then $\mathbb{R} d\Phi(V) + \operatorname{span}\{\Psi_j | j =$

$1, \cdots, 2m\} = \mathbb{R}d\Phi_U(\partial/\partial\theta, 0) + \{d\Phi_U(0, X) | X \in T_q(\Sigma)\}$. Denote σ^Φ the second fundamental form of the immersion $\Phi : \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n(1)$. Since each fiber $\pi_\varphi^{-1}(q)$ is a great circle of $S^n(1)$, we have

$$\sigma^\Phi(V, V) = 0. \tag{11}$$

On the other hand, if we denote D the Euclidean connection of $\mathbb{R}^{n+1}$, then using (9) we get

$$D_{\partial/\partial\theta}\Psi_{2k-1} = -\sin\theta\mathbf{p}_{2k-1} - \cos\theta\mathbf{p}_{2k},$$
$$D_{\partial/\partial\theta}\Psi_{2k} = -\sin\theta\mathbf{p}_{2k} + \cos\theta\mathbf{p}_{2k-1},$$

$(k = 1, \cdots, m)$ and both of these terms are contained in the tangent space of $\Phi(\varphi^* V_2(\mathbb{R}^{n+1}))$. Hence we obtain

$$\sigma^\Phi(V, d\phi_U^{-1}(0, X)) = 0 \quad \text{for} \quad X \in T_q(\Sigma). \tag{12}$$

From these, we obtain a generalization of Lemma 2.5:

Theorem 3.2. *(6.4 of [19]) Let $\varphi : \Sigma^m \to Q^{n-1}$ $(m < n-1)$ be a holomorphic immersion from a Kähler manifold Σ^m to the complex quadric for which (10) holds. Then with respect to the immersion $\Phi : \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n(1)$ given by (5), any tangent line of the fiber $\pi_\varphi^{-1}(p)$ at each $u \in \pi_\varphi^{-1}(p)$, $p \in \Sigma^m$ lies in the kernel of differential of the Gauss mapping of φ Hence Φ is tangentially degenerate.*

Finally, we give examples obtained in this context. Let $M(m+1, \mathbb{K})$ be the space of all $(m+1) \times (m+1)$ matrices, and let

$$\mathrm{Sym}^{\mathbb{K}}(m+1) = \{A \in M(m+1, \mathbb{K}) \mid {}^tA = A\},$$
$$\mathrm{Sym}_0^{\mathbb{K}}(m+1) = \{A \in \mathrm{Sym}^{\mathbb{K}}(m+1) \mid \mathrm{trace}\, A = 0\},$$

where $\mathbb{K} = \mathbb{C}, \mathbb{R}$. The inner product is given by

$$\langle A_1, A_2 \rangle = \frac{1}{2}\mathrm{Re}(\mathrm{trace}(A_1 A_2^*)). \tag{13}$$

Define a mapping $\widetilde{\varphi}_m : S^{2m+1}(\sqrt{2}) \to \mathrm{Sym}^{\mathbb{C}}(m+1)$ as

$$\widetilde{\varphi}_m(\mathbf{z}) = \frac{1}{\sqrt{2}}\mathbf{z}^t\mathbf{z} = \frac{1}{\sqrt{2}}\begin{pmatrix} z_0^2 & z_0 z_1 & \cdots & z_0 z_m \\ z_1 z_0 & z_1^2 & \cdots & z_1 z_m \\ \vdots & \vdots & \ddots & \vdots \\ z_m z_0 & z_m z_1 & \cdots & z_m^2 \end{pmatrix}. \tag{14}$$

for $\mathbf{z} = (z_j) \in S^{2m+1}(\sqrt{2})$. Then it can be verified that $\langle \widetilde{\varphi}_m(\mathbf{z}), \widetilde{\varphi}_m(\mathbf{z}) \rangle = 1$ and $\widetilde{\varphi}_m$ induces a mapping φ_m of $\mathbb{CP}^m$ into $\mathbb{P}(\mathrm{Sym}^{\mathbb{C}}(m+1)) \cong \mathbb{CP}^{m(m+3)/2}$ such that the following diagram is commutative:

$$\begin{array}{ccc} S^{2m+1}(\sqrt{2}) & \xrightarrow{\widetilde{\varphi}_m} & S_1(\mathrm{Sym}^{\mathbb{C}}(m+1)) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{CP}^m & \xrightarrow{\varphi_m} & \mathbb{P}(\mathrm{Sym}^{\mathbb{C}}(m+1)). \end{array}$$

φ_m is nothing but the *complex Veronese embedding*.

If we restrict $\widetilde{\varphi}_m$ to the submanifold $\widetilde{Q}(\mathbb{C}^{m+1})$ which is given by (2), then the image $\widetilde{\varphi}_m(\widetilde{Q}(\mathbb{C}^{m+1}))$ is contained in the submanifold

$$\widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) = \{A \in S_1(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \mid \mathrm{trace}\, A^2 = 0\}. \tag{15}$$

Putting $W = \mathrm{Sym}_0^{\mathbb{C}}(m+1)$ in the beginning of this section, we can identify

$$\begin{array}{ccc} \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) & \xrightarrow{\sim} & V_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1)) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) & \xrightarrow{\sim} & \widetilde{G}_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1)), \end{array}$$

where we can write

$$Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) = \{\pi(A) \in \mathbb{P}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \mid \mathrm{trace}\, A^2 = 0.\}$$

Considering a suitable Riemannian metric induced from (13), we have the following commutative diagram :

$$\begin{array}{ccc} \widetilde{Q}(\mathbb{C}^{m+1}) & \xrightarrow{\widetilde{\varphi}_m} & \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Q^{m-1} & \xrightarrow{\varphi_m} & Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1)). \end{array} \tag{16}$$

Then we can show that the holomorphic embedding $\varphi_m : Q^{m-1} \to Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ satisfies (10). Therefore, applying Theorem 3.2, we obtain homogeneous examples of t angentially degenerate submanifolds : $\Phi : \varphi_m^* V_2(\mathbb{R}^{m+1}) \to S^{(m^2+3m-2)/2}(1)$.

4. Austere submanifolds in spheres.

A submanifold M in a Riemannian manifold is called *austere* [15] if for each normal vector ξ, the set of eigenvalues of the shape operator A_ξ is invariant under multiplication of -1. Clearly austere submanifolds are minimal, and they are closely related to *special Lagrangian submanifolds* (see also [7]). In fact, Harvey and Lawson showed (Theorem 3.17 in [15]) that from any compact austere submanifold of S^n, one can construct an n-dimensional cone of least mass in $\mathbb{R}^{2n+2}$. In this section we will show that if Σ is a complex submanifold of *first order isotropic* in Q^{n-1}, then the corresponding submanifold M in S^n with 2-parameter family of great circles is austere, as well as tangentially degenerate.

We will use notations of §3. Let $(\Sigma^m, \langle\ ,\ \rangle, J)$ be a Kähler manifold of $\dim_{\mathbb{C}} \Sigma = m$ and let $\varphi : \Sigma^m \to Q^{n-1} \cong G_2(\mathbb{R}^{n+1})$ $(m < n-1)$ be a holomorphic isometric immersion. We say that holomorphic immersion $\varphi : \Sigma \to Q^{n-1}$ is *first order isotropic* if

$$\langle \mathbf{p}(X), \mathbf{p}(Y) \rangle = \frac{1}{2}\langle X, Y \rangle. \tag{17}$$

This condition is independent of the choice of local cross section $\eta' : U \to \varphi^2 V_2(\mathbb{R}^{n+1})$ for an open subset $U \subset \Sigma$. Moreover, we see that

Proposition 4.1. *(8.1 of* [19]*) Let $\varphi : \Sigma \to Q^{n-1}$ be a holomorphic isometric immersion of a Kähler manifold and let $\iota : Q^{n-1} \to \mathbb{CP}^n$ be the inclusion. Then φ is first order isotropic if and only if $d(\iota \circ \varphi)(X)$ is an isotropic vector for each $X \in T\Sigma$.*

Moreover, we can show

Proposition 4.2. *(8.2 of* [19]*) Let $\varphi : \Sigma \to Q^{n-1}$ be a first order isotropic holomorphic isometric immersion from a Kähler manifold to the complex quadric, and let $\Phi : \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n$ be the corresponding immersion defined by (5). Then restriction of the differential of the projection $\pi_\varphi : \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$ to the horizontal subspaces is a homothety with respect to the metric on $\varphi^* V_2(\mathbb{R}^{n+1})$ induced by Φ.*

Theorem 4.3. *(8.3 of* [19]*) Let $\varphi : \Sigma \to Q^{n-1}$ be a first order isotropic holomorphic isometric immersion from a Kähler manifold to the complex quadric. Then the corresponding immersion $\Phi : \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n$ defined by (5) is austere.*

Remark 4.4 : It is well-known (cf. [8], [21]) that there is a one-to-one correspondence between *totally isotropic holomorphic curves* in Q^{2m-1} and *pseudoholomorphic surfaces (superminimal surfaces, or isotropic minimal surfaces)* in S^{2m}. Hence from minimal 2-spheres in S^{2m}, we can construct 3-dimensional austere submanifolds in S^{2m} [22].

5. Further examples of tangentially degenerate submanifolds

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and let $\mathbf{x} \in \mathbb{K}^{n+1}$ be a column vector. The usual inner product on $\mathbb{K}^{n+1} = \mathbb{R}^{(n+1)d}$, $d = 1, 2, 4$, respectively, is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathrm{Re}(\mathbf{x}^* \mathbf{y}) \quad \text{for} \quad \mathbf{x}, \mathbf{y} \in \mathbb{K}^{n+1}$$

where $\mathrm{Re}(\mathbf{x}^*\mathbf{y})$ denotes the real part of $\mathbf{x}^*\mathbf{y}$. Let $\pi : S^{(n+1)d-1} \to \mathbb{KP}^n$ be the Hopf fibration and denote $\pi(\mathbf{x}) = [\mathbf{x}] \in \mathbb{KP}^n$ for $\mathbf{x} \in S^{(n+1)d-1}$. Then the canonical metric in $\mathbb{KP}^n$ is the invariant metric such that π is a Riemannian submersion. Let

$$V_2(\mathbb{K}^{n+1}) = \{(\mathbf{u}_1, \mathbf{u}_2) \in S^{(n+1)d-1} \times S^{(n+1)d-1} \mid \mathbf{u}_1^* \mathbf{u}_2 = 0\}$$

be the Stiefel manifold over $\mathbb{K}$. Then the tangent space $T_{(\mathbf{u}_1,\mathbf{u}_2)}(V_2(\mathbb{K}^{n+1}))$ at $(\mathbf{u}_1,\mathbf{u}_2)\in V_2(\mathbb{K}^{n+1})$ is

$$\begin{aligned} T_{(\mathbf{u}_1,\mathbf{u}_2)}(V_2(\mathbb{K}^{n+1})) = \{(\mathbf{x}_1,\mathbf{x}_2)\in \mathbb{K}^{n+1}\times\mathbb{K}^{n+1} \mid \\ \langle \mathbf{x}_1,\mathbf{u}_1\rangle = \langle \mathbf{x}_2,\mathbf{u}_2\rangle = 0,\ \mathbf{x}_1^*\mathbf{u}_2+\mathbf{u}_1^*\mathbf{x}_2=0\}. \end{aligned}$$

Define the subspaces $T_0(\mathbf{u}_1,\mathbf{u}_2)$, $T_\lambda(\mathbf{u}_1,\mathbf{u}_2)$, $T_\mu(\mathbf{u}_1,\mathbf{u}_2)$ of $T_{(\mathbf{u}_1,\mathbf{u}_2)}(V_2(\mathbb{K}^{n+1}))$ as

$$\begin{aligned} T_0(\mathbf{u}_1,\mathbf{u}_2) &= \{(\mathbf{x}_1,\mathbf{x}_2)\in\mathbb{K}^{n+1}\times\mathbb{K}^{n+1} \mid \mathbf{x}_\alpha^*\mathbf{u}_\beta = 0 \text{ for } \alpha,\beta=1,2\} \\ T_\lambda(\mathbf{u}_1,\mathbf{u}_2) &= \{(-\mathbf{u}_2\bar\lambda,\mathbf{u}_1\lambda)\in\mathbb{K}^{n+1}\times\mathbb{K}^{n+1} \mid \lambda\in\mathbb{K}\} \\ T_\mu(\mathbf{u}_1,\mathbf{u}_2) &= \{(\mathbf{u}_1\mu_1,\mathbf{u}_2\mu_2)\in\mathbb{K}^{n+1}\times\mathbb{K}^{n+1} \mid \mu_1,\mu_2\in\operatorname{Im}\mathbb{K}\}. \end{aligned}$$

Then we have $T_{(\mathbf{u}_1,\mathbf{u}_2)}(V_2(\mathbb{K}^{n+1})) = T_0(\mathbf{u}_1,\mathbf{u}_2)\oplus T_\lambda(\mathbf{u}_1,\mathbf{u}_2)\oplus T_\mu(\mathbf{u}_1,\mathbf{u}_2)$.

Put

$$F_2(\mathbb{K}^{n+1}) = \{([\mathbf{u}_1],[\mathbf{u}_2])\in\mathbb{KP}^n\times\mathbb{KP}^n \mid \mathbf{u}_1^*\mathbf{u}_2=0\}.$$

Then $V_2(\mathbb{K}^{n+1})$ is a principal fiber bundle over $F_2(\mathbb{K}^{n+1})$ with structure group $S^{d-1}\times S^{d-1}$ and projection map $\pi : V_2(\mathbb{K}^{n+1})\to F_2(\mathbb{K}^{n+1})$ defined by

$$\pi(\mathbf{u}_1,\mathbf{u}_2) = ([\mathbf{u}_1],[\mathbf{u}_2]).$$

We mention that $F_2(\mathbb{K}^{n+1})$ is considered as the total space of "tautological $\mathbb{KP}^1$-bundle" over 2-plane Grassmann manifold $\widetilde{G}_2(\mathbb{K}^{n+1})$ with projection $\pi(\mathbf{u}_1,\mathbf{u}_2)\mapsto \operatorname{span}_{\mathbb{K}}\{\mathbf{u}_1,\mathbf{u}_2\}$.

Let $\mathfrak{M}^{\mathbb{K}}(n+1)$ be the space of all $(n+1)\times(n+1)$ matrices over $\mathbb{K}$. The inner product on $\mathfrak{M}^{\mathbb{K}}(n+1)=\mathbb{R}^{(n+1)^2d}$ is defined as

$$\langle A,B\rangle = \frac{1}{2}\operatorname{Re}\operatorname{trace}(AB^*) \quad \text{for} \quad A,B\in\mathfrak{M}^{\mathbb{K}}(n+1).$$

Let

$$\begin{aligned} \operatorname{Herm}^{\mathbb{K}}(n+1) &= \{A\in\mathfrak{M}^{\mathbb{K}}(n+1) \mid A^*=A\} \\ \operatorname{Herm}_0^{\mathbb{K}}(n+1) &= \{A\in\operatorname{Herm}^{\mathbb{K}}(n+1) \mid \operatorname{trace}A=0\} \\ S(\operatorname{Herm}_0^{\mathbb{K}}(n+1)) &= \{A\in\operatorname{Herm}_0^{\mathbb{K}}(n+1) \mid \langle A,A\rangle=1\} \\ U^{\mathbb{K}}(n+1) &= \{P\in\mathfrak{M}^{\mathbb{K}}(n+1) \mid P^*P=E\} \end{aligned}$$

where $A^*={}^t\bar{A}$ and E is the identity matrix. $U^{\mathbb{K}}(n+1)$ acts on $V_2(\mathbb{K}^{n+1})$, $F_2(\mathbb{K}^{n+1})$ and $\widetilde{G}_2(\mathbb{K}^{n+1})$ transitively as

$$\begin{aligned} P\cdot(\mathbf{u}_1,\mathbf{u}_2) &= (P\mathbf{u}_1,P\mathbf{u}_2) \quad \text{for } (\mathbf{u}_1,\mathbf{u}_2)\in V_2(\mathbb{K}^{n+1}) \\ P\cdot\pi(\mathbf{u}_1,\mathbf{u}_2) &= \pi(P\mathbf{u}_1,P\mathbf{u}_2) \quad \text{for } \pi(\mathbf{u}_1,\mathbf{u}_2)\in F_2(\mathbb{K}^{n+1}) \\ P\cdot\operatorname{span}_{\mathbb{K}}\{\mathbf{u}_1,\mathbf{u}_2\} &= \operatorname{span}_{\mathbb{K}}\{P\mathbf{u}_1,P\mathbf{u}_2\} \quad \text{for } \operatorname{span}_{\mathbb{K}}\{\mathbf{u}_1,\mathbf{u}_2\}\in\widetilde{G}_2(\mathbb{K}^{n+1}) \end{aligned}$$

and $P \in U^{\mathbb{K}}(n+1)$. Hence as homogeneous spaces, we have

$$V_2(\mathbb{K}^{n+1}) = U^{\mathbb{K}}(n+1)/U^{\mathbb{K}}(n-1)$$
$$F_2(\mathbb{K}^{n+1}) = U^{\mathbb{K}}(n+1)/U^{\mathbb{K}}(1) \times U^{\mathbb{K}}(1) \times U^{\mathbb{K}}(n-1)$$
$$\widetilde{G}_2(\mathbb{K}^{n+1}) = U^{\mathbb{K}}(n+1)/U^{\mathbb{K}}(2) \times U^{\mathbb{K}}(n-1).$$

Define a map $\widetilde{\varphi}_n^{\mathbb{K}} : V_2(\mathbb{K}^{n+1}) \to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1))$ as follows:

$$\widetilde{\varphi}_n^{\mathbb{K}}(\mathbf{u}_1, \mathbf{u}_2) = \mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^* \quad \text{for} \quad (\mathbf{u}_1, \mathbf{u}_2) \in V_2(\mathbb{K}^{n+1}). \tag{18}$$

Then it is easily verified that $\widetilde{\varphi}_n^{\mathbb{K}}$ gives a map $\varphi_n^{\mathbb{K}} : F_2(\mathbb{K}^{n+1}) \to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1))$ such that $\widetilde{\varphi}_n^{\mathbb{K}} = \varphi_n^{\mathbb{K}} \circ \pi$. For simplicity, we denote $\varphi = \varphi_n^{\mathbb{K}}$.

$U^{\mathbb{K}}(n+1)$ acts on $\mathrm{Herm}_0^{\mathbb{K}}(n+1)$ orthogonally as

$$P(A) = PAP^* \quad \text{for} \quad P \in U^{\mathbb{K}}(n+1),\ A \in \mathrm{Herm}_0^{\mathbb{K}}(n+1).$$

The map φ is equivariant, and we can show that it is an embedding. Moreover, we see that $T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$ is the kernel of all the shape operators, where $\dim T_\lambda(\mathbf{u}_1, \mathbf{u}_2) = d$.

Our examples $\varphi : F_2(\mathbb{K}^{n+1}) \to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1)) = S^m$ satisfying Ferus equality $r = F(\ell)$ (put $\nu = \ell - r$) for $\ell = \dim_{\mathbb{R}} F_2(\mathbb{K}^{n+1}) \leq 32$ are as follows:

Embedding	ℓ	r	ν	m
$\varphi_2^{\mathbb{R}} : F_2(\mathbb{R}^3) \to S(\mathrm{Herm}_0^{\mathbb{R}}(3))$	3	2	1	4
$\varphi_3^{\mathbb{R}} : F_2(\mathbb{R}^4) \to S(\mathrm{Herm}_0^{\mathbb{R}}(4))$	5	4	1	8
$\varphi_2^{\mathbb{C}} : F_2(\mathbb{C}^3) \to S(\mathrm{Herm}_0^{\mathbb{C}}(3))$	6	4	2	7
$\varphi_5^{\mathbb{R}} : F_2(\mathbb{R}^6) \to S(\mathrm{Herm}_0^{\mathbb{R}}(6))$	9	8	1	19
$\varphi_3^{\mathbb{C}} : F_2(\mathbb{C}^4) \to S(\mathrm{Herm}_0^{\mathbb{C}}(4))$	10	8	2	14
$\varphi_2^{\mathbb{H}} : F_2(\mathbb{H}^3) \to S(\mathrm{Herm}_0^{\mathbb{H}}(3))$	12	8	4	13
$\varphi_9^{\mathbb{R}} : F_2(\mathbb{R}^{10}) \to S(\mathrm{Herm}_0^{\mathbb{R}}(10))$	17	16	1	53
$\varphi_5^{\mathbb{C}} : F_2(\mathbb{C}^6) \to S(\mathrm{Herm}_0^{\mathbb{C}}(6))$	18	16	2	33
$\varphi_3^{\mathbb{H}} : F_2(\mathbb{H}^4) \to S(\mathrm{Herm}_0^{\mathbb{H}}(4))$	20	16	4	26
$\varphi_{13}^{\mathbb{R}} : F_2(\mathbb{R}^{14}) \to S(\mathrm{Herm}_0^{\mathbb{R}}(14))$	25	24	1	103
$\varphi_7^{\mathbb{C}} : F_2(\mathbb{C}^8) \to S(\mathrm{Herm}_0^{\mathbb{C}}(8))$	26	24	2	62
$\varphi_4^{\mathbb{H}} : F_2(\mathbb{H}^5) \to S(\mathrm{Herm}_0^{\mathbb{H}}(5))$	28	24	4	43

Note that $\varphi_2^{\mathbb{K}} : F_2(\mathbb{K}^3) \to S(\mathrm{Herm}_0^{\mathbb{K}}(3)) = S^{3d+1}$ is nothing but the Cartan's isoparametric (minimal) hypersurfaces, while other examples are not obtained from isoparametric hypersurfaces.

Moreover, we obtain

Proposition 5.1. *The embedding φ is tangentially degenerate, austere and minimal.*

Remark 5.2 : When $\mathbb{K} = \mathbb{R}$, the submanifold $F_2(\mathbb{R}^{n+1})$ is given in §3. Also when $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{H}$, the submanifold $F_2(\mathbb{K}^{n+1})$ is obtained as a pull-back bundle over

2-plane Grassmannian $G_2(\mathbb{K}^{n+1})$ of canonical d-dimensional sphere bundle over real Grassmannian $G^{\mathbb{R}}_{d+1}(\mathrm{Herm}^{\mathbb{K}}_0(n+1))$ with respect to the embedding:

$$\begin{gathered}
\text{Case } \mathbb{K} = \mathbb{C}: \qquad G_2(\mathbb{C}^{n+1}) \to G^{\mathbb{R}}_3(\mathrm{Herm}^{\mathbb{C}}_0(n+1)),\\
\mathrm{span}_{\mathbb{C}}\{\mathbf{u}, \mathbf{v}\} \mapsto \mathrm{span}_{\mathbb{R}}\{\mathbf{uu}^* - \mathbf{vv}^*, \mathbf{uv}^* + \mathbf{vu}^*, \mathbf{i}(\mathbf{uv}^* - \mathbf{vu}^*)\}\\
\text{Case } \mathbb{K} = \mathrm{H}: \qquad G_2(\mathrm{H}^{n+1}) \to G^{\mathbb{R}}_5(\mathrm{Herm}^{\mathrm{H}}_0(n+1)),\\
\mathrm{span}_{\mathrm{H}}\{\mathbf{u}, \mathbf{v}\} \mapsto \mathrm{span}_{\mathbb{R}}\{\mathbf{uu}^* - \mathbf{vv}^*, \mathbf{uv}^* + \mathbf{vu}^*,\\
\mathbf{uiv}^* - \mathbf{viu}^*, \mathbf{ujv}^* - \mathbf{vju}^*, \mathbf{ukv}^* - \mathbf{vku}^*\},
\end{gathered}$$

where $\mathbf{u}^*\mathbf{u} = \mathbf{v}^*\mathbf{v} = 1$ and $\mathbf{u}^*\mathbf{v} = 0$. For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and H, $\varphi : F_2(\mathbb{K}^{n+1}) \to S(\mathrm{Herm}^{\mathbb{K}}_0(n+1))$ satisfies B. Y. Chen's equality (cf. Theorem 4.1 in [10])

$$\begin{gathered}
\delta((n-1)d, (n-1)d) = (n-1)(nd+1)d\\
(n = (2n-1)d,\ n_1 = n_2 = (n-1)d,\ H = 0,\ \epsilon = 1).
\end{gathered}$$

6. Hypersurfaces with degenerate Gauss mappings in the four dimensional sphere

In what follows, we study the simplest case $n = 4, \ell = 3, r = F(3) = 2$.

Recall that the Cartan hypersurface $M^3 \subset S^4$ is a homogeneous space of $SO(3)$ and written as $M = O(3)/(O(1) \times O(1) \times O(1))$. The Gauss mapping $\gamma : M^3 \to G_4(\mathbb{R}^5) \cong G_1(\mathbb{R}^{5*}) = \mathbb{RP}^{4*}$ into the dual projective space, has the constant rank 2. Moreover its image $\gamma(M)$, that is the projective dual in this case, is a linear projection $\mathbb{RP}^2 \subset \mathbb{RP}^{4*}$ of the Veronese surface $\mathbb{RP}^2 \subset \mathbb{RP}^{5*}$ in the sense of algebraic geometry [18]: The Veronese surface has the crucial property that its secant variety is of positive codimension in $\mathbb{RP}^{5*}$ (cf. [30]). Notice that it lifts to the Veronese surface $i : \mathbb{RP}^2 \hookrightarrow \widetilde{G}_4(\mathbb{R}^5) \cong \widetilde{G}_1(\mathbb{R}^{5*}) = S^{4*}$, in the sense of differential geometry. The liftability means just that M is orientable. Consider the double covering $\pi : S^2 \to \mathbb{RP}^2$ and take the fiber product $\widetilde{M}$ of π and $\gamma : M \to \mathbb{RP}^2$:

$$\begin{array}{ccccc}
\widetilde{M} & \longrightarrow & S^2 & & \\
{\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \pi} & & \\
M & \xrightarrow{\ \gamma\ } & \mathbb{RP}^2 & \xrightarrow{\ i\ } & S^{4*}.
\end{array}$$

We call $\widetilde{M}$ the *doubled Cartan hypersurface.* Then we have the tangentially degenerate immersion $\widetilde{M} \to S^4$, that is the composition of the double covering $\Pi : \widetilde{M} \to M$ and the inclusion $M \subset S^4$. Remark that $\widetilde{M}$ is connected and realized by $O(3)/(SO(1) \times O(1) \times O(1))$ as a homogeneous space. Also remark, by the spherical-projective duality, that $\widetilde{M}$ is the total space of the associated $\widetilde{G}_1(\mathbb{R}^2) = S^1$ bundle over S^2 to the normal bundle of the immersion $i \circ \pi : S^2 \to S^{4*}$.

Then we give the following characterization of the diffeomorphism type, up to finite coverings, of compact connected tangentially degenerate hypersurfaces in S^4, using the result of Asperti [3].

Theorem 6.1. *(10.1 of* [19]*) Let M' be a compact connected* 3*-dimen- sional manifold, and $f' : M'^3 \to S^4$ a tangentially degenerate immersion. Assume that the rank of the Gauss mapping of f' is everywhere* 2*. Then, there exists a finite covering mapping $\widetilde{M} \to M'$ from the doubled Cartan hypersurface $\widetilde{M}$, and therefore, there exists a tangentially degenerate immersion $f : \widetilde{M} \to S^4$ with $f(\widetilde{M}) = f'(M')$.*

Lastly we proceed to construct an example of tangentially degenerate immersions from a compact submanifold M of dimension 3 to S^4, the rank of whose Gauss mapping is not constant 2.

Recall that for a Riemannian surface Σ, a holomorphic immersion $\varphi : \Sigma \to Q^{n-1}$ is called *first-order isotropic* if the complex derivative $\varphi' : \Sigma \to \mathbb{CP}^n$ lies in Q^{n-1} again: $q(\varphi') = 0$. This condition is equivalent to that the *tangent developable*, the union of tangent lines, to φ is contained in Q^{n-1} (cf. Proposition 4.1). A holomorphic immersion $\varphi : \Sigma \to Q^{n-1}$ *has no real point* if $\varphi'(\Sigma) \cap \mathbb{RP}^n = \emptyset$, which is the case for first-order isotropic immersion. Using the notation in §4, let $M = \varphi^* V_2(\mathbb{R}^{n+1})$ be the pull-back bundle over the Riemannian surface Σ, and let Φ be given by (5).

Theorem 6.2. *(10.2 of* [19]*)* ([22]) *If $\varphi : \Sigma \to Q^{n-1}$ has no real point, then $f : M^3 \to S^n$ is a tangentially degenerate immersion. If $\varphi : \Sigma \to Q^{n-1}$ is a first-order isotropic immersion, then $f : M^3 \to S^n$ is a minimal tangentially degenerate immersion, with respect to the ordinary metric on S^n.*

Now, in the case $n = 4$, there exist first-order isotropic holomorphic immersions (unramified) $\varphi : S^2 (= \mathbb{CP}^2) \to Q^3 \subset \mathbb{CP}^4$, [6]. Thus we have

Proposition 6.3. *(10.3 of* [19]*) There exist a minimal tangentially degenerate immersion $f : M^3 \to S^4$ such that M is a circle bundle over S^2, and that the oriented Gauss mapping $\widetilde{\gamma} : M \to \widetilde{G}_4(\mathbb{R}^5) = S^4$ splits into a fibration $M \to S^2$ and a ramified minimal immersion $X : S^2 \to S^4$. The rank of $\widetilde{\gamma}$ is not constant 2.*

Proof: Take $\gamma_3 : S^2 \to Q^3$ of [6], page 237. The corresponding complex contact curve $\lambda_3 : S^2 \to \mathbb{CP}^3$ has the ramification degree 2. Therefore the induced minimal immersion $X = \pi \circ \lambda_3 : S^2 \to S^4$ is ramified as well, and X is a parameterization of the image of $\widetilde{\gamma}$. □

Remark that, in [5], it is proved that there exist minimal immersions $\Sigma \to S^4$ from any *compact* Riemann surface Σ of arbitrary genus. Then, by taking their directrix, we have first-order isotropic homomorphic mappings $\varphi : \Sigma \to Q^3$, however, in general, ramified.

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