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FRONT PROPAGATION AND BLOCKING FOR THE COMPETITION-DIFFUSION SYSTEM IN A DOMAIN OF HALF-LINES WITH A JUNCTION

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Abstract. The two-component Lotka-Volterra competition-diffusion system is well accepted as a model describing the invasion of a superior species into a new habitat. Under a bistable condition, we deal with the system in a domain of half-lines with a single junction and investigate the condition for the invasion from some of the half-lines beyond the junction or blocking the propagation of the superior species. We first give a sufficient condition for the invasion in the whole domain by a subsolution. Then, making use of sub- and supersolutions, we construct a standing front solution blocking the propagation if the number of half-lines occupied by the inferior species is sufficiently larger than that occupied by the superior species.

1. **Introduction.** Lotka-Volterra(LV) competition equations are widely accepted as a model describing the competitive exclusion principle in population dynamics. The dynamics of the system demonstrate that one of two competing species prevails over the other except in a weak competitive case. More precisely, parameter regimes distinguish three cases: Case (i) the stronger species always prevails over the weaker one, Case (ii) the two species compete strongly and the winner depends on the initial

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condition, and Case (iii) the competition is mild so that the two species coexist. We mathematically describe these cases for the following Lotka-Volterra equations:

$$\begin{aligned} \dot{U} &= U(1-U-k_1V), \\ \dot{V} &= rV(1-V-k_2U), \end{aligned} \tag{1}$$

where k_1, k_2 and r are positive parameters. The variables U and V stand for the density of the competing species. We note that the system is normalized so that the U equation with $k_1 = 0$ (resp. the V equation with $k_2 = 0$) has the stable equilibrium $U = 1$ (resp. $V = 1$). When $k_1 < 1 < k_2$ (resp. $k_2 < 1 < k_1$), there is no positive equilibrium, while non-negative equilibrium $E_1 := (1, 0)$ (resp. $E_2 := (0, 1)$) is asymptotically stable. This implies that the stronger species always prevails over the weaker one (Case (i)). If $k_1, k_2 > 1$, then the system has an unstable equilibrium

$$E_3 := \left(\frac{k_1-1}{k_1k_2-1}, \frac{k_2-1}{k_1k_2-1}\right),$$

and E_1 and E_2 are both asymptotically stable. Thus, in this case, the initial condition determines to which equilibrium the solution converges (Case (ii)). As for the case $k_1, k_2 < 1$, E_3 becomes asymptotically stable and corresponds to the coexistence of the two species (Case (iii)).

It is also interesting and important to study how a new species invades a habitat occupied by a native species, and the LV competition system with diffusion has been employed as a model describing the invasion of the new species. With (1), the competition-diffusion system reads

$$\begin{aligned} \partial_t U &= \partial_x^2 U + U(1-U-k_1V), \\ \partial_t V &= d\partial_x^2 V + rV(1-V-k_2U), \end{aligned} \qquad -\infty < x < \infty,\ t > 0, \tag{2}$$

where d is a positive constant. Since $0 \le U(x,t), V(x,t) \le 1$ for initial data $0 \le U_0(x), V_0(x) \le 1$, we consider solutions of (2) in this region.

By using (2), the invasion phenomenon can be discussed by the existence of traveling front solutions connecting the two equilibria E_1 and E_2, namely, solutions of the form $(U, V) = (\Phi(x+ct), \Psi(x+ct))$, where $(\Phi(z), \Psi(z))$ and c satisfy

$$\begin{cases} \Phi'' - c\Phi' + \Phi(1-\Phi-k_1\Psi) = 0, \\ d\Psi'' - c\Psi' + r\Psi(1-k_2\Phi-\Psi) = 0, \qquad -\infty < z = x+ct < \infty, \\ (\Phi(-\infty), \Psi(-\infty)) = (0,1),\ (\Phi(\infty), \Psi(\infty)) = (1,0), \end{cases} \tag{3}$$

and the prime stands for the derivative with respect to z. The existence of the traveling front solutions of (2) has been extensively studied by [1], [5], [9], [10], [17], [19], [20] and references therein (see also related works [2] and [21]).

We note that if the speed c is positive, then U eventually prevails, while V prevails if c is negative. In other words, the sign of the speed indicates the success of the invasion. As a matter of fact, in Case (i), the speed is always positive or negative corresponding to the condition $k_1 < 1 < k_2$ or $k_2 < 1 < k_1$. On the other hand, for Case (ii), it is not so simple to identify the sign. Fortunately, the recent works [4], [13], and [14] provided several parameter regimes in which the speed c is positive. Throughout the present paper, we engaged stick with Case (ii).

Although the results showing the sign of the speed c are crucial, the circumstance of the invasion is idealized, i.e., the domain is the homogenous straight line. For instance, [16, 11] showed that the LV competition-diffusion system allows a stable nonconstant equilibrium solution in a multi-dimensional nonconvex bounded domain, though the nonexistence of stable nonconstant equilibria in a bounded convex domain is known ([12]). That is, the geometry of the domain could affect the propagation of the front wave. On the other hand, in the domain of multi-half lines with a junction, called an unbounded star-graph, the front propagation for the scalar bistable reaction-diffusion equation was recently investigated in [6] and [7]. Specifically, these works show that, under a suitable condition, blocking takes place by the emergence of a standing front solution.

In this paper, motivated by [6] and [7], we deal with the system (2) in the unbounded star-graph defined by

$$\Omega = \left(\bigcup_{n=1}^{m} \Omega_n \right) \cup \{\mathcal{O}\}, \qquad m \geq 3, \tag{4}$$

where $\{\Omega_n : \ n = 1, 2, \ldots, m\}$ are a family of disjoint half-lines and these endpoints are joined at the origin $\mathcal{O}$, that is,

$$\Omega_i \cap \Omega_j = \emptyset \ (i \neq j), \quad \mathrm{cl}(\Omega_n) \setminus \Omega_n = \{\mathcal{O}\} \ (n = 1, \ldots, m).$$

In order to handle a function defined in Ω, we use the local coordinate x_n in each Ω_n as follows:

$$\Omega_n = \begin{cases} \{-\infty < x_n < 0\} & (1 \leq n \leq \ell), \\ \{0 < x_n < \infty\} & (\ell + 1 \leq n \leq m). \end{cases} \tag{5}$$

We distinguish Ω_i $(1 \leq i \leq \ell)$ and the remaining half-lines Ω_j $(\ell + 1 \leq j \leq m)$ by setting the condition that in each Ω_j $(\ell + 1 \leq j \leq m)$ the front wave comes from $x_j \approx \infty$ toward the junction.

For a given function $u(x)$ $(x \in \Omega)$, we denote the restriction of u on Ω_n as

$$u_n(x_n) = u_{|\Omega_n}(x_n).$$

Then Equation (2) on Ω is expressed by m equations on Ω_n $(1 \leq n \leq m)$ glued at the junction $\mathcal{O}$ as follows:

$$\begin{aligned} &\partial_t U_n = \partial_{x_n}^2 U_n + f_1(U_n, V_n), \\ &\partial_t V_n = d\partial_{x_n}^2 V_n + f_2(U_n, V_n), \end{aligned} \qquad (x_n \in \Omega_n, \ t > 0), \qquad n = 1, \ldots, m, \tag{6}$$

with

$$\begin{aligned} &(U_1(0,t), V_1(0,t)) = (U_n(0,t), V_n(0,t)), \qquad n = 2, \ldots, m, \\ &\sum_{i=1}^{\ell} (\partial_{x_i} U_i(-0,t), \partial_{x_i} V_i(-0,t)) = \sum_{j=\ell+1}^{m} (\partial_{x_j} U_j(+0,t), \partial_{x_j} V_j(+0,t)), \end{aligned} \tag{7}$$

where we put

$$f_1(U,V) := U(1 - U - k_1 V), \qquad f_2(U,V) := rV(1 - k_2 U - V). \tag{8}$$

We note that the second condition of (7) implies the conservation of the total flux at the junction.

Under the condition that (2) has the traveling front solution $(U,V) = (\Phi(x + ct), \Psi(x + ct))$ with speed $c > 0$, we investigate how the front propagation could be blocked by the presence of the junction. We briefly state a biological motivation of our study using Figure 1.

We set a weaker species V_n in Ω_n for each $n \in \{1, 2, 3, 4\}$. The stronger species U invades in Ω_4 from $x_4 \approx \infty$ and it is expected that (U_4, V_4) approximately behaves as $(\Phi(x + ct), \Psi(x + ct))$ (up to phase-shift) as long as the front of (U_4, V_4) is sufficiently far from the junction. If U were strong enough, it could invade other habitats Ω_n $(n = 1, 2, 3)$ of V, and eventually, it would occupy the whole domain Ω. On the other hand, if the reinforcement of V from Ω_n $(n = 1, 2, 3)$ were strong enough against the invasion of U, the propagation of the front of (U_4, V_4) would be blocked.

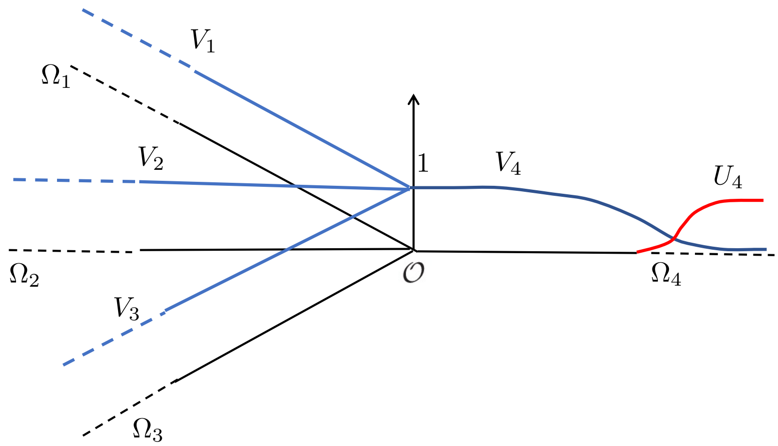

Figure 1. Occupation of the species U and V in the domain Ω with $m = 4, \ell = 3$. The vertical line indicates the values of U and V in the domain.

For the purpose of our study, we first establish the parameter regime for the existence of the traveling front solutions to (2) with $c > 0$ in Theorem 2.2 in §2.

Next, we observe that the function $(\widetilde{\Phi}(x, t), \widetilde{\Psi}(x, t))$ $(x \in \Omega)$ defined by

$$(\widetilde{\Phi}_n(x_n, t), \widetilde{\Psi}_n(x_n, t)) := (\Phi(x_n + ct), \Psi(x_n + ct)) \quad (x_n \in \mathrm{cl}(\Omega_n)), \quad n = 1, \dots, m$$

gives a subsolution to (6)-(7) if $\ell \le m/2$. Then, arbitrarily given p, the solution $(U(x, t), V(x, t))$ to (6)-(7) satisfying

$$(U_n(x_n, 0), V_n(x_n, 0)) = (\Phi(x_n + p), \Psi(x_n + p)) \quad (x_n \in \mathrm{cl}(\Omega_n)), \quad n = 1, \dots, m$$

converges to the steady state $(U, V) = (1, 0)$ uniformly in Ω_j $(j = \ell + 1, \dots, m)$ and locally uniformly in Ω_i $(i = 1, \dots, \ell)$ as $t \to \infty$ if $\ell \le m/2$.

This implies the success of the invasion of superior species U. Even if $\ell > m/2$, it is shown in Theorem 3.1 in §3 that this invasion takes place by appropriate k_1 and k_2. On the other hand, there is a chance to block the invasion by tuning ℓ and m for the fixed other parameters. We let $\mu, \tilde{\mu} > 0$ be numbers such that

$$0 < \lim_{z \to -\infty} e^{-\mu z} \Phi(z) < \infty, \quad 0 < \lim_{z \to -\infty} e^{-\tilde{\mu} z} (1 - \Psi(z)) < \infty \tag{9}$$

hold for the solution $(\Phi(z), \Psi(z))$ of (3) with $c > 0$.

Theorem 1.1. *In addition to the parameter conditions for the existence of the traveling solution* $(\Phi(x + ct), \Psi(x + ct))$ *with* $c > 0$*, assume* $(d - 1)\mu^2 + k_1 - 1 > 0$*,* *where μ is as in (9). Then there is $\rho_*\in (0,1)$ such that (6)-(7) have a positive stationary solution $(U^*(x), V^*(x))$ $(x \in \Omega)$ satisfying*

$$\lim_{x_i\to-\infty}(U_i^*(x_i), V_i^*(x_i)) = (0,1), \quad \lim_{x_j\to\infty}(U_j^*(x_j), V_i^*(x_j)) = (1,0),$$

if $(m-\ell)/\ell \le \rho_*$ holds.

The proof of the theorem is carried out in §4 by employing an appropriate time-independent subsolution and supersolution. The idea found in [11] was very useful for the construction of the subsolution.

Regarding the result of Theorem 1.1, although the existence of standing front solutions to the scalar reaction-diffusion equation in Ω is shown in [6] and [7], their argument does not work in the LV system. Indeed, as for the scalar case, since the stationary problem has a 2-dimensional Hamiltonian structure, the authors used a phase-plane analysis effectively in finding the stationary solution. On the other hand, we needed to construct the sub- and supersolutions based on the maximal principle. As a consequence, the condition to guarantee the existence of the solution is not as simple as the one in [6] or [7].

We can also expect the existence of a front-like entire solution as in [6] and [7], which behaves as the profile (Φ, Ψ) in Ω_j $(\ell+1 \le j \le m)$ as $t \approx -\infty$. However, there are difficulties involved in constructing a suitable subsolution, and this therefore remains for future study.

We organize the rest of the paper as follows: in the next section, Theorem 2.2 gives a sufficient condition for the positive speed, $c > 0$, of the traveling front solution to (2). In §3, we discuss the condition under which fronts can pass through the junction. In addition to the case in which the number, ℓ, of the branches occupied by the inferior species is smaller than or equal to $m/2$, we show for any $\ell > m/2$ that the superior species can penetrate the junction by adapting the parameter k_1 close 1 in Theorem 3.1. In the final section, we show the existence of a standing front. Using suitable sub- and supersolutions, we obtain a standing front solution if ℓ is large enough. The main result is stated in Theorem 4.7.

2. **Speed of traveling front solutions.**

In this section we give a sufficient condition under that the traveling wave solution $(\Phi(x+ct), \Psi(x+ct))$ of (2) has positive speed c. Although several such conditions have been obtained in [4, 13, 14], we take a different approach here for our later purpose.

By introducing the new variables $u = U, v = 1-V$, the system (2) is transformed into the cooperative system

$$\begin{aligned} \partial_t u &= \partial_x^2 u + f(u,v), \\ \partial_t v &= d\partial_x^2 v + g(u,v), \end{aligned} \qquad -\infty < x < \infty,\ t > 0, \tag{10}$$

where we put

$$f(u,v) := u(1-u-k_1(1-v)), \qquad g(u,v) := r(1-v)(k_2 u - v). \tag{11}$$

This system has the monotone traveling wave solution $(u,v) = (\phi(x+ct), \psi(x+ct))$, $z = x+ct$, enjoying

$$\begin{aligned} \phi'' - c\phi' + f(\phi,\psi) &= 0, \\ d\psi'' - c\psi' + g(\phi,\psi) &= 0, \end{aligned} \qquad -\infty < z < \infty, \tag{12}$$

with

$$(\phi(-\infty), \psi(-\infty)) = (0,0), \quad (\phi(\infty), \psi(\infty)) = (1,1), \tag{13}$$

where the prime stands for the derivative with respect to z.

Let

$$X = \{h \in C^2(\mathbb{R}) \mid h'(z) > 0 \ (z \in \mathbb{R}),\ h(-\infty) = 0,\ h(\infty) = 1\}.$$

For a pair $(\underline{\phi}, \underline{\psi}) \in X \times X$, we define

$$I[\underline{\phi}, \underline{\psi}](z) := \frac{\underline{\phi}''(z) + f(\underline{\phi}(z), \underline{\psi}(z))}{\underline{\phi}'(z)}, \tag{14}$$

$$J[\underline{\phi}, \underline{\psi}](z) := \frac{d\underline{\psi}''(z) + g(\underline{\phi}(z), \underline{\psi}(z))}{\underline{\psi}'(z)}, \tag{15}$$

where f and g are as in (11).

Lemma 2.1. *If there exists a pair $(\underline{\phi}, \underline{\psi}) \in X \times X$ satisfying*

$$s_1 := \inf_{z \in \mathbb{R}} I[\underline{\phi}, \underline{\psi}](z) > 0, \quad s_2 := \inf_{z \in \mathbb{R}} J[\underline{\phi}, \underline{\psi}](z) > 0, \tag{16}$$

then the monotone solution $(\phi(z), \psi(z))$ of (12) with (13) has $c > 0$.

Proof. Set $\underline{c} = \min\{s_1, s_2\} > 0$. Then, $(\underline{\phi}, \underline{\psi})$ satisfies

$$\underline{\phi}'' - \underline{c}\underline{\phi}' + f(\underline{\phi}, \underline{\psi}) \geq 0, \quad d\underline{\psi}'' - \underline{c}\underline{\psi}' + g(\underline{\phi}, \underline{\psi}) \geq 0.$$

Hence, $(\underline{\phi}(x + \underline{c}t), \underline{\psi}(x + \underline{c}t))$ is a subsolution of (10) with a monotone increasing profile connecting $(0, 0)$ and $(1, 1)$. Applying a comparison argument ([13, Theorem 3.2]), we obtain $c \geq \underline{c} > 0$. □

Using the above lemma, we give a sufficient condition for $c > 0$.

Theorem 2.2. *The speed c is positive if*

$$1 < k_1 < 2, \quad k_2 > \max\left\{\frac{5}{3}, 2 - \frac{2d}{r}, 1 + \frac{4d}{r}(k_1 - 1)\right\}. \tag{17}$$

Proof. By Lemma 2.1 it suffices to find $(\underline{\phi}, \underline{\psi}) \in X \times X$ satisfying $s_1, s_2 > 0$ under (17), where s_1 and s_2 are the constants defined in (16).

Let $\varsigma(z) = (1 + e^{-az})^{-1}$ be a sigmoid function with parameter $a > 0$. Then, we easily verify

$$\varsigma \in X \quad \text{and} \quad \varsigma' = a\varsigma(1 - \varsigma).$$

We take $(\underline{\phi}, \underline{\psi}) = (\varsigma, 2\varsigma - \varsigma^2) \in X \times X$. By simple calculation we have

$$I[\underline{\phi}, \underline{\psi}](z) = a(1 - 2\varsigma(z)) + \frac{1}{a}(1 - k_1 + k_1\varsigma(z)) =: F(\varsigma(z)),$$

$$J[\underline{\phi}, \underline{\psi}](z) = da(1 - 3\varsigma(z)) + \frac{r}{2a}(k_2 - 2 + \varsigma(z)) =: G(\varsigma(z)).$$

Since F and G are linear in ς,

$$s_1 = \min\{F(0), F(1)\} = \min\left\{a - \frac{1}{a}(k_1 - 1), -a + \frac{1}{a}\right\},$$

$$s_2 = \min\{G(0), G(1)\} = \min\left\{da + \frac{r}{2a}(k_2 - 2), -2da + \frac{r}{2a}(k_2 - 1)\right\}.$$

Since

$$\frac{4d}{r}(k_1 - 1) < k_2 - 1, \quad 2 - k_2 < \frac{2d}{r},$$

by (17), we can take $a > 0$ such that

$$\max\left\{k_1 - 1, \frac{r}{2d}(2 - k_2)\right\} < a^2 < \min\left\{1, \frac{r}{4d}(k_2 - 1)\right\}.$$

Thus, in view of

$$\frac{1}{a}(k_1-1)<a<\frac{1}{a},\quad \frac{r}{2a}(2-k_2)<da<\frac{r}{4a}(k_2-1),$$

we obtain $s_1>0$ and $s_2>0$. □

Remark 1. We compare our condition (17) with the one found in the previous work [13]. Condition (17) is equivalent to

$$1<k_1<2,\;\; k_2>\frac{5}{3},\;\; \frac{2-k_2}{2}<\frac{d}{r}<\frac{k_2-1}{4(k_1-1)}. \tag{18}$$

Applying the formula (see (6.6) in [13])

$$c(d,r,k_1,k_2)=-\sqrt{dr}c(1/d,1/r,k_2,k_1), \tag{19}$$

for the speed $c=c(d,r,k_1,k_2)$ in (3) (or equivalently (12)) to (18) yields that the speed c is negative if

$$k_1>\frac{5}{3},\;\; 1<k_2<2,\;\; \frac{2-k_1}{2}<\frac{r}{d}<\frac{k_1-1}{4(k_2-1)}. \tag{20}$$

Since $(2-k_1)/2<(k_1-1)/4$ for $k_1>5/3$, (20) improves the condition for negative wave speed in [13, Theorem 4.4].

Remark 2. As for a symmetric competition case where $k_1=k_2(=:k>1)$ and $r=1$, the readers can refer to a nice survey [3] on the sign of the wave speed. In view of (20), we see that $c_{k,d}:=c(d,1,k,k)$ is negative if

$$\frac{5}{3}<k<2,\;\; 4<d<\frac{2}{2-k}. \tag{21}$$

This extends the known result stated in [3] and enlarges the shaded areas of Figure 2 therein.

Remark 3. If both $I[\underline{\phi},\underline{\psi}]$ and $J[\underline{\phi},\underline{\psi}]$ are independent of z and if $s_1=s_2$, then $(\underline{\phi}(x+ct),\underline{\psi}(x+ct))$ with $c=s_1$ becomes a monotone traveling wave solution of (10). For example, under the condition (17), if

$$k_1=\frac{r}{3d}>1,\quad k_2=\frac{5}{3}+\frac{2}{r}-\frac{1}{3d}>\frac{5}{3}, \tag{22}$$

then we can take $a=\sqrt{k_1/2}$ such that $(\underline{\phi}(x+ct),\underline{\psi}(x+ct))$ with $(\underline{\phi},\underline{\psi})=(\varsigma,2\varsigma-\varsigma^2)$ is the monotone traveling wave solution of (10) with speed

$$c=\frac{2-k_1}{\sqrt{2k_1}}.$$

Therefore, the original competition-diffusion system (2) has a traveling wave solution $(\Phi(x+ct),\Psi(x+ct))$ with $\Phi=\underline{\phi}$, $\Psi=1-\underline{\psi}$. This traveling wave is essentially the same as one of the exact solutions obtained by Rodrigo and Mimura [20]. The above argument gives an alternative proof for the existence of such an exact solution.

3. **Conditions for passing through the junction.** In this section we give some sufficient conditions for successful invasion of the superior species beyond the junction $\mathcal{O}$. Roughly speaking, if ℓ, the number of branches occupied by the inferior species, is not so large, the superior species can pass through the junction and eventually occupy the whole domain Ω.

We deal with Equation (10) on Ω defined by (5) as

$$\begin{aligned} &\partial_t u_n = \partial_{x_n}^2 u_n + f(u_n, v_n), \\ &\partial_t v_n = d\partial_{x_n}^2 v_n + g(u_n, v_n), \end{aligned} \qquad \text{on} \quad \Omega_n \times (t_0, t_1), \qquad n = 1, 2, \ldots, m, \tag{23}$$

with

$$\begin{aligned} &(u_1(0,t), v_1(0,t)) = (u_n(0,t), v_n(0,t)) \quad (n = 2, 3, \ldots m), \\ &\sum_{i=1}^{\ell} (\partial_{x_i} u_i(-0,t), \partial_{x_i} v_i(-0,t)) = \sum_{j=\ell+1}^{m} (\partial_{x_j} u_j(+0,t), \partial_{x_j} v_j(+0,t)). \end{aligned} \tag{24}$$

If $(\underline{u}(x,t), \underline{v}(x,t)))$ in Ω satisfies

$$\begin{aligned} &\partial_t \underline{u}_n \le \partial_{x_n}^2 \underline{u}_n + f(\underline{u}_n, \underline{v}_n), \\ &\partial_t \underline{v}_n \le d\partial_{x_n}^2 \underline{v}_n + g(\underline{u}_n, \underline{v}_n), \end{aligned} \qquad \text{on} \quad \Omega_n \times (t_0, t_1), \qquad n = 1, 2, \ldots, m, \tag{25}$$

with

$$\begin{aligned} &(u_1(0,t), v_1(0,t)) = (u_n(0,t), v_n(0,t)) \quad (n = 2, 3, \ldots, m), \\ &\sum_{i=1}^{\ell} (\partial_{x_i} u_i(-0,t), \partial_{x_i} v_i(-0,t)) \le \sum_{j=\ell+1}^{m} (\partial_{x_j} u_j(+0,t), \partial_{x_j} v_j(+0,t)), \end{aligned} \tag{26}$$

for $t \in (t_0, t_1)$, then $(\underline{u}(x,t), \underline{v}(x,t))$ is a subsolution of (23) with (24) in $\Omega \times (t_0, t_1)$. On the other hand, if the inequalities in (25) and (26) are reverse, then it is a supersolution.

Suppose the condition (17) in Theorem 2.2 and let $(\phi(x+ct), \psi(x+ct))$ be the monotone traveling wave solution of (10) connecting $(0,0)$ and $(1,1)$ with positive speed c. In order to show that the superior species can pass through the junction $\mathcal{O}$, we will construct a subsolution $(\underline{u}(x,t), \underline{v}(x,t))$ in $\Omega \times \mathbb{R}$ satisfying

$$\lim_{t \to -\infty} (\underline{u}(x,t), \underline{v}(x,t)) = (0,0), \quad \lim_{t \to \infty} (\underline{u}(x,t), \underline{v}(x,t)) = (1,1)$$

and

$$\begin{aligned} &\lim_{x_i \to -\infty} (\underline{u}_i(x_i,t), \underline{v}_i(x_i,t)) = (0,0), \quad i = 1, \ldots, \ell, \\ &\lim_{x_j \to \infty} (\underline{u}_j(x_j,t), \underline{v}_j(x_j,t)) = (1,1), \quad j = \ell+1, \ldots, m. \end{aligned}$$

This subsolution serves to sweep out the inferior species as $t \to \infty$.

First we consider the case where $\ell \le m - \ell$, namely, $\ell \le m/2$. We define

$$(\underline{u}_n(x,t), \underline{v}_n(x,t)) = (\phi(x+ct), \psi(x+ct)) \quad \text{on } \Omega_n \times \mathbb{R}, \quad n = 1, 2, \ldots, m.$$

Then we easily see that $(\underline{u}(x,t), \underline{v}(x,t)) = (\underline{u}_n(x,t), \underline{v}_n(x,t))$ on $\Omega_n \times \mathbb{R}$ $(n = 1, \ldots, m)$ becomes a subsolution of (23)-(24) in $\Omega \times \mathbb{R}$. Since $c > 0$, $(\underline{u}, \underline{v})$ passes through the junction $\mathcal{O}$ with constant speed c. This means that when $\ell \le m/2$, the superior species U can invade the branches occupied by the inferior one V $(= 1 - v)$ beyond the junction and that the inferior species will be swept out.

Next we consider the case where $\ell > m/2$. In other words, the inferior species initially occupies more branches than that the superior species does. Set

$$\alpha := \frac{m-\ell}{\ell} \in (0,1). \tag{27}$$

As in the proof of Therorem 2.2, $(\underline{\phi}, \underline{\psi}) = (\varsigma, 2\varsigma - \varsigma^2)$ with $\varsigma(z) = (1+e^{-az})^{-1}$ satisfies

$$\begin{aligned} &\underline{\phi}'' - s_1\underline{\phi}' + f(\underline{\phi}, \underline{\psi}) \geq 0, \\ &d\underline{\psi}'' - s_2\underline{\psi}' + g(\underline{\phi}, \underline{\psi}) \geq 0, \qquad -\infty < z < \infty, \\ &\underline{\phi}' > 0, \ \underline{\psi}' > 0, \end{aligned} \tag{28}$$

and

$$(\underline{\phi}(-\infty), \underline{\psi}(-\infty)) = (0,0), \quad (\underline{\phi}(\infty), \underline{\psi}(\infty)) = (1,1), \tag{29}$$

with

$$\begin{aligned} s_1 &= \min\left\{a - \frac{1}{a}(k_1 - 1), -a + \frac{1}{a}\right\}, \\ s_2 &= \min\left\{da + \frac{r}{2a}(k_2 - 2), -2da + \frac{r}{2a}(k_2 - 1)\right\}. \end{aligned}$$

Under the condition (17), we can find $a > 0$ such that $\underline{c} := \min\{s_1, s_2\}$ is positive.

Assume (27) and define a pair of functions $(\underline{u}_n, \underline{v}_n)$ on $\Omega_n \times \mathbb{R}$ $(n = 1, \dots, m)$ by

$$(\underline{u}_n(x,t), \underline{v}_n(x,t)) = \begin{cases} (\underline{\phi}(\alpha x + st), \underline{\psi}(\alpha x + st)), & n = 1, 2, \dots, \ell, \\ (\underline{\phi}(x + st), \underline{\psi}(x + st)), & n = \ell + 1, \dots, m. \end{cases} \tag{30}$$

Then $(\underline{u}_n, \underline{v}_n)$ $(n = 1, \dots, m)$ satisfy (26). The following proposition gives a sufficient condition for penetration of the front through the junction $\mathcal{O}$.

Proposition 1. *If*

$$\sup_{-\infty<z<\infty} \frac{\underline{\phi}''(z)}{\underline{\phi}'(z)} < \frac{s_1}{1-\alpha^2}, \qquad \sup_{-\infty<z<\infty} \frac{\underline{\psi}''(z)}{\underline{\psi}'(z)} < \frac{s_2}{d(1-\alpha^2)}, \tag{31}$$

then there exists some $s \in (0, \underline{c})$ *such that* $(\underline{u}(x,t), \underline{v}(x,t)) = (\underline{u}_n(x,t), \underline{v}_n(x,t))$ *on* $\Omega_n \times \mathbb{R}$ $(n = 1, \dots, m)$ *becomes a subsolution of* (23)-(24).

Proof. If $s < \underline{c}$, then (25) with $(t_0, t_1) = (-\infty, \infty)$ holds for $n = \ell + 1, \dots, m$. For $n = 1 \dots, \ell$, we set

$$I_n := \partial_t \underline{u}_n - \partial_{x_n}^2 \underline{u}_n - f(\underline{u}_n, \underline{v}_n), \quad J_n := \partial_t \underline{v}_n - d\partial_{x_n}^2 \underline{v}_n - g(\underline{u}_n, \underline{v}_n).$$

By (28),

$$\begin{aligned} I_n &= s\underline{\phi}'(z) - \alpha^2 \underline{\phi}''(z) - f(\underline{\phi}(z), \underline{\psi}(z)) \\ &\leq (1-\alpha^2)\underline{\phi}''(z) - (s_1 - s)\underline{\phi}'(z) = \underline{\phi}'(z)\left\{(1-\alpha^2)\underline{\phi}''(z)/\phi'(z) - (s_1 - s)\right\}, \end{aligned}$$

where $z = \alpha x + st$. Therefore, if

$$b_1 := (1-\alpha^2) \sup_{-\infty<z<\infty} \frac{\underline{\phi}''(z)}{\underline{\phi}'(z)} < s_1,$$

then $I_n < 0$ for $s \in (0, s_1 - b_1)$. Similarly, if

$$(1-\alpha^2) \sup_{-\infty<z<\infty} \frac{d\underline{\psi}''(z)}{\underline{\psi}'(z)} < s_2,$$

then $J_n < 0$ for some $s \in (0, \underline{c})$. Thus $(\underline{u}, \underline{v})$ satisfies (25) and (26) with $(t_0, t_1) = (-\infty, \infty)$. □

The above proposition yields that if (31) is satisfied, then the propagation of the subsolution $(\underline{u}, \underline{v})$ of the form (30) cannot be blocked by the junction $\mathcal{O}$ and once the fronts pass through the junction, the propagation speed will recover.

In the rest of this section we seek for a condition that allows a subsolution $(\underline{u}, \underline{v})$ of the form (30), where $(\underline{\phi}, \underline{\psi})$ satisfies (31) in addition to (28) and (29). Suppose that the condition (17) in Theorem 2.2 is fulfilled. Assume further

$$k_2 \geq \max\left\{2 - \frac{2d}{r}(k_1 - 1), 1 + \frac{d}{r}(k_1 + 2)\right\}, \tag{32}$$

then taking $a = \sqrt{k_1/2}$, we obtain $s_1 = (2 - k_1)/\sqrt{2k_1} \leq s_2/d$. In view of

$$\underline{\phi}''/\underline{\phi}' = a(1 - 2\varsigma), \qquad \underline{\psi}''/\underline{\psi}' = a(1 - 3\varsigma),$$

the condition (31) is fulfilled if

$$1 - \alpha^2 < \min\left\{\frac{s_1}{a}, \frac{s_2}{da}\right\} = \frac{s_1}{a} = \frac{2 - k_1}{k_1}, \quad i.e., \quad k_1 < \frac{2}{2 - \alpha^2}.$$

This condition is met by (17).

In summary, we obtain the following:

Theorem 3.1. *Suppose $\ell > m/2$ and set $\alpha = (m - \ell)/\ell \in (0, 1)$. Then, if*

$$1 < k_1 < 2, \quad k_2 \geq \max\left\{2 - \frac{2d}{r}(k_1 - 1), 1 + \frac{d}{r}(k_1 + 2)\right\}, \tag{33}$$

and

$$\sqrt{\frac{2(k_1 - 1)}{k_1}} < \alpha < 1, \tag{34}$$

then there exists some $s > 0$ such that with

$$(\underline{\phi}, \underline{\psi}) = (\varsigma, 2\varsigma - \varsigma^2), \quad \varsigma(z) = (1 + e^{-az})^{-1}, \quad a = \sqrt{k_1/2},$$

$(\underline{u}, \underline{v}) = (\underline{u}_n, \underline{v}_n)$ on $\Omega_n \times \mathbb{R}$ $(n = 1, \ldots, m)$ defined by (30) is a subsolution of (23)-(24) and the solution $(u(x,t), v(x,t))$ enjoying $(u(\cdot, t_0), v(\cdot, t_0)) \geq (\underline{u}(\cdot, t_0), \underline{v}(\cdot, t_0))$ for given t_0 converges to $(1, 1)$ uniformly in Ω_j $(\ell+1 \leq j \leq m)$ and locally uniformly in Ω_i $(1 \leq i \leq \ell)$.

Proof. It suffices to show that (33) implies the condition (17). It is obvious that

$$2 - \frac{2d}{r}(k_1 - 1) > 2 - \frac{2d}{r}, \quad 1 + \frac{d}{r}(k_1 + 2) > 1 + \frac{4d}{r}(k_1 - 1)$$

hold for $1 < k_1 < 2$. Furthermore, for $1 < k_1 < 2$, we have

$$2 - \frac{2d}{r}(k_1 - 1) > \frac{5}{3} \quad \text{if } \frac{d}{r} < \frac{1}{6},$$

$$\max\left\{2 - \frac{2d}{r}(k_1 - 1), 1 + \frac{d}{r}(k_1 + 2)\right\} \geq \frac{4}{3} + \frac{2d}{r} \geq \frac{5}{3} \quad \text{if } \frac{d}{r} \geq \frac{1}{6}.$$

Thus (17) holds if (33) is fulfilled. □

Remark 4. As in Remark 3, if (22) is satisfied, then the traveling wave solution $(\phi(x + ct), \psi(x + ct))$ of (10) is written in the following form:

$$(\phi, \psi) = (\varsigma, 2\varsigma - \varsigma^2), \; \varsigma = (1 + e^{-az})^{-1}, \; a = \sqrt{\frac{k_1}{2}}, \; c = \frac{2 - k_1}{\sqrt{2k_1}}.$$

Note that (ϕ, ψ) satisfies (28) with $s_1 = s_2 = c$. In view of

$$\phi''/\phi' = a(1-2\varsigma), \qquad \psi''/\psi' = a(1-3\varsigma),$$

the condition (31) is fulfilled if

$$1-\alpha^2 < \min\left\{\frac{c}{a}, \frac{c}{da}\right\} = \min\left\{1, \frac{1}{d}\right\} \frac{2-k_1}{k_1},$$

namely,

$$1 < k_1 < \frac{2}{1+(1-\alpha^2)\max\{1,d\}}.$$

However, this condition is included in (34) and does not hold for large $d > 1$.

4. **Blocking of the propagation.** As seen in the previous section, if ℓ is small enough, the superior species U $(= u)$ eventually occupies the whole domain while V $(= 1 - v)$ be swept out. However, if the inferior species V occupies in a larger number of the branches of Ω i.e., $\ell \gg m - \ell$, it would have a chance to survive. In order to verify it, we investigate the condition for the existence of the standing front of (23)-(24) in Ω.

We aim to prove the existence of a positive solution to the following stationary problem of (23)-(24):

$$\begin{aligned} &\partial_{x_n}^2 u_n + f(u_n, v_n) = 0, \\ &d\partial_{x_n}^2 v_n + g(u_n, v_n) = 0, \end{aligned} \qquad \text{on} \quad \Omega_n, \qquad n = 1, 2, \ldots, m, \tag{35}$$

with

$$\begin{aligned} &(u_1(0), v_1(0)) = (u_n(0), v_n(0)) \qquad (n = 2, 3, \ldots m), \\ &\sum_{i=1}^{\ell} (\partial_{x_i} u_i(-0), \partial_{x_i} v_i(-0)) = \sum_{j=\ell+1}^{m} (\partial_{x_j} u_j(+0), \partial_{x_j} v_j(+0)). \end{aligned} \tag{36}$$

We let (u^*, v^*) be the unstable constant equilibrium of the system, namely,

$$(u^*, v^*) = \left(\frac{k_1-1}{k_1k_2-1}, \frac{k_2(k_1-1)}{k_1k_2-1}\right), \tag{37}$$

satisfying

$$f(u^*, v^*) = g(u^*, v^*) = 0.$$

Consider the problem

$$\begin{cases} \mathcal{F}_1(u,v) := u_{xx} + f(u,v) = 0, \\ \mathcal{F}_2(u,v) := dv_{xx} + g(u,v) = 0, \end{cases} \qquad -\infty < x < 0, \tag{38}$$

with

$$\begin{aligned} &(u(-\infty), v(-\infty)) = (0, 0), \\ &(u(0), v(0)) = (u_\delta, v_\delta) := \left(\frac{k_1 - 1 - \delta}{k_1(k_2+\delta)-1}, \frac{(k_2+\delta)(k_1-1-\delta)}{k_1(k_2+\delta)-1}\right), \end{aligned} \tag{39}$$

where

$$k_1 - 1 > \delta > 0 \tag{40}$$

is assumed. We note that

$$u_\delta < u^*, \qquad v_\delta < v^*.$$

We define

$$(\overline{Y}_1(x), \overline{Z}_1(x)) := (u_\delta, v_\delta)e^{\mu_s x} \tag{41}$$

for $\mu_s > 0$. Then we have

Lemma 4.1. *As for* $(\overline{Y}_1(x), \overline{Z}_1(x))$ *defined in* (41),

$$\mathcal{F}_1(\overline{Y}_1, \overline{Z}_1) < 0, \qquad \mathcal{F}_2(\overline{Y}_1, \overline{Z}_1) < 0 \qquad (-\infty < x < 0)$$

hold if

$$\mu_s \le \min\left\{1, \left(\frac{r(1-v_\delta)}{d(k_2+\delta)}\right)^{1/2}\right\}\delta^{1/2}.$$

Proof. We compute

$$\begin{aligned}\mathcal{F}_1(\overline{Y}_1, \overline{Z}_1) &= u_\delta e^{\mu_s x}[\mu_s^2 - (k_1-1) + (k_1 v_\delta - u_\delta)e^{\mu_s x}] \\ &= u_\delta e^{\mu_s x}[\mu_s^2 - (k_1-1) + (k_1 - 1 - \delta)e^{\mu_s x}] \\ &< u_\delta e^{\mu_s x}(\mu_s^2 - \delta) \le 0 \qquad (-\infty < x < 0),\end{aligned}$$

if $\mu_s \le \delta^{1/2}$. On the other hand,

$$\begin{aligned}\mathcal{F}_2(\overline{Y}_1, \overline{Z}_1) &= e^{\mu_s x}[d v_\delta \mu_s^2 + r(1 - v_\delta e^{\mu_s x})(k_2 u_\delta - v_\delta)] \\ &= e^{\mu_s x}[d v_\delta \mu_s^2 - r\delta u_\delta(1 - v_\delta e^{\mu_s x})] \\ &< e^{\mu_s x}[d\mu_s^2 v_\delta - r\delta u_\delta(1-v_\delta)] \le 0 \qquad (-\infty < x < 0),\end{aligned}$$

if

$$\mu_s \le \left(\frac{r\delta u_\delta(1-v_\delta)}{d v_\delta}\right)^{1/2} = \left(\frac{r\delta(1-v_\delta)}{d(k_2+\delta)}\right)^{1/2},$$

where we used $v_\delta = (k_2+\delta)u_\delta$. This leads to the assertion of the lemma. □

Next consider the problem

$$\begin{cases}\mathcal{F}_1(u,v) = u_{xx} + f(u,v) = 0, \\ \mathcal{F}_2(u,v) = d v_{xx} + g(u,v) = 0,\end{cases} \qquad 0 < x < \infty, \tag{42}$$

with

$$(u(0), v(0)) = (u_\delta, v_\delta), \qquad (u(\infty), v(\infty)) = (1,1). \tag{43}$$

We define

$$(\overline{Y}_2(x), \overline{Z}_2(x)) := (1 - (1-u_\delta)e^{-\lambda_s x}, 1 - (1-v_\delta)e^{-\lambda_s x}), \tag{44}$$

for $\lambda_s > 0$. Then we have the following lemma:

Lemma 4.2. *As for* $(\overline{Y}_2(x), \overline{Z}_2(x))$*, defined in* (44),

$$\mathcal{F}_1(\overline{Y}_2, \overline{Z}_2) < 0, \qquad \mathcal{F}_2(\overline{Y}_2, \overline{Z}_2) < 0 \qquad (0 < x < \infty),$$

for $\lambda_s \ge \sqrt{r(k_2-1)/d}$.

Proof. By simple calculation, we obtain

$$\begin{aligned}\mathcal{F}_1(\overline{Y}_2, \overline{Z}_2) &= e^{-\lambda_s x}\left[-\lambda_s^2(1-u_\delta) + \{1-(1-u_\delta)e^{-\lambda_s x}\}\{1 - u_\delta - k_1(1-v_\delta)\}\right] \\ &= e^{-\lambda_s x}\left[-\lambda_s^2(1-u_\delta) - \delta\{1-(1-u_\delta)e^{-\lambda_s x}\}\right], \\ \mathcal{F}_2(\overline{Y}_2, \overline{Z}_2) &= e^{-\lambda_s x}(1-v_\delta)\left[-d\lambda_s^2 + r(k_2-1) - r\{k_2(1-u_\delta) - (1-v_\delta)\}e^{-\lambda_s x}\right] \\ &= e^{-\lambda_s x}(1-v_\delta)\left[-d\lambda_s^2 + r(k_2-1) - r(k_2 - 1 + \delta u_\delta)e^{-\lambda_s x}\right] \\ &< e^{-\lambda_s x}(1-v_\delta)\left[-d\lambda_s^2 + r(k_2-1)\right] \qquad (0 < x < \infty).\end{aligned}$$

Hence, $\mathcal{F}_1(\overline{Y}_2, \overline{Z}_2) < 0$ and $\mathcal{F}_2(\overline{Y}_2, \overline{Z}_2) < 0$ for $x > 0$ if $\lambda_s \ge \sqrt{r(k_2-1)/d}$. □

We let

$$\mu_s = \min\left\{1,\ \left(\frac{r(1-v_\delta)}{d(k_2+\delta)}\right)^{1/2}\right\}\delta^{1/2}, \quad \lambda_s = \left(\frac{r(k_2-1)}{d}\right)^{1/2}, \tag{45}$$

for $\delta \in (0, k_1 - 1)$. In view of

$$(\overline{Y}_1{}'(0), \overline{Z}_1{}'(0)) = \mu_s(u_\delta, v_\delta),$$
$$(\overline{Y}_2{}'(0), \overline{Z}_2{}'(0)) = \lambda_s(1-u_\delta, 1-v_\delta),$$

we see that $(\overline{u}, \overline{v})$ defined by

$$\begin{aligned}(\overline{u}_i, \overline{v}_i) &:= (\overline{Y}_1, \overline{Z}_1) \quad \text{in } \Omega_i\ (1 \le i \le \ell),\\ (\overline{u}_j, \overline{v}_j) &:= (\overline{Y}_2, \overline{Z}_2) \quad \text{in } \Omega_j\ (\ell+1 \le j \le m)\end{aligned} \tag{46}$$

provides a supersolution to (35) with (36) if

$$\ell\mu_s u_\delta \ge (m-\ell)\lambda_s(1-u_\delta), \qquad \ell\mu_s v_\delta \ge (m-\ell)\lambda_s(1-v_\delta). \tag{47}$$

The latter inequality reads

$$\ell\mu_s u_\delta \ge (m-\ell)\lambda_s(1/(k_2+\delta) - u_\delta).$$

Hence, the conditions of (47) reduce to

$$\frac{m-\ell}{\ell} \le \frac{\mu_s}{\lambda_s}\left(\frac{u_\delta}{1-u_\delta}\right). \tag{48}$$

We can fix the value of the right-hand side of (48), for instance, by taking $\delta = (k_1-1)/3$. In the consequence we have

Lemma 4.3. *Define (u_δ, v_δ) as in (37). Let μ_s and λ_s be the numbers defined in (45). Set $\delta = (k_1-1)/3$. Then there is a constant $\rho_* \in (0,1)$ such that if the positive integers m and ℓ satisfy $\alpha = (m-\ell)/\ell \le \rho_*$, then $(\overline{u}, \overline{v})$ defined by (46) is a supersolution to (35) with (36).*

We remark that the condition $\alpha \le \rho_*$ can be realized as follows: Put $\ell = m - n_1$ and fix n_1. Then $\alpha = n_1/(m-n_1)$ implies that sufficiently large m, namely ℓ, enjoys the condition.

Before discussing the existence of a subsolution, we state some properties of the monotone traveling solution $(\phi(x+ct), \psi(x+ct))$ satisfying (12)-(13). See [18, Lemma 2.2] for details.

Lemma 4.4. *Let μ and λ be positive solutions of $\mu^2 - c\mu - (k_1-1) = 0$ and $d\lambda^2 - c\lambda - r = 0$, respectively.*

(i) *If $\mu > \lambda$, then for some $A_1, A_2 > 0$,*

$$\begin{pmatrix}\phi(z)\\ \phi'(z)\end{pmatrix} = A_1 e^{\mu z}(1+o(1))\begin{pmatrix}1\\ \mu\end{pmatrix}, \quad \begin{pmatrix}\psi(z)\\ \psi'(z)\end{pmatrix} = A_2 e^{\lambda z}(1+o(1))\begin{pmatrix}1\\ \lambda\end{pmatrix}$$

hold as $z \to -\infty$.

(ii) *If $\mu < \lambda$, then for some $A > 0$,*

$$\begin{pmatrix}\phi(z)\\ \phi'(z)\end{pmatrix} = A e^{\mu z}(1+o(1))\begin{pmatrix}1\\ \mu\end{pmatrix}, \quad \begin{pmatrix}\psi(z)\\ \psi'(z)\end{pmatrix} = -Ase^{\mu z}(1+o(1))\begin{pmatrix}1\\ \mu\end{pmatrix}$$

hold as $z \to -\infty$, where $s = rk_2/(d\mu^2 - c\mu - r) < 0$.

(iii) *If $\mu = \lambda$, then for some $A > 0$,*

$$\begin{pmatrix} \phi(z) \\ \phi'(z) \end{pmatrix} = A\tau e^{\mu z}(1 + o(1)) \begin{pmatrix} 1 \\ \mu \end{pmatrix}, \quad \begin{pmatrix} \psi(z) \\ \psi'(z) \end{pmatrix} = -Aze^{\mu z}(1 + o(1)) \begin{pmatrix} 1 \\ \mu \end{pmatrix}$$

hold as $z \to -\infty$, where $\tau = (2d\mu - c)/(rk_2) > 0$.

By Lemma 4.4, there exist positive constants M_0 and z_0 satisfying

$$\frac{\phi(z)}{\phi'(z)}, \ \frac{\psi(z)}{\psi'(z)} \le M_0 \quad (z \le -z_0). \tag{49}$$

Set

$$\eta := \sqrt{2M_0/c}. \tag{50}$$

Then we can take $M_1, z_1 > 0$ such that

$$\frac{\phi(z + \eta)}{\phi(z)} \le M_1 \quad (z \le -z_1). \tag{51}$$

For later purpose, we investigate the behavior of $1 - k_2\phi(z)/\psi(z)$ as $z \to -\infty$. If $\mu \ge \lambda$, then Lemma 4.4 (i) and (iii) imply that

$$1 - k_2\frac{\phi(z)}{\psi(z)} \to 1 \quad (z \to -\infty).$$

On the other hand, if $\mu < \lambda$, then by Lemma 4.4 (ii),

$$1 - k_2\frac{\phi(z)}{\psi(z)} \to 1 + \frac{k_2}{s} = \frac{\mu(d\mu - c)}{r} \quad (z \to -\infty).$$

Therefore, if we further assume $d\mu > c$, or equivalently,

$$(d - 1)\mu^2 + k_1 - 1 > 0, \tag{52}$$

the above limit is positive. Consequently, under the assumption (52), we have

$$1 - k_2\phi(z)/\psi(z) > 0 \quad (z \le -z_2), \tag{53}$$

for some $z_2 > 0$. We remark that (52) is true for $d \ge 1$.

Define

$$\underline{Y}(x) := \begin{cases} \phi(x + p) & (\eta \le x < \infty), \\ \phi(x + p) - b(x/\eta)\phi(p) & (0 \le x \le \eta), \end{cases}$$
$$\underline{Z}(x) := \begin{cases} \psi(x + p) & (\eta \le x < \infty), \\ \psi(x + p) - b(x/\eta)\psi(p) & (0 \le x \le \eta), \end{cases} \tag{54}$$

where

$$b(y) := (y - 1)^2, \tag{55}$$

$\eta > 0$ is defined in (50) and $p < 0$ are taken later so that $(\underline{Y}(x), \underline{Z}(x))$ becomes a subsolution. We easily verify that $\underline{Y}(x)$ and $\underline{Z}(x)$ are positive, increasing and of class C^1 in $x \in (0, \infty)$, in addition, continuous on $[0, \infty)$ with $\underline{Y}(0) = \underline{Z}(0) = 0$.

Lemma 4.5. *Let (ϕ, ψ) be the traveling front solution stated in §2. Assume (52) and set η as in (50). Then, there is a constant $p_m < 0$ such that $\underline{Y}(x)$ and $\underline{Z}(x)$ of (54) with $p \le p_m$ satisfy*

$$\mathcal{F}_1(\underline{Y}, \underline{Z}) > 0, \quad \mathcal{F}_2(\underline{Y}, \underline{Z}) > 0 \quad (x > 0, x \ne \eta).$$

Proof. We easily see from (54) that

$$\mathcal{F}_1(\underline{Y},\underline{Z})(x) = c\phi'(x+p) > 0, \quad \mathcal{F}_2(\underline{Y},\underline{Z})(x) = c\psi'(x+p) > 0 \quad (\eta \le x < \infty).$$

We will verify the inequalities for $0 < x < \eta$. We abbreviate to write b instead of $b(x/\eta)$ below if there is no confusion. Let $0 < x < \eta$. We compute

$$\begin{aligned}
&\mathcal{F}_1(\underline{Y},\underline{Z})(x) \\
&= \phi''(x+p) - (2/\eta^2)\phi(p) + f(\phi(x+p) - b\phi(p), \psi(x+p) - b\psi(p)) \\
&= c\phi'(x+p) - (2/\eta^2)\phi(p) - f(\phi(x+p), \psi(x+p)) \\
&\quad + f(\phi(x+p) - b\phi(p), \psi(x+p) - b\psi(p)) \\
&= c\phi'(x+p) - (2/\eta^2)\phi(p) + (k_1 - 1)b\phi(p) + 2\phi(x+p)b\phi(p) \\
&\quad - k_1\psi(x+p)b\phi(p) - k_1\phi(x+p)b\psi(p) - b^2(\phi(p))^2 + k_1 b^2\phi(p)\psi(p) \\
&> \phi'(x+p)[c - (2/\eta^2)\phi(x+p)/\phi'(x+p)] \\
&\quad + b\phi(p)[k_1 - 1 - k_1\psi(x+p) - k_1\phi(x+p)\psi(p)/\phi(p) - b\phi(p)].
\end{aligned}$$

Take $p_1 < 0$ so that

$$\eta + p_1 < -z_0, \quad p_1 < -z_1, \quad k_1\{\psi(\eta + p_1) + M_1\psi(p_1)\} + b\phi(p_1) < k_1 - 1$$

hold, where z_0 and z_1, M_1 are as in (49) and (51), respectively. Then for $p \le p_1$ we have

$$\mathcal{F}_1(\underline{Y},\underline{Z})(x) > 0 \qquad (0 \le x \le \eta).$$

Next we compute $\mathcal{F}_2(\underline{Y},\underline{Z})$ as follows:

$$\begin{aligned}
&\mathcal{F}_2(\underline{Y},\underline{Z})(x) \\
&= d\psi''(x+p) - (2/\eta^2)\psi(p) + g(\phi(x+p) - b\phi(p), \psi(x+p) - b\psi(p)) \\
&= c\psi'(x+p) - (2/\eta^2)\psi(p) - g(\phi(x+p), \psi(x+p)) \\
&\quad + g(\phi(x+p) - b\phi(p), \psi(x+p) - b\psi(p)) \\
&= c\psi'(x+p) - (2/\eta^2)\psi(p) - rk_2 b\phi(p) + rb\psi(p) + rk_2\psi(x+p)b\phi(p) \\
&\quad + rk_2\phi(x+p)b\psi(p) - 2r\psi(x+p)b\psi(p) - rk_2 b^2\phi(p)\psi(p) + rb^2(\psi(p))^2 \\
&> \psi'(x+p)[c - (2/\eta^2)\psi(x+p)/\psi'(x+p)] \\
&\quad + rb\psi(p)[1 - k_2\phi(p)/\psi(p) - 2\psi(x+p) - k_2 b\phi(p)].
\end{aligned}$$

In view of (53), we can take $p_2 \le p_1$ such that

$$1 - k_2\frac{\phi(p_2)}{\psi(p_2)} - 2\psi(\eta + p_2) - k_2\phi(p_2) > 0. \tag{56}$$

Then for $p \le p_2$

$$\mathcal{F}_2(\underline{Y},\underline{Z})(x) > 0 \qquad (0 < x < \eta)$$

holds.

Consequently, taking $p_m = p_2$, we obtain the desired assertion of the lemma. □

Define $(\underline{u}, \underline{v})$ on Ω by

$$\begin{aligned}
(\underline{u}_i, \underline{v}_i) &:= (0, 0) && \text{in} \quad \Omega_i \ (1 \le i \le \ell), \\
(\underline{u}_j, \underline{v}_j) &:= (\underline{Y}, \underline{Z}) && \text{in} \quad \Omega_j \ (\ell + 1 \le j \le m).
\end{aligned} \tag{57}$$

The following lemma is a corollary to Lemma 4.5:

Lemma 4.6. *Assume* (52). *Then* $(\underline{u}, \underline{v})$ *defined by* (57) *is a subsolution to* (35) *with* (36).

Proof. It is obvious that $(\underline{u}, \underline{v})$ is continuous in Ω and

$$\mathcal{F}_1(\underline{u}_n, \underline{v}_n) \geq 0, \ \ \mathcal{F}_2(\underline{u}_n, \underline{v}_n) \geq 0 \quad \text{on } \Omega_n \ (n = 1, \ldots, m).$$

Furthermore, considering

$$\underline{Y}'(+0) = \phi'(p) + \frac{2}{\eta}\phi(p) > 0, \quad \underline{Z}'(+0) = \psi'(p) + \frac{2}{\eta}\psi(p) > 0,$$

we find that

$$\sum_{i=1}^{\ell}(\partial_{x_i} u_i(-0), \partial_{x_i} v_i(-0)) = 0 < \sum_{j=\ell+1}^{m} (\partial_{x_j} u_j(+0), \partial_{x_j} v_j(+0)).$$

This completes the proof of Lemma 4.6. □

Combining Lemmas 4.3 and 4.6, we obtain

Theorem 4.7. *Suppose the same assumptions in Lemmas 4.3 and 4.6. Then there is a positive solution* $(\tilde{u}(x), \tilde{v}(x))$ *to* (35) *with* (36) *satisfying*

$$(\tilde{u}(-\infty), \tilde{v}(-\infty)) = (0, 0) \quad \text{and} \quad (\tilde{u}(\infty), \tilde{v}(\infty)) = (1, 1).$$

Proof. In terms of the super and subsolutions obtained in Lemmas 4.3 and 4.6 we apply the standard maximum principle for the order-preserving flow to obtain the assertion. The details is left to the readers (see [15]). □

Proof of Theorem 1.1. The assertion immediately follows from Theorem 4.7 for $(U^*(x), V^*(x)) := (\tilde{u}(x), 1 - \tilde{v}(x))$. □

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