制御手術理論とその応用の研究

(研究課題番号 11640093)

平成 11 年度～平成 13 年度科学研究費補助金 (基盤研究 (C)(2))

研究成果報告書

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研究代表者　山　崎　正　之
(城西大学理学部教授)

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はしがき

これは平成11年度〜平成13年度の科学研究費補助金による研究成果報告書である。3年間の研究テーマは制御手術理論とその応用に関する研究である。

位相幾何学の最も重要な問題は空間の分類である。手術理論は多様体の分類を行うための強力な道具であり、30年前にはほぼ完成された。最も重要な成果はいわゆる手術の完全列であり、この完全列を理解することが、与えられたホモトピー型の多様体の分類を理解することに等しい。つまり、本来幾何学的な問題が、代数的な問題に帰着されたわけである。しかし、代数的な問題が代数の手法を持っていつも解決できるわけではない。代数的な対象に対する幾何学的考察が代数的な問題に対する解決を与え、ひいてはもともとの幾何学的問題の解決を与えることが、実際にはありうる。制御手術理論はまさにそのような理論であり、多様体に対する深い理解を与えてくれるものである。

さて、その制御手術理論であるが、研究されはじめてからすでに20年の歴史を持つが、誰もが使いやすい形で整備されているわけではない。その傾向は古典的手術理論でも同様なのであるが、制御手術理論においては、いわゆるフォークロアが多く、局所基本群が自明な場合を除き、きちんと定理 (予想) そのものが明確に定式化されているとは言えない。現在の専門家に続く若い研究者を育てるためにも、理論の整備が必要である。

制御手術理論における手術の完全列が一般的な場合にもなりたつかどうかは、いまだわかっていないが、それを証明できるとすれば、その最も重要なステップは制御手術群の安定性である。そして、その安定性が証明にむけて最も重要な技巧は、幾何的ポアンカレ2次複体の分割である。実は、その分割は一般的には不可能であり、それが問題を著しく困難にしている。我々は、分割を実行するためのひとつの十分条件 (残念ながら必要条件ではない) をみつけることができた。またそれをもちいてごく特殊な場合の安定性を証明することができた。今後、一般的な場合の証明を与えることを目標としてさらに努力したい。

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CONTROLLED SURGERY THEORY

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INTRODUCTION

Although algebraic topology is a very powerful tool for studying global properties of spaces, it has some limitations:

1. Topological conditions are difficult to handle, compared with homotopical conditions, that is, the condition that a given map is a homeomorphism is much stronger than the condition that it is a homotopy equivalence, but it is difficult to reflect this in algebraic form (for example, topological invariance of Whitehead torsion, topological invariance of rational Pontryagin classes, etc.).
2. Local conditions or geometric properties of spaces are difficult to handle.

Following Chapman and Ferry, Quinn started a project to overcome these difficulties in [20] and subsequent papers. This is what is called the "controlled topology" or the "Chapman-Ferry-Quinn theory".

Regarding the basis elements of free modules to be points on a space with a control map to some metric space, Quinn introduced the notion of size for homomorphisms between them. Controlled topology works well when objects of small sizes split into pieces lying over small subsets of the space. If this is the case, each piece may reflect the local data. Also, if the space has a good local property, the split pieces can usually be deformed to even smaller objects. Such an operation is called "squeezing".

In this article, I describe some aspects of "controlled surgery theory". We actually need some knowledge on "controlled K-theory", but this will complicate the exposition too much; so we only consider cases in which we can disregard K-theoretic problems.

The plan is as follows. We review the classical surgery theory in Section 1. In Section 2, we describe the "spacification" of surgery theory due to Casson-Sullivan-Quinn-Ranicki. In Section 3, we introduce control into surgery theory. Quinn used such a theory in [21] to study the resolution problem of homology manifolds. Here we follow the version of controlled surgery theory given by Ferry and Pedersen ([12], [13], and [2]). We will assume that the control map $p : K \to X$ to a metric space X is UV^1. This roughly means that the point inverses $p^{-1}(*)$ are simply-connected. Under this assumption, the local Whitehead and $\widetilde{K}_0$ groups vanish, and the controlled L-groups are homology groups. In general, controlled L-groups are not homology, but $L^{(-\infty)}$-groups are known to be homology groups ([33]). Section 4 describes a result of Bryant-Ferry-Mio-Weinberger [2] on resolutions of homology

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manifolds as an application. As other applications, we mention results on the Novikov Conjecture and the finiteness theorem of Grove-Petersen-Wu.

Controlled topology has made, and is still making, a lot of progress through the efforts of many people. There are now various kinds of control methods besides the original ϵ control. There is also a sheaf-theoretic approach as in [18]. But I could not include them in this article. Please refer to the original papers that are referred to in Section 5. It will be necessary to make even further modifications to use control topolgy in different applications.

1. Classical Surgery Theory

We review the classical surgery theory (in the topological category). Suppose a topological space K is given. Let us consider the following questions.

(Q1) Does K have the homotopy type of a closed topological manifold?

(Q2) Suppose the answer to the above question is "yes". How many different closed topological manifolds are there that are homotopy equivalent to K?

To simplify the argument, we assume that K is a finite CW complex and we consider only oriented closed topological manifolds.

Let $\widetilde{K}$ be a covering space of K with the group of covering translations π. Let us denote the group ring $\mathbb{Z}[\pi]$ by Λ, and the integral coefficient cellular chain complex $C_*(\widetilde{K})$ of $\widetilde{K}$ by $C_*(K;\Lambda)$. It is a Λ-module chain complex via the (left) action of π on $\widetilde{K}$ (the simplices of K form a natural basis as a Λ-module). The dual $\mathrm{Hom}_\Lambda(C_*(\widetilde{K}),\Lambda)$ as a Λ-module chain complex is denoted by $C^*(K;\Lambda)$. Here the action of Λ from the left is given by:

$$((\sum n_g g)f)(x) = f(x)(\sum n_g g^{-1}) \qquad (\sum n_g g \in \Lambda,\ f \in C^r(K;\Lambda),\ x \in C_r(K;\Lambda)).$$

Also, for an integer n, the n-dual $C^{n-*}(K;\Lambda)$ is defined by:

$$(C^{n-*}(K;\Lambda))_r = C^{n-r}(K;\Lambda),$$
$$d_r = (-1)^r(d_{C_*(K,\Lambda)})^* : C^{n-r}(K;\Lambda) \to C^{n-r+1}(K;\Lambda)\ .$$

The homology group $H_r(C^{n-*}(K;\Lambda))$ of this chain complex coincides with the compact-support integral cohomology group $H^{n-r}_{\mathrm{cpt}}(\widetilde{K})$ of $\widetilde{K}$.

If K is homotopy equivalent to a topological manifold, then Poincaré duality must hold for K just as for manifolds. So we assume that K is a Poincaré complex in the following sense:

Definition 1.1. A finite CW complex K is an *n-dimensional Poincaré complex* if there exists an n-cycle $\xi \in C_n(K)$ such that the following is a chain equivalence:

$$\tilde{\xi} \cap - \ :\ C^{n-*}(K;\mathbb{Z}[\pi_1(K)]) \to C_*(K;\mathbb{Z}[\pi_1(K)])\ .$$

Here $\tilde{\xi}$ denotes the possibly infinite transfer of ξ in $C^{\mathrm{lf}}_n(\widetilde{K})$. If the Whitehead torsion of this map ($\in \mathrm{Wh}(\pi)$) is trivial, the Poincaré complex is said to be *simple.*

One of the differences between Poincaré complexes and topological manifolds appears in the bundle theory. The bundles for topological manifolds are the topological fiber bundles (or simply Top bundles); for a locally-flat embedding of a topological manifold M in a Euclidean space, there is a normal Top bundle, denoted ν_M, which is stably unique with respect to connected sums with trivial bundles. On the other hand, for Poincaré complexes embedded in Euclidean spaces, we have the following:

Theorem 1.2. (Spivak) *Suppose an n-dimensional Poincaré complex K is embedded in a sufficiently high-dimensional Euclidean space E^N. Then the retraction to K of a regular neighborhood of K defines a spherical fibration with homotopy fiber S^{N-n} when restricted to the boundary. Furthermore, this fibration is stably unique with respect to taking a fiberwise join with trivial spherical fibrations, i.e., taking a fiberwise suspension.*

Such a stable spherical fibration is called the *Spivak normal fibration* of K and is denoted by ν_K.

The classifying space for stable spherical fibrations is denoted by BG. The i-th homotopy group of BG is the stable homotopy group $\lim_{k\to\infty} \pi_{i+k-1}(S^k)$ of the spheres. The classifying space for the stable Top bundles is denoted by $B\mathrm{Top}$, and the homotopy fiber of the natural map $J : B\mathrm{Top} \to BG$ is denoted by G/Top. If an n-dimensional Poincaré complex K is an n-dimensional closed manifold, then its Spivak normal fibration ν_K has a reduction to a Top bundle, that is, the classifying map for ν_K has a lift to $B\mathrm{Top}$.

So let us assume that ν_K has a Top reduction. Then the set of all homotopy classes of the lifts to $B\mathrm{Top}$ can be identified with $[K : G/\mathrm{Top}]$, and there is a one-to-one correspondence between $[K : G/\mathrm{Top}]$ and the set of all normal bordism classes of degree 1 normal maps from closed manifolds to K. (See the definition below.)

Definition 1.3. (1) A *normal map* (f, b) $M \to K$ from an n-dimensional manifold M to an n-dimensional Poincaré complex K is a pair consisting of a map $f : M \to K$ and a stable bundle map $b : \nu_M \to \eta$ covering f from the stable normal bundle $\nu_M : M \to B\mathrm{Top}$ of M to some Top bundle $\eta : K \to B\mathrm{Top}$ over K.

(2) Two normal maps $(f : M \to K,\ b : \nu_M \to \eta)$ and $(f : M' \to K,\ b : \nu_{M'} \to \eta')$ are *normally bordant* if there exist a bordism $F : W^{n+1} \to K \times [0, 1]$ between f and f', a Top bundle H over $K \times [0, 1]$, and a stable bundle map $B : \nu_W \to H$ covering F whose restrictions to the two ends are b and b'.

Surgery is a method to deform a degree 1 normal map; the new normal map is normally bordant to the original map and, conversely, normally bordant normal maps can be deformed by surgery to each other.

We can answer Q1 if we have a method to detect whether a given normal map is normally bordant to one whose underlying map is a homotopy equivalence. Such an obstruction is defined in the L^h-group (surgery obstruction group). The decoration h stands for *h*omotopy equivalence. In general, for a ring A with involution containing 1, abelian groups $L_n^h(A)$ $(n \geq 0)$ are defined. The surgery obstruction $\sigma(f, b)$ of a degree 1 normal map $(f, b) : M^n \to K$ is defined in the L^h-group $L_n^h(\mathbb{Z}[\pi_1(K)])$ of the group ring $\mathbb{Z}[\pi_1(K)]$, and the following holds:

Theorem 1.4. ([31], [15]) *Let $(f, b) : M \to K$ be a normal map, and suppose $n \geq 4$. If $n = 4$, further suppose that $\pi_1(K)$ is good in the sense of Freedman-Quinn. Then (f, b) is normally bordant to a homotopy equivalence if and only if the surgery obstruction $\sigma(f, b)$ vanishes.*

To answer Q2, we first need to fix the classification scheme. Let us consider the classification up to h-cobordisms. For this it suffices to verify that the surgery obstruction for a normal map from a manifold with boundary to $K \times I$ which restricts to a homotopy equivalence of boundary can be defined in $L^h_{n+1}(\mathbb{Z}[\pi_1(K)])$ and that a similar result holds as above.

Definition 1.5. The *homotopy structure set* $\mathcal{S}^h(K)$ of an n-dimensional Poincaré complex K is defined by:

$$\mathcal{S}^h(K) = \{f : M \to K \mid M : \text{an } n\text{-manifold}, \ f : \text{a homotopy equivalence}\} / \sim ,$$

where two homotopy equivalences $f : M \to K$ and $f' : M' \to K$ are equivalent if there exist an h-cobordism W between M and M' and an extension $F : W \to K$ of f and f'.

Theorem 1.6. (Surgery Exact Sequence) *Suppose $n \geq 5$ and K is a connected n-dimensional Poincaré complex. If ν_K has a* Top *bundle reduction, then there is an exact sequence of sets:*

$$L^h_{n+1}(\mathbb{Z}[\pi_1(K)]) \xrightarrow{\omega} \mathcal{S}^h(K) \xrightarrow{\eta} [K : G/\mathrm{Top}] \xrightarrow{\sigma} L^h_n(\mathbb{Z}[\pi_1(K)]) .$$

Remark. The map σ sends a normal map to its surgery obstruction. The map η sends a homotopy equivalence $f : M \to K$ to the induced normal map $(f, b : \nu_M \to (f^{-1})^*\nu_M)$. The exactness at $[K : G/\mathrm{Top}]$ is in the previous theorem (i.e., $\mathrm{im}(\eta) = \sigma^{-1}(0)$). One can introduce an abelian group structure in $[K : G/\mathrm{Top}]$, but σ may not be a homomorphism. In fact, $0 \in L^h_n(\mathbb{Z}[\pi_1(K)])$ is in the image of σ if and only if $\mathcal{S}^h(K)$ is nonempty (i.e., K has the homotopy type of a closed n-dimensional manifold). Assuming $\mathcal{S}^h(K)$ is nonempty, ω represents an action of $L^h_{n+1}(\mathbb{Z}[\pi_1(K)])$ on $\mathcal{S}^h(K)$, and the exactness at $\mathcal{S}^h(K)$ means that the inverse image of an element by η coincides with some orbit of the action ω.

If we deal with simple homotopy equivalence instead of homotopy equivalence, similar objects $\mathcal{S}^s(-)$ and $L^s_i(\mathbb{Z}[-])$ can be defined analogously, and a similar exact sequence holds. For example, $\mathcal{S}^s(K^n)$ can be defined using simple homotopy equivalence and s-cobordism. Note that, if $n \geq 5$, the s-cobordism theorem implies that two simple homotopy equivalences $f_i : M \to K$ $(i = 1, 2)$ are equivalent if and only if there exists a homeomorphism $h : M_1 \to M_2$ satisfying $f_1 \simeq f_2 \circ h$.

Since these two theories are quite parallel to each other, we omit the decorations for $\mathcal{S}$ and L from now on. If the Whitehead group $\mathrm{Wh}(\pi_1(K))$ vanishes, that is, $\pi_1(K)$ is trivial, or isomorphic to $\mathbb{Z}^n$, then there is no difference between them anyway.

Both L-groups have periodicity of period 4. For example, the L-groups of $A = \mathbb{Z}[\{1\}] = \mathbb{Z}$ are given by:

Theorem 1.7.

$$L_i(\mathbb{Z}) = \begin{cases} \mathbb{Z} & i \equiv 0 \bmod 4, \\ \mathbb{Z}/2 & i \equiv 2 \bmod 4 \ (i \geq 0), \\ 0 & i \text{ odd}. \end{cases}$$

This coincides with the homotopy group $\pi_i(G/\mathrm{Top})$ of G/Top for $i > 0$:

Theorem 1.8. G/Top *is connected, and* $\pi_i(G/\mathrm{Top}, *) \cong L_i(\mathbb{Z})$ $(i > 0)$.

When K is a closed topological manifold, we can define the relative structure set $\mathcal{S}^h(K \times I^i, \partial)$ using homotopy equivalence $(W, \partial) \to (K \times I^i, \partial)$ that restrict to homeomorphisms on the boundary, and we can extend the surgery exact sequence to the left:

$$\cdots \to \mathcal{S}(K \times I^i, \partial) \to [K \times I^i, \partial : G/\mathrm{Top}, *] \to L_{n+i}(\mathbb{Z}[\pi_1(K)]) \\ \to \mathcal{S}(K \times I^{i-1}, \partial) \to \cdots.$$

2. Spectra and Homology

A collection $\mathbb{E} = \{\mathbb{E}_k,\ \epsilon_k \mid k \in \mathbb{Z}\}$ of based spaces $\mathbb{E}_k$ and based homotopy equivalences $\epsilon_k : \mathbb{E}_k \to \Omega\mathbb{E}_{k+1}$ to the loop spaces is called an *Ω-spectrum.* In the following we actually consider a Δ-set [30] satisfying the Kan condition instead of spaces, but we pretend that these are ordinary spaces.

The *homotopy groups* of an Ω-spectrum $\mathbb{E}$ are defined to be $\pi_n(\mathbb{E}) = \pi_{n+k}(\mathbb{E}_k)$ ($n, k \in \mathbb{Z}$, $n + k \geq 0$). When $\pi_i(\mathbb{E}) = 0$ for all $i < q$, $\mathbb{E}$ is said to be *q-connective.*

The surgery obstruction groups $L_n^h(\mathbb{Z}[\pi_1(K)])$ ($n \geq 0$) are the homotopy groups of a certain 0-connective Ω-spectrum. In fact, there is a functor $\mathbb{L}\langle 0\rangle(-)$ that sends a path-connected space K to an Ω-spectrum $\mathbb{L}\langle 0\rangle(K)$ satisfying

$$\pi_n(\mathbb{L}\langle 0\rangle(K)) \ = \ \begin{cases} L_n(\mathbb{Z}[\pi_1(K)]) & n \geq 0\ , \\ 0 & n < 0\ . \end{cases}$$

This Ω-spectrum is called the *0-connective periodic L-spectrum.* Quinn constructed this spectrum using a geometric method ([19], [31, 17A], [32, Chapter 3]). Since surgery obstruction groups can be identified with the set of cobordism classes of certain chain complexes with duality structures ([25], [26]) and there are notions of n-ads (pairs, triads, 4-ads, ...) of such chain complexes, one can algebraically construct a similar functor that sends a ring R to a spectrum $\mathbb{L}(R)$. Or one can algebraically construct "$\mathbb{L}(K)$" using geometric modules on K (cf. [27], [33]). Geometric modules will be introduced in the next section.

By imposing some restriction on the chain complexes used in the construction above, one can kill the homotopy group $\pi_0(\mathbb{L}\langle 0\rangle(K))$ and construct the *1-connective L-spectrum* $\mathbb{L}\langle 1\rangle(K)$ that satisfies

$$\pi_n(\mathbb{L}\langle 1\rangle(K)) = \begin{cases} L_n(\mathbb{Z}[\pi_1(K)]) & n \geq 1\ , \\ 0 & n < 1\ . \end{cases}$$

There is a map of spectra $\mathbb{L}\langle 1\rangle(K) \to \mathbb{L}\langle 0\rangle(K)$ that induces isomorphisms on π_n ($n > 0$).

When $K = \{*\}$ or K is simply-connected, we use the following notation:

$$\mathbb{L}\langle 0\rangle \ = \ \mathbb{L}\langle 0\rangle(\{*\}), \qquad \mathbb{L}\langle 1\rangle \ = \ \mathbb{L}\langle 1\rangle(\{*\}).$$

There are homotopy equivalences $\mathbb{L}\langle 0\rangle_0 \simeq \mathbb{Z} \times G/\mathrm{Top}$, $\mathbb{L}\langle 1\rangle_0 \simeq G/\mathrm{Top}$ (Casson, Sullivan).

(Co)homology groups of a pair (K, L) with the coefficient an Ω-spectrum $\mathbb{E}$ are defined by:

$$H_n(K, L; \mathbb{E}) = \lim_{k \to \infty} \pi_{n+k}(K/L \wedge \mathbb{E}_k), \qquad H^n(K, L; \mathbb{E}) = [K/L, * : \mathbb{E}_n, *].$$

(If $L = \emptyset$, then $K/\emptyset = K \sqcup \{*\}$.) These are the homotopy groups of certain Ω-spectra: $H_n(K, L; \mathbb{E}) = \pi_n(\mathbb{H}_\bullet(K, L; \mathbb{E}))$, $H^n(K, L; \mathbb{E}) = \pi_n(\mathbb{H}^\bullet(K, L; \mathbb{E}))$.

For each functor $\mathbb{E}$ that associates an Ω-spectrum to a space, Quinn defined the "*assembly map*" between spectra:

$$A : \mathbb{H}_\bullet(K; \mathbb{E}(\{*\})) \to \mathbb{E}(K).$$

The induced homomorphism $A : H_*(K; \mathbb{E}(\{*\})) \to \pi_*(\mathbb{E}(K))$ is also called the *assembly map*.

Let $q = 0,\ 1$. (There are analogous results for any $q \in \mathbb{Z}$.) For a polyhedron K, Ranicki defined the *algebraic structure spectrum* $\mathbb{S}\langle q\rangle(K)$ *in the q-connective L-theory* as the relative term of the assembly map for the q-connective L-spectrum, and obtained a fibration of spectra ([25]):

$$\mathbb{H}_\bullet(K; \mathbb{L}\langle q\rangle) \xrightarrow{A} \mathbb{L}\langle q\rangle(K) \to \mathbb{S}\langle q\rangle(K) .$$

Its homotopy exact sequence

$$\begin{aligned} \cdots \to \mathcal{S}_{n+1}\langle q\rangle(K) \to H_n(K; \mathbb{L}\langle q\rangle) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(K)]) \xrightarrow{\partial} \mathcal{S}_n\langle q\rangle(K) \\ \to H_{n-1}(K; \mathbb{L}\langle q\rangle) \to \dots \end{aligned}$$

is called the *q-connective algebraic surgery exact sequence* for K. Here $\mathcal{S}_n\langle q\rangle(K)$ denotes the homotopy group $\pi_n(\mathbb{S}\langle q\rangle(K))$ and is called the *algebraic structure group* in the q-connective L-theory.

Theorem 2.1. (Ranicki, [25], [26], [27]) *Let K be an n-dimensional closed manifold, and let $n \geq 5$. For each $i \geq 0$, there are isomorphisms*

$$\begin{aligned} &\mathcal{S}(K \times I^i, \partial) \cong \mathcal{S}_{n+i+1}\langle 1\rangle(K), \\ &[K \times I^i, \partial : G/\mathrm{Top}, *] = [K \times I^i, \partial : \mathbb{L}\langle 1\rangle_0, *] = H^{-i}(K; \mathbb{L}\langle 1\rangle) \cong H_{n+i}(K; \mathbb{L}\langle 1\rangle), \end{aligned}$$

and the surgery exact sequence for K coincides with the 1-connective algebraic surgery exact sequence under these identifications.

As the last topic of this section, we mention the total surgery obstruction of Ranicki. Let K be an n-dimensional Poincaré complex. In the previous section, the question Q1 had two-step obstructions: (1) Does ν_K have a Top bundle reduction? (2) Is there a degree 1 normal map to K with trivial surgery obstruction? Ranicki unified these and defined a single obstruction $s(K)$, called the *total surgery obstruction*, for K to have the homotopy type of a closed topological manifold as an element of $\mathcal{S}_n\langle 1\rangle(K)$, where $n \geq 5$. The image of $s(K)$ in $\mathcal{S}_n\langle 0\rangle(K)$ is denoted $\bar{s}(K)$, and is called the *0-connective total surgery obstruction*. In the algebraic surgery exact sequence,

- the image $t(K) \in H_{n-1}(K; \mathbb{L}\langle 1\rangle)$ (resp. $\bar{t}(K) \in H_{n-1}(K; \mathbb{L}\langle 0\rangle)$) of $s(K) \in \mathcal{S}_n\langle 1\rangle(K)$ (resp. $\bar{s}(K) \in \mathcal{S}_n\langle 0\rangle(K)$) is the obstruction for ν_K to have a lift to $B\widetilde{\mathrm{Top}}(k)$ (resp. $B\mathrm{Top}$), and
- if $t(K) = 0$ (resp. $\bar{t}(K) = 0$), ∂ maps the surgery obstruction $\sigma(f, b) \in L_n(\mathbb{Z}[\pi_1(K)])$ of any degree 1 normal map $(f, b) : M \to K$ to $s(K)$ (resp. $\bar{s}(K)$),

where $B\widetilde{\mathrm{Top}}(k) \to BG(k)$ is the pull-back of $B\mathrm{Top} \to BG$ via the obvious map $BG(k) \to BG$.

3. Controlled Topology

Let X be a metric space, M and K topological spaces, and ϵ a positive number. Suppose that a continuous map $p : K \to X$ is given.

Definition 3.1. A homotopy $H : M \times [0,1] \to K$ is said to be a $p^{-1}(\epsilon)$ *homotopy* if the diameter of $p \circ H(a \times [0,1])$ is less than or equal to ϵ for each $a \in M$.

Definition 3.2. A continuous map $f : M \to K$ is a $p^{-1}(\epsilon)$ *homotopy equivalence* if there exist a continuous map $g : K \to M$ and homotopies $h : g \circ f \simeq 1_M$, $k : f \circ g \simeq 1_K$ such that both $f \circ h$ and k are $p^{-1}(\epsilon)$ homotopies.

Now we consider the ϵ version of surgery theory. Let K be an n-dimensional Poincaré complex and $p : K \to X$ a map from K to a metric space (X, d). When does there exist a $p^{-1}(\epsilon)$ homotopy equivalence from an n-dimensional topological manifold? Obviously ν_K must have a Top bundle reduction, and therefore there must exist a degree 1 normal map (f, b) from a topological manifold. To define an obstruction in the "controlled surgery obstruction group", we need more assumptions. The difference between the Poincaré duality of manifolds (or homology manifolds) and that of Poincaré complexes is that the former is something assembled from local dualities. Note that the definition of manifolds *is* local. Suppose there is a $p^{-1}(\epsilon)$ homotopy equivalence with small ϵ. Since the Poincaré duality for a manifold is "small", the Poincaré duality for K must also be "small".

To clarify the meaning of the previous sentence, we introduce the notion of geometric modules ([4], [20], [22], [28]). To simplify the definition we assume that $p : K \to X$ is a UV^1-map; i.e., p is proper and onto, and for any $\epsilon > 0$, any map $\alpha : P^2 \to X$ from a 2-complex P, and any lift $\alpha_0 : P_0 \to K$ of α defined on a subcomplex P_0 of P, there exists an extension $\bar{\alpha} : P \to K$ of α_0 such that $d(p \circ \bar{\alpha}(s), \alpha(s)) < \epsilon$ (for all $s \in P$). For example, the identity map $1 : X \to X$ is a UV^1-map.

Definition 3.3. (1) Let S be a set. The pair $(\mathbb{Z}[S], \phi)$ of the free module $\mathbb{Z}[S]$ generated by S and a map $\phi : S \to K$ is called a *geometric module* on K.

(2) A *morphism* $f : (\mathbb{Z}[S], \phi : S \to K) \to (\mathbb{Z}[T], \psi : T \to K)$ between two geometric modules on K is a formal sum $\sum_{\lambda \in \Lambda} n_\lambda (s_\lambda, t_\lambda)$ of pairs $(s, t) \in S \times T$ with integral coefficients such that, for each $s \in S$, there are only finitely many λ's with $s_\lambda = s$ and that, for each $t \in T$, there are only finitely many λ's with $t_\lambda = t$. A morphism obtained by reduction of like terms is regarded to be a different morphism from the original.

(3) A morphism $f : (\mathbb{Z}[S], \phi) \to (\mathbb{Z}[T], \psi)$ *has radius* $p^{-1}(\epsilon)$ if the coefficient n_λ in f of a pair (s_λ, t_λ) satisfying $d(p \circ \phi(s_\lambda), p \circ \psi(t_\lambda)) > \epsilon$ is 0.

(4) The *dual* $(\mathbb{Z}[S], \phi)^*$ of $(\mathbb{Z}[S], \phi)$ is defined to be $(\mathbb{Z}[S], \phi)$ itself; thus $\mathbb{Z}[S]$ can be regarded as a submodule of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[S], \mathbb{Z})$. The *dual* $f^* : (\mathbb{Z}[T], \psi)^* \to (\mathbb{Z}[S], \phi)^*$ of a morphism $f : (\mathbb{Z}[S], \phi) \to (\mathbb{Z}[T], \psi)$ with $f = \sum_{\lambda \in \Lambda} n_\lambda (s_\lambda, t_\lambda)$ is defined by $f^* = \sum_{\lambda \in \Lambda} n_\lambda (t_\lambda, s_\lambda)$. If f has radius $p^{-1}(\epsilon)$, then so does f^*.

(5) By $\sim_\epsilon$ we denote the equivalence relation on the set of morphisms $f : (\mathbb{Z}[S], \phi) \to (\mathbb{Z}[T], \psi)$ with radius ϵ generated by reductions of like terms satisfying $d(p \circ \phi(s_\lambda), p \circ \psi(t_\lambda)) \leq \epsilon$ together with the inverse operations.

Remarks on definition. (1) If X is bounded and the sets S, T are finite, then the set of the equivalence classes with respect to $\sim_\epsilon$ of the morphisms of radius $p^{-1}(\epsilon)$

from $(\mathbb{Z}[S], \phi)$ to $(\mathbb{Z}[T], \psi)$ can be identified with $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[S], \mathbb{Z}[T])$ for sufficiently large ϵ.

(2) Let $(\mathbb{Z}[S], \phi : S \to K)$ be a geometric module and let $\widetilde{S} \to S$ be the pull-back of the universal cover $\widetilde{K} \to K$ of K via ϕ. Then the free module $\mathbb{Z}[\widetilde{S}]$ generated by $\widetilde{S}$ has an action of $\pi_1(K)$ and can be regarded as a free $\mathbb{Z}[\pi_1(K)]$-module. In general, a morphism $f : (\mathbb{Z}[S], \phi) \to (\mathbb{Z}[T], \psi)$ does not necessarily define a $\mathbb{Z}[\pi_1(K)]$-module homomorphism $\mathbb{Z}[\widetilde{S}] \to \mathbb{Z}[\widetilde{T}]$. But, if K and X are finite polyhedra and f has radius $p^{-1}(\epsilon)$ for sufficiently small ϵ, then it defines a unique $\mathbb{Z}[\pi_1(K)]$-module homomorphism $\mathbb{Z}[\widetilde{S}] \to \mathbb{Z}[\widetilde{T}]$ since p is UV^1. (Note that non-contractible loops in K have images in X of diameter bounded below by some positive number.)

(3) For the definition of morphisms for a more general map p, please refer to [28].

Suppose that K is a finite polyhedron and that a UV^1-map $p : K \to X$ to a polyhedral metric space X is given. The simplicial chain complex $C_*(K)$ of K can be regarded as a "chain complex" of geometric modules. Pick a representative point (e.g., the barycenter) from each simplex and think of it as a basis element. For any $\epsilon > 0$, we may assume that the boundary morphisms of $C_*(K)$ have radius $p^{-1}(\epsilon)$ and that the compositions of consecutive boundary morphisms are $p^{-1}(\epsilon)$ homotopic to 0 (the empty morphism) by taking a sufficiently fine triangulation. As we noted above, we can construct $C_*(\widetilde{K})$ from $C_*(K)$ by taking the pull-back of the universal cover $\widetilde{K} \to K$.

Definition 3.4. K above is an *n-dimensional $p^{-1}(\epsilon)$ Poincaré complex* with respect to p if there exists a cycle $\xi \in C_n(K)$ for which

$$\xi \cap - \ : \ C^{n-*}(K) \to C_*(K)$$

is a $p^{-1}(\epsilon)$ chain homotopy equivalence for some sufficiently fine subdivision of K. Here a "$p^{-1}(\epsilon)$ chain homotopy equivalence" is defined by throwing in appropriate $p^{-1}(\epsilon)$'s in the ordinary definition and using $\sim_\epsilon$'s in place of the equalities ([28]).

Remark. Although one can consider the controlled torsion of $p^{-1}(\epsilon)$ chain homotopy equivalences, such a torsion is known to vanish for sufficiently small ϵ when the control map p is UV^1. So we do not define $p^{-1}(\epsilon)$ simple Poincaré complexes here.

Theorem 3.5. (S. C. Ferry and E. K. Pedersen, [13], [2, § 2]) *Let X be a finite polyhedron and $n \geq 5$. Then there exist $\epsilon_0 > 0$ and $T \geq 1$ satisfying the following: For any ϵ ($\epsilon_0 > \epsilon > 0$), any n-dimensional $p^{-1}(\epsilon)$ Poincaré complex $p : K \to X$, and any degree 1 normal map $(f, b) : M^n \to K$, there is defined a well-defined obstruction class $\sigma^c(f, b)$ in $H_n(X; \mathbb{L}\langle 0\rangle)$, and (f, b) is normally bordant to a $p^{-1}(T\epsilon)$ homotopy equivalence if and only if $\sigma^c(f, b)$ vanishes.*

The data on K seems to be missing from the obstruction class, but this is due to the assumption that the fiber of p is approximately simply-connected. The image of $\sigma^c(f, b)$ by the assembly map $A : H_n(X; \mathbb{L}\langle 0\rangle) \to L_n(\mathbb{Z}[\pi_1(K)])$ coincides with the ordinary surgery obstruction $\sigma(f, b)$.

To state the epsilon surgery exact sequence of Ferry-Pedersen, we need to define a $p^{-1}(\epsilon)$ homotopy structure set $\mathcal{S}_\epsilon(p : K \to X)$. This will be done in Theorem 3.7 below. We first introduce the unstable structure set as a preparation:

Definition 3.6. Let $p : K \to X$ be a UV^1-map. Define $\mathcal{S}'_\epsilon(p : K \to X)$ by

$$\{f : M \to K \mid M : n\text{-manifold}, f : \text{a } p^{-1}(\epsilon) \text{ homotopy equivalence } \} / \sim ,$$

where two maps $f : M \to K$, $f' : M' \to K$ are equivalent if there exists a homeomorphism $h : M \to M'$ such that f and $f' \circ h$ are $p^{-1}(\epsilon)$ homotopic. (Note that, for $n \geq 5$, a controlled h-cobordism is a controlled product in this situation. [20])

Theorem 3.7. (S. C. Ferry and E. K. Pedersen, [13], [2, § 2]) *Let X be a finite polyhedron and $n \geq 5$. Then there exist $\epsilon_0 > 0$ and $T \geq 1$ satisfying the following: If K is an n-dimensional closed topological manifold and $p : K \to X$ is UV^1, then for any ϵ ($\epsilon_0 > \epsilon > 0$) there is a functorial surgery exact sequence*

$$\cdots \to H_{n+1}(X; \mathbb{L}\langle 0\rangle) \to \mathcal{S}_\epsilon(p : N \to X) \to H_n(N; \mathbb{L}\langle 1\rangle) \to H_n(X; \mathbb{L}\langle 0\rangle) ,$$

where $\mathcal{S}_\epsilon(p) = \operatorname{im}(\mathcal{S}'_\epsilon(p) \to \mathcal{S}'_{T\epsilon}(p))$ and $\mathcal{S}_\epsilon(p) \cong \mathcal{S}_{\epsilon_0}(p)$.

Remark. When K is only a $p^{-1}(\epsilon)$ Poincaré complex, there is an exact sequence up to terms that make sense, but the maps may not be homomorphisms just as in the classical case.

Note that, for sufficiently small ϵ, the controlled surgery obstruction groups are independent of the value of ϵ. This sort of phenomenon happens often in controlled topology: we can squeeze sufficiently small objects to get objects as small as we like. The key idea is to split objects into pieces and to use induction ([29]). See also the remark on the size of the images of non-contractible loops above.

4. Homology Manifolds

In this section, we describe the work of Bryant-Ferry-Mio-Weinberger [2] on homology manifolds. The *ANR* (absolute neighborhood retract) X is a *homology n-manifold* if $H_i(X, X - \{x\}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{O\})$ holds for each $x \in X$. An n-dimensional closed topological manifold is a homology n-manifold, and the following gives a partial converse:

Theorem 4.1. (Edwards, [5]) *Let $n \geq 5$ and let X be a homology n-manifold satisfying DDP. Then any resolution $f : M \to X$ can be approximated by a homeomorphism.*

We say that X satisfies DDP (*Disjoint Disks Property*) if any two continuous maps from a 2-disk to X can be approximated so that they have disjoint images, and $f : M \to X$ is a *resolution* if M is an n-dimensional topological manifold and f is a proper cell-like surjection. (A map is *cell-like* if every inverse image of a point is contractible in any neighborhood. If a map is an ϵ homotopy equivalence for any $\epsilon > 0$ with respect to the identity map of the target, then it is cell-like.)

In [21], [23], Quinn found the obstruction for a connected homology n-manifold X ($n \geq 5$) to have a resolution:

Theorem 4.2. (Quinn) *There is an integer-valued invariant $I(X)$ of connected homology n-manifolds X ($n \geq 5$) satisfying:*

(1) $I(X) \equiv 1 \pmod 8$;
(2) *if $U \subset X$ is a connected open subset, then $I(X) = I(U)$;*
(3) $I(X \times Y) = I(X) \times I(Y)$;
(4) *X has a resolution if and only if $I(X) = 1$.*

Define an integer $i(X)$ by $i(X) = (I(X)-1)/8$. This integer has the following interpretation: There is a degree 1 normal map to the homology manifold X ([12, Theorem 16.6]). Since X is ϵ Poincaré with respect to the identity control map $1_X : X \to X$ for every $\epsilon > 0$, one can associate an ϵ surgery obstruction class $\sigma^c(f,b) \in H_n(X;\mathbb{L}\langle 0\rangle) \cong [X : \mathbb{Z} \times G/\mathrm{Top}]$ to any degree 1 normal map $(f,b) : M \to X$. The $[X:\mathbb{Z}] = \mathbb{Z}$ component of $\sigma^c(f,b)$ is the integer $i(X)$.

The existence of resolution can be derived from the vanishing $i(X) = 0$ in the following way. For simplicity we assume that X is a polyhedron. By changing $[f,b] \in [X : G/\mathrm{Top}]$ if necessary, we may assume that $\sigma^c(f,b) = 0 \in [X : \mathbb{Z} \times G/\mathrm{Top}]$. Therefore $\mathcal{S}_\epsilon(1_X : X \to X) \neq \emptyset$ for every $\epsilon > 0$. In the exact sequence

$$H_{n+1}(X;\mathbb{L}\langle 1\rangle) \to H_{n+1}(X;\mathbb{L}\langle 0\rangle) \to \mathcal{S}_\epsilon(1_X) \to H_n(X;\mathbb{L}\langle 1\rangle) \to H_n(X;\mathbb{L}\langle 0\rangle)\ ,$$

the first map is an isomorphism and the last map is an injection, because $H_i(X;\mathbb{Z}) = 0$ for $i > n$. Therefore there exist a closed topological manifold M^n and a sequence of maps $f_i : M \to X$ such that (1) f_i is a $1/i^2$ homotopy equivalence, and (2) f_i and f_{i+1} are $1/i^2$ homotopic. The limit is the desired resolution of X.

Let X be a homology n-manifold and pick a degree 1 normal map $f : M \to X$. As above, f determines $\sigma^c(f,b) \in H_n(X;\mathbb{L}\langle 0\rangle)$. From $A(\sigma^c(f,b)) = \sigma(f,b)$ and $\partial(\sigma(f,b)) = \bar{s}(X)$, we can deduce $\bar{s}(X) = 0 \in \mathcal{S}_n\langle 0\rangle(X)$. The following is the main result in [2].

Theorem 4.3. (Bryant-Ferry-Mio-Weinberger) *Let X be an n-dimensional Poincaré complex and $n \geq 6$. X has the homotopy type of an n-dimensional topological manifold if and only if $\bar{s}(X) = 0 \in \mathcal{S}_n\langle 0\rangle(X)$. If $\bar{s}(X) = 0$, then there exists a covariantly functorial 4-periodic exact sequence of abelian groups:*

$$\cdots H_{n+1}(X;\mathbb{L}\langle 0\rangle) \to L_{n+1}(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}^H(X) \to H_n(X;\mathbb{L}\langle 0\rangle) \to L_n(\mathbb{Z}[\pi_1(X)])\ .$$

Remarks. (1) $\mathcal{S}^H(X)$ is defined by replacing "manifolds" with "homology manifolds" in the definition of $\mathcal{S}(X)$.

(2) When X is a homology n-manifold, there is an isomorphism $\mathcal{S}^H(X) \xrightarrow{\cong} \mathcal{S}_{n+1}\langle 0\rangle(X)$ that sends a homotopy equivalence $f : Z \to X$ from another homology manifold Z to its relative total surgery obstruction of the mapping cylinder of f. Via this isomorphism, the above exact sequence can be identified with the 0-connective algebraic surgery exact sequence (Ranicki [27]).

Example. Take the n-dimensional sphere S^n ($n \geq 6$) as X. In the surgery exact sequence for homology manifolds, we have

$$H_{n+1}(S^n;\mathbb{L}\langle 0\rangle) = H_0(S^n, L_{n+1}(\mathbb{Z})) \oplus H_n(S^n; L_1(\mathbb{Z})) = L_{n+1}(\mathbb{Z})\ ,$$
$$H_n(S^n;\mathbb{L}\langle 0\rangle) = H_0(S^n; L_n) \oplus H_n(S^n; L_0) = L_n(\mathbb{Z}) \oplus L_0(\mathbb{Z})\ ,$$

and we can deduce that $\mathcal{S}^H(S^n) \cong L_0(\mathbb{Z}) = \mathbb{Z}$. Here the equivalence class of a homology manifold Z that is homotopy equivalent to S^n corresponds to its resolution obstruction $i(Z)$. Therefore it implies the existence of such homology manifolds that do not admit resolutions. Similar results also hold for simply-connected manifolds other than S^n.

5. Other Applications

Let Γ be an infinite discrete group, and let $B\Gamma$ be the classifying space for Γ. The Novikov Conjecture on higher signatures can be rephrased in the following way:

Novikov Conjecture. *The assembly map* $A : H_*(B\Gamma; \mathbb{L}\langle 0\rangle) \to L_*(\mathbb{Z}[\Gamma])$ *is split injective when tensored with* $\mathbb{Q}$.

For torsion-free groups, there is a stronger conjecture:

Integral Novikov Conjecture. *If* Γ *is torsion-free, the assembly map* $A : H_*(B\Gamma; \mathbb{L}\langle 0\rangle) \to L_*(\mathbb{Z}[\Gamma])$ *is an isomorphism.*

If Γ is the fundamental group of an aspherical manifold M^n $(n \geq 5)$ (i.e., $M \simeq B\Gamma$), the Integral Novikov Conjecture is true for Γ, and $\mathrm{Wh}(\Gamma) = 0$, then the Borel Conjecture below holds true. It is conjectured that $\mathrm{Wh}(\Gamma) = 0$ for torsion-free groups Γ.

Borel Conjecture. *If a closed manifold* M *is* $K(\Gamma, 1)$, *then any homotopy equivalence* $f : N \to M$ *from a closed manifold is homotopic to a homeomorphism.*

Farrell and Hsiang were the first to attack the Novikov Conjecture and the Borel Conjecture using controlled topology ([6], [7], [8], ...). They showed the topological rigidity for flat and almost-flat manifolds using geometric methods, Dress induction, etc. Then Farrell and Jones introduced the notion of foliated control [9], and showed the rigidity for compact non-positively curved manifolds, etc. ([10]); a lot of improvements have been (and are still being) made by them since then. On the other hand, bounded control [11] and continuous control [1] have been playing an important and central role in controlled topology. For example, Carlsson and Pedersen verified the Novikov Conjecture for a class of groups containing word-hyperbolic groups [3].

There are other topics, such as C^*-algebra and its relation to controlled topology [17] and so on; see [14] for these and for more details on the Novikov Conjecture.

Another application is the finiteness theorem of Grove-Petersen-Wu:

Theorem 5.1. ([16]) $\mathfrak{M}^{\bullet D \bullet}_{k \bullet v}(n)$ *denote the set of all the* n*-dimensional Riemannian manifolds with sectional curvature* $\geq k$, *diameter* $\leq D$, *volume* $\geq v$. *For* $n \neq 3$, *there are only finitely many topological types of manifolds in* $\mathfrak{M}^{\bullet D \bullet}_{k \bullet v}(n)$. *For* $n \neq 3, 4$, *there are only finitely many diffeomorphism types of such manifolds.*

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MODULI SPACE OF THE POLYNOMIALS WITH DEGREE N

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1 Introduction

The subject of this paper is a new coordinate system, so-called multiplier cooridinates, introduced into the moduli space, $\mathbb{M}_n$, of the polynomial maps $\mathrm{Poly}_n(\mathbb{C})$ from the Riemann sphere, $\widehat{\mathbb{C}}$, to itself, with degree n.

In study of its geometry and topology from a viewpoint of complex dynamical systems, we make use of this system in order to express singular part, and dynamical loci as algebraic curves or surfaces. And to exhibit the moduli space with a higher degree under this system deserves particular attention: for example, a problem of characterization of exceptinal part, $\mathcal{E}_n(=\mathbb{C}^{n-1}\setminus\mathbb{M}_n)$. This problem is our main subject.

The initiator of the use of multiplier cooridinates is J. Milnor ([Mil93]), to the case of the quadratic rational maps.

First, we investigate the moduli space $\mathbb{M}_n$ consisting of all holomorphic (affine) conjugacy classes of $\mathrm{Poly}_n(\mathbb{C})$. A polynomial map p of degree n is monic and centered if it has the form $p(z) = z^n + c_{n-2}z^{n-2} + \cdots + c_1 z + c_0$. Every polynomial map from $\widehat{\mathbb{C}}$ to itself is conjugate under an affine change of variable to a monic centered one, and this is uniquely determined up to conjugacy under the action of the group $G(n-1)$ of $(n-1)$-st roots of unity. Hence the affine space $\mathcal{P}_1(n)$ of all monic centered polynomials of degree n with coordinate $(c_0, c_1, \cdots, c_{n-2})$ is regarded as an $(n-1)$-sheeted covering space of $\mathbb{M}_n$ Thus we can use $\mathcal{P}_1(n)$ as a coordinate space for the moduli space $\mathbb{M}_n$, though it remains the ambiguity up to the group $G(n-1)$. This coordinate space has the advantages of being easy to be treated.

However, it would be also worthwhile to introduce another coordinate system having any merit different from $\mathcal{P}_1(n)$'s. In fact, Milnor successfully introduced coordinates in the moduli space of the space of all quadratic rational maps using the elementary symmetric functions of the multipliers at the fixed points of a map ([Mil93]). In the case of $\mathrm{Poly}_n(\mathbb{C})$, we try to explore an analogy to this in section 2.

2 Polynomials of degree n

2.1 Moduli space of polynomial maps

Let $\widehat{\mathbb{C}}$ be the Riemann sphere, and $\mathrm{Poly}_n(\mathbb{C})$ be the space of all polynomial maps of degree n from $\widehat{\mathbb{C}}$ to itself: $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \quad (a_n \neq 0)$. The group $\mathfrak{A}(\mathbb{C})$ of all affine transformations acts on $\mathrm{Poly}_n(\mathbb{C})$ by conjugation:

$$g \circ p \circ g^{-1} \in \mathrm{Poly}_n(\mathbb{C}) \quad \text{for} \quad g \in \mathfrak{A}(\mathbb{C}),\ p \in \mathrm{Poly}_n(\mathbb{C}).$$

Two maps $p_1, p_2 \in \mathrm{Poly}_n(\mathbb{C})$ are **holomorphically conjugate** if and only if there exists $g \in \mathfrak{A}(\mathbb{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. Under the conjugacy of the action of $\mathfrak{A}(\mathbb{C})$, it can be assumed that any map in $\mathrm{Poly}_n(\mathbb{C})$ is "monic" and "centered", i.e.,

$$p(z) = z^n + c_{n-2}z^{n-2} + c_{n-3}z^{n-3} \cdots + c_0.$$

This p is determined up to the action of the group $G(n-1)$ of $(n-1)$-st roots of unity, where each $\eta \in G(n-1)$ acts on $p \in \mathrm{Poly}_n(\mathbb{C})$ by the transformation $p(z) \mapsto p(\eta z)/\eta$. For example, in the case of $n = 4$ the following three monic and centered polynomials belong to the same conjugacy class:

$$\begin{aligned} &z^4 + az^2 + bz + c \\ &z^4 + a\omega z^2 + bz + c\omega^2 \\ &z^4 + a\omega^2 z^2 + bz + c\omega \end{aligned}$$

where ω is a third root of unity.

The quotient space of $\mathrm{Poly}_n(\mathbb{C})$ under this action will be denoted by $\mathbb{M}_n$, and called the **moduli space** of holomorphic conjugacy classes $\langle p \rangle$ of polynomial maps p of degree n. Let $\mathcal{P}_1(n)$ be the affine space of all monic centered polynomials of degree n with coordinate $(c_0, c_1, \cdots, c_{n-2})$. Then we have an $(n-1)$-to-one canonical projection Φ from $\mathcal{P}_1(n)$ onto $\mathbb{M}_n$. Thus we can use $\mathcal{P}_1(n)$ as coordinate space for $\mathbb{M}_n$ though there remains the ambiguity up to the group $G(n-1)$.

2.2 Multiplier coordinates

Now we intend to explore another coordinate space for $\mathbb{M}_n$. For each $p(z) \in \mathrm{Poly}_n(\mathbb{C})$, let $z_1, \cdots, z_n, z_{n+1}(=\infty)$ be the fixed points of p and μ_i the multipliers of z_i ; $\mu_i = p'(z_i)$ $(1 \leq i \leq n)$, and $\mu_{n+1} = 0$. Consider the elementary symmetric functions of the n multipliers,

$$
\begin{aligned}
&\sigma_{n,1} = \mu_1 + \cdots + \mu_n,\\
&\sigma_{n,2} = \mu_1\mu_2 + \cdots + \mu_{n-1}\mu_n = \textstyle\sum_{i=1}^{n-1} \mu_i \sum_{j>i}^{n} \mu_j,\\
&\quad \cdots\\
&\sigma_{n,n} = \mu_1\mu_2\cdots\mu_n,\\
&\sigma_{n,n+1} = 0.
\end{aligned}
$$

Note that these are well defined on the moduli space $\mathbb{M}_n$, since μ_i's are invariant by affine conjugacy.

2.2.1 The holomorphic index fixed point formula

For an isolated fixed point $f(x_0) = x_0, \quad x_0 \neq \infty$ we define the holomorphic index of f at x_0 to be the residue

$$\iota(f,\ x_0) = \frac{1}{2\pi i}\oint \frac{1}{z - f(z)}dz$$

For the point at infinity, we define the residue of f at ∞ to be equal to the residue of $\phi \circ f \circ \phi$ at origin, where $\phi(z) = \frac{1}{z}$. The Fatou index theorem (see [?]) is as follows: For any rational map $f : \mathbf{C} \to \mathbf{C}$ with $f(z)$ not identically equal to z, we have the relation $\sum_{f(z)=z} \iota(f,\ z) = 1$. This theorem can be applied to these μ_i's ; $\sum_{i=1}^{n} \frac{1}{1-\mu_i} + \frac{1}{1-0} = 1$, provided $\mu_i \neq 1 (1 < i < n)$. Arranging this equation for the form of elementary symmetric functions, we have

$$\gamma_0 + \gamma_1\sigma_{n,1} + \gamma_2\sigma_{n,2} + \cdots + \gamma_{n-1}\sigma_{n,n-1} = 0$$

where

$$\gamma_k = (-1)^k n \binom{n-1}{k} \Big/ \binom{n}{k} = (-1)^k(n-k).$$

Note that $\mu_i = 1$ $(1 \leq i \leq n)$ is allowable here. Then we have the following Linear Relation :

Theorem 1 *Among $\sigma_{n,i}$'s, there is a linear relation*

$$\sum_{k=0}^{n-1} (-1)^k (n-k)\sigma_{n,k} = 0, \tag{1}$$

where we put $\sigma_{n,0} = 1$.

- For the cubic case ($n = 3$),we have $3 - 2\sigma_{3,1} + \sigma_{3,2} = 0$
- For the quartic case ($n = 4$), we have $4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0$

In view of Theorem , we have the natural map Ψ from $\mathbb{M}_n$ to $\mathbb{C}^{n-1}$ corresponding to $\Psi(p) = (\sigma_{n,1}, \sigma_{n,2}, \cdots, \sigma_{n,n-2}, \sigma_{n,n})$.

Let $\Sigma(n)$ be the image $\Psi(\mathbb{M}_n)(\subset \mathbb{C}^{n-1})$.

2.2.2 Characterization of exceptional set

To investigate whether this map Ψ is surjective or not is our main subject: a problem of characterization of the part of $\mathbb{C}^{n-1} \setminus \Sigma(n)$.

We call this set **exceptional set** and denote it by $\mathcal{E}_n = \mathbb{C}^{n-1} \setminus \Sigma(n)$.

our main subject is as follows:
For a given $(s_1, s_2, \cdots, s_{n-2}, s_n) \in \mathbb{C}^{n-1}$, we set s_{n-1} a solution of $\Sigma_{k=0}^{n-1}(-1)^k(n-k)s_k = 0$, $s_0 = 1$. Then for the point $(s_1, \cdots, s_n) \in \mathbb{C}^{n-1}$, we set a polynomial

$$m(z) = z^n + s_1 z^{n-1} + s_2 z^{n-2} + \cdots + s_{n-1}z + s_n$$

Then w denote the roots of this polynomial by

$$\mu_1, \mu_2, \cdots, \mu_{n-1}, \mu_n,$$

. **Can we obtain a polynomial $p(z) \in \mathcal{P}_1(n)$ corresponding to $(s_1, \cdots, s_n)$ as $(\sigma_1, \cdots, \sigma_n)$?**

Namely can we find a polynomial satisfying that for fixed points z_i

$$p(z_i) = z_i, \quad (i = 1, \cdots, n) \ \text{ with } \ \mu_i = p'(z_i).$$

The case $n = 3$ is nicely solved:Ψ is surjective. (This fact is mentioned in [Mil93] without any details.) We solved this problem for the case $n = 4$: Ψ is no longer surjective.

As for the cases of general n, we expect analogous results.

We have a following result:

Theorem 2 *(M.FUJIMURA)*
If a polynomial $m(z)$ has n roots $\mu_i \neq 1$ satisfying $\sum_i \frac{1}{b_i} = 0 b_i = 1 - \mu_i$, and for any proper subset S of roots, $\sum_{s \in S} \frac{1}{b_s} \neq 0$, then there exists a polynomial $p(z) \in \mathcal{P}_1(n)$ such that

$$p(z_i) = z_i, \quad (i = 1, \cdots, n) \text{ with } \quad \mu_i = p'(z_i).$$

Examples
- For a set $\{\mu, 2-\mu, \lambda, 2-\lambda\}$, $\mu \neq \lambda$, $\mu \neq 1$ a corresponding polynomial exits.
- For a set $\{\mu, 2-\mu, \mu, 2-\mu\}$ $\mu \neq 1$, no corresponding polynomial exits.
- For a set $\{\mu, \mu, \mu, \lambda, \lambda\}$, $\mu \neq 1$, $5 - 2\mu - 3\lambda = 0$ a corresponding polynomial exits.
- For a set $\{\mu, \mu, \mu, 2-\mu, \frac{3-\mu}{2}\}$, $\mu \neq 1$, no corresponding polynomial exits.

2.3 Polynomials of degree 3

2.3.1 Moduli space $\mathbb{M}_3(\mathbb{C})$

Here we abbreviate $\sigma_{3,i}$ as σ_i. These σ_i $i = 1, 2, 3$ are defined on $\mathbb{M}_3(\mathbb{C})$, with the linear relation: $3 - 2\sigma_1 + \sigma_2 = 0$

For the cubic case, we can show that the excetional set is empty: namely for any point $(s_1, s_3) \in \mathbb{C}^2$, we can regard it as a point of $(\sigma_1, \sigma_3) \in \Sigma(3)$ satisfying the above relation $3 - 2\sigma_1 + \sigma_2 = 0$. Therefore, $(s_1, s_3) \in \mathbb{C}^2$ uniquely determines $\langle p \rangle \in \mathbb{M}_3(\mathbb{C})$.

In fact, a map in $\mathrm{Poly}_3(\mathbb{C})$ is conjugate to a normal form $z^3 + az + b$, whose parameter (a, b^2) is unique to the class $\langle p \rangle$. (a, b^2) relates to (σ_1, σ_3) as follows:

Translation Formula for Cubic Polynomials

$$\begin{aligned} \sigma_1 &= -3a + 6, \\ \sigma_3 &= 27b^2 + a(2a - 3)^2, \end{aligned} \tag{2}$$

Inverse Formula for Cubic Polynomials

$$\begin{aligned} a &= (6 - \sigma_1)/3, \\ b^2 &= (4\sigma_1^3 - 36\sigma_1^2 + 81\sigma_1 + 27\sigma_3 - 54)/729. \end{aligned} \tag{3}$$

Proposition 1 *(σ_1, σ_3) is a coordinate system of $\mathbb{M}_3(\mathbb{C})$.*

2.4 Polynomials of degree 4

2.4.1 Moduli space $\mathbb{M}_4(\mathbb{C})$

In the case of $\mathrm{Poly}_4(\mathbb{C})$, we can go on further analysis by using a symbolic and algebraic computation systems. Here we write $\sigma_{4,i} = \sigma_i$ $(i = 1, \cdots, 4)$ for brevity. Set $\mathrm{Poly}_4(\mathbb{C}) \ni p(z) = a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0$, $\mathcal{P}_1(4) \ni p(z) = z^4 + c_2z^2 + c_1z + c_0$, $\mathbb{M}_4 \ni \langle p \rangle$,

$$\begin{aligned} p &= z^4 + c_2z^2 + c_1z + c_0 \\ &\sim z^4 + \omega c_2z^2 + c_1z + \omega^2c_0 \\ &\sim z^4 + \omega^2c_2z^2 + c_1z + \omega c_0 \\ &\qquad\qquad\qquad (\omega^3 = 1) \end{aligned}$$

There are natural projections:

$$\begin{aligned} \Phi &: \mathcal{P}_1(4) \longrightarrow \mathbb{M}_4 \quad \text{three-to-one map} \\ \Psi &: \mathbb{M}_4 \longrightarrow \Sigma(4) \subset \mathbb{C}^{n-1} \quad \text{two-to-one in general.} \end{aligned}$$

2.4.2 Excetional set

For a polynomial $p(z) = a_4z^4+a_3z^3+a_2z^2+a_1z+a_0$, we chose $z^4+c_2z^2+c_1z+c_0 \in < p >$ and set $\Sigma(4) \ni (\sigma_1, \sigma_2, \sigma_4)$. For the quartic case, a linear relation is as follows; $4 - 3\sigma_1 + 2\sigma_2 - \sigma_3 = 0$. We have a following transformation formula:

$$\begin{aligned}
\sigma_1 &= -8c_1 + 12 \\
\sigma_2 &= 4c_2^3 - 16c_0c_2 + 18c_1^2 - 60c_1 + 48 \\
\sigma_4 &= 16c_0c_2^4 + (-4c_1^2 + 8c_1)c_2^3 - 128c_0^2c_2^2 + \\
&\quad (144c_0c_1^2 - 288c_0c_1 + 128c_0)c_2 - \\
&\quad 27c_1^4 + 108c_1^3 - 144c_1^2 + 64c_1 + 256c_0^3
\end{aligned}$$

We have the following result :

Theorem 3 *Exception set is a puncuted curve:*

$$\mathcal{E}_4 = \{(4, s, \frac{s^2}{4} - 2s + 4), \ s \neq 6, \ \ s \in \mathbb{C}\}$$

2.4.3 On $\mathcal{E}_4$

To a point

$$(\sigma_1, \sigma_2, \sigma_4) = (4, s, \frac{s^2}{4} - 2s + 4),$$

we set a polynomial

$$m(z) = z^4 - \sigma_1z^3 + \sigma_2z^2 - \sigma_3z + \sigma_4$$

where

$$4 - 3\sigma_1 + 2\sigma_2 - \sigma_3 = 0.$$

Let roots of this polynomial $m(z)$ be $\mu,\ \mu,\ 2-\mu,\ 2-\mu,$ and $(\sigma_1, \sigma_2, \sigma_4) = (4,\ -2(\mu^2 - 2\mu - 2),\ \mu^4 - 4\mu^3 + 4\mu\ ),\ \mu \neq 1$. Then we consider that on the excetional set $\mathcal{E}_4$, quadratic polynomials

$$z^2 - \frac{1}{4}\mu^2 + \frac{1}{2}\mu$$

are doubled.

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THEORY AND METHODS

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Spiral traveling wave solutions of some parabolic equations on annuli

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Abstract

This paper deals with spiral traveling wave solutions of some parabolic equations on annuli related to a model of the motion of screw dislocations. We prove the existence, stability and uniqueness of spiral traveling wave solutions. Next we consider a model equation for screw dislocations and study the properties of spiral solutions for the equation of interface motion which is formally derived in the singular limit of the model equation.

1 Introduction

In this paper we shall investigate a semilinear parabolic equation on a two-dimensional annulus:

$$\begin{cases} u_t = \Delta u + g(u-\theta), & x \in \Omega,\ t>0, \\ u_r = 0, & x \in \partial\Omega,\ t>0, \end{cases} \tag{1}$$

where $\Omega = \{x \in \mathbf{R}^2 \mid a < |x| < b\}$, (r,θ) denotes the polar coordinates of $x \in \overline{\Omega}$ and g is the derivative of a multi-well potential.

Our motivation for studying problem (1) originates from crystallization processes in material sciences. Screw dislocations are observed on the surface of actual crystals such as silicon carbide, calcogen, paraffin and polyethylene ([19]). Frank [6], [3] originally proposed the following mechanism of the formation of screw dislocations: Crystals generally contain lattice defects. Once a lattice defect reaches the surface of a crystal, the defect creates a mononuclear layer (or a step) on it. Since the velocity of progress of the step is assumed to be the same at any point, the angular velocity near the corner of the defect is faster than that at the edge. Thus, the dislocation proceeds in a spiral shape.

Recently Kobayashi [10] has proposed the following reaction-diffusion equation as a model of the motion of screw dislocations:

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$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f(u-\theta\,;\varepsilon), & x \in \Omega,\ t>0 \\ u_r = 0, & x \in \partial\Omega,\ t>0, \end{cases} \tag{2}$$

where the parameter $\varepsilon > 0$ is sufficiently small and $f(\cdot\,;\varepsilon)$ is the derivative of a multi-well potential for each ε. The unknown function $u(x,t)$ represents the normalized height of the crystal. Some numerical experiments imply that equation (2) has a rotating and growing solution with a spiral shape. The purpose of the present paper is to show the existence, uniqueness and stability of such a solution, which we call a *spiral traveling wave solution*. More precisely, a solution $\overline{u}(x,t)$ of (2) or (1) is called a spiral traveling wave solution with growth speed ω if it is written in the form

$$\overline{u}(x,t) = \varphi(r, \theta - \omega t) + \omega t, \quad x \in \Omega,\ t>0. \tag{3}$$

Since the reaction term is very large, equation (2) gives rise to sharp internal layers (or interfaces). As we will see later, the motion of such interfaces is driven by their curvature. To be more precise, each interface moves according to the equation

$$V = c - \kappa \tag{4}$$

in the singular limit as $\varepsilon \to 0$, where V and κ denote the normal velocity and the curvature of the interface respectively, and c is a positive constant determined by the nonlinearity f. Equation (4) also arises from the kinematic theory in excitable media as Belousov-Zhabotinskii reagent. For mathematical results in this area we refer to [9], [12] and references therein.

Our paper is organized as follows: In Section 2 we introduce basic notation and state our main results (Theorem A — on the existence — and Theorem B — on the uniqueness and the stability —). We prove Theorems A and B in Section 3. In Section 4 we present a formal derivation of the equation of interface motion corresponding to equation (2). In Section 5 we study spiral solutions with constant angular speed for the interface equation (Theorem C). In Appendix we recall monotonicity and convergence results in order-preserving dynamical systems in the presence of symmetry obtained by Ogiwara and Matano [16, Propositions B1 and B2]. These results play a crucial role in the proof of Theorems A and B.

By (3) a spiral traveling wave solution $\overline{u}$ with growth speed ω satisfies

$$\overline{u}(x, t+T_0) = \overline{u}(x,t) + 2\pi, \qquad x \in \overline{\Omega},\ t>0, \tag{5}$$

where $T_0 = 2\pi/\omega$. Solutions with property (5) have been studied for other equations such as systems of ordinary differential equations ([11], [7], [2]) and parabolic equations in the whole space $\mathbf{R}^N$ ([15]). The methods of these literatures are based on the theory of dynamical systems and are, in essence, same as that of [16]. For our problem (1), as we will see in Lemma 5, if a solution $\overline{u}$ satisfies (5) for some T_0 then it is a spiral traveling wave solution with growth speed $2\pi/T_0$.

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2 Main results

Throughout this paper, we assume that the nonlinearity $g(v)$ satisfies the following:

(A1) g is a smooth, 2π-periodic function on $\mathbf{R}$;

(A2) g has three zeroes $0 < \zeta < 2\pi$ in the interval $[0, 2\pi]$;

(A3) $\displaystyle\int_0^{2\pi} g(v)dv > 0$.

It is known that, for any $u_0 \in C(\overline{\Omega})$, there exists a solution $u(x,t)$ of (1) with initial data $u(\cdot, 0) = u_0$ (see [13]). Here $C(\overline{\Omega})$ denotes Banach space of continuous functions on $\overline{\Omega}$ endowed with the norm $\|u_0\|_{C(\overline{\Omega})} = \sup\{|u_0(x)| \mid x \in \overline{\Omega}\}$. For u_1, $u_2 \in C(\overline{\Omega})$ we write

$$\begin{array}{lll} u_1 \le u_2 & \text{if} & u_1(x) \le u_2(x), \quad x \in \overline{\Omega}, \\ u_1 < u_2 & \text{if} & u_1(x) \le u_2(x) \ \text{and} \ u_1(x) \not\equiv u_2(x), \quad x \in \overline{\Omega}, \\ u_1 \ll u_2 & \text{if} & u_1(x) < u_2(x), \quad x \in \overline{\Omega}. \end{array} \tag{6}$$

Let $\{\Phi_t\}_{t\in[0,\infty)}$ be the local semiflow on $C(\overline{\Omega})$ generated by (1). In other words, the map Φ_t on $C(\overline{\Omega})$ is defined by

$$\Phi_t(u_0) = u(\cdot, t) \qquad \text{for each } t \in [0, \infty),$$

where $u(x,t)$ is a solution of (1) with initial data $u(\cdot, 0) = u_0$. The strong maximum principle ([17]) shows that Φ_t is strongly order-preserving, that is, $u_1 < u_2$ implies $\Phi_t(u_1) \ll \Phi_t(u_2)$ for each $t > 0$. Further the standard parabolic estimate ([13]) shows that Φ_t is a compact map on $C(\overline{\Omega})$ for each $t > 0$.

Definition 1
A solution $\overline{u}(x,t)$ of (1) is called a spiral traveling wave solution if it is written in the form

$$\overline{u}(x,t) = \varphi(r, \theta - \omega t) + \omega t, \quad x \in \Omega,\ t > 0$$

for some function $\varphi(r, \xi)$ and some constant ω. We call the constant ω the growth speed of the spiral traveling wave solution $\overline{u}$.

Remark 1
Clearly, if $\overline{u}(x,t) = \overline{u}(r, \theta, t)$ is a spiral traveling wave solution of (1), then $\overline{u}(x, t+\tau)$ is also a spiral traveling wave solution for any constant τ. Further, $\overline{u}(r, \theta - \alpha, t) + \alpha$ is also a spiral traveling wave solution for any constant α.

It is easily seen that if $\varphi(r, \theta - \omega t) + \omega t$ is a spiral traveling wave solution of (1) then $\varphi(r, \xi)$ satisfies

$$-\omega\varphi_\xi + \omega = \Delta\varphi + g(\varphi - \xi). \tag{7}$$

Lemma 2
If a spiral traveling wave solution for (1) exists, then its growth speed is positive.

Proof Let $\varphi(r, \theta - \omega t) + \omega t$ be a spiral traveling wave solution. Then (7) is fulfilled. Multiplying both sides of (7) by $\varphi_\xi - 1$ and integrating over Ω, we have

$$\begin{aligned} -\omega \int_\Omega (\varphi_\xi - 1)^2 \, dx &= \int_\Omega \{\Delta\varphi \cdot (\varphi_\xi - 1) + g(\varphi - \xi) \cdot (\varphi_\xi - 1)\} dx \\ &= -\frac{b^2 - a^2}{2} \int_0^{2\pi} g(v) \, dv, \end{aligned}$$

and hence

$$\omega = \frac{(b^2 - a^2) \displaystyle\int_0^{2\pi} g(v) \, dv}{2 \displaystyle\int_\Omega (\varphi_\xi - 1)^2 \, dx}.$$

This proves the lemma. ∎

Definition 3
A spiral traveling wave solution $\overline{u}$ of (1) is called stable if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\|u(\cdot, t) - \overline{u}(\cdot, t)\|_{C(\overline{\Omega})} < \varepsilon, \qquad t > 0$$

holds for any solution u of (1) satisfying $\|u(\cdot, 0) - \overline{u}(\cdot, 0)\|_{C(\overline{\Omega})} < \delta$.

Concerning the existence, stability and uniqueness of spiral traveling wave solutions, we obtain the following:

Theorem A
For any $b > a > 0$, (1) possesses a spiral traveling wave solution.

Theorem B

(i) *A spiral traveling wave solution $\overline{u}$ of (1) is stable and is monotone increasing in t, that is, $\overline{u}_t(x, t) > 0$ for all $x \in \Omega$, $t > 0$. Further it is unique up to translation to the t-direction, namely, if u is a spiral traveling wave solution of (1) then there exists some $\tau_0 \in \mathbf{R}$ such that $u(\cdot, t) = \overline{u}(\cdot, t + \tau_0)$ for $t > 0$.*

(ii) *For any solution u of (1), there exists some τ_0 such that*

$$\lim_{t \to \infty} \|u(\cdot, t) - \overline{u}(\cdot, t + \tau_0)\|_{C(\overline{\Omega})} = 0. \tag{8}$$

Remark 2

From Theorem B, we see that a spiral traveling wave solution $\overline{u}$ of (1) is stable with asymptotic phase, namely, it is stable and, for any solution u of (1) with initial data sufficiently close to $\overline{u}$, there exists some τ_0 such that (8) holds.

3 Proof of Theorems A and B

In this section, we prove Theorems A and B. In what follows $x \in \overline{\Omega}$ will be often identified with (r, θ), the polar coordinates of x.

Lemma 4

Let $v(x,t)$ be a solution of (1) with initial data $v(\cdot, 0) \equiv 0$. Then there exists some constant $M > 0$ such that

$$\max\{v(x,t) \mid x \in \overline{\Omega}\} - \min\{v(x,t) \mid x \in \overline{\Omega}\} < M$$

for all $t > 0$.

Proof Differentiating (1) by θ, we see that the function $w(x,t) = v_\theta(x,t) - 1$ satisfies

$$\begin{cases} w_t = \Delta w + g'(v-\theta)w, & x \in \Omega,\ t > 0, \\ w_r = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{9}$$

Since $w(\cdot, 0) = -1 < 0$, from the strong maximum principle it follows that $w(\cdot, t) < 0$, namely

$$v_\theta(\cdot, t) < 1, \qquad t > 0.$$

Hence, using the fact that $v(r, 0, t) = v(r, 2\pi, t)$, we have

$$\theta - 2\pi < v(r,\theta,t) - v(r,0,t) < \theta, \qquad a \le r \le b,\ 0 \le \theta \le 2\pi,\ t > 0.$$

Thus

$$v(r,\theta,t) - v(a,\theta,t) - 2\pi < v(r,0,t) - v(a,0,t) < v(r,\theta,t) - v(a,\theta,t) + 2\pi \tag{10}$$

holds for $a \le r \le b$, $0 \le \theta \le 2\pi$, $t > 0$. Now fix $t_0 > 0$ arbitrarily and take a small constant $\delta > 0$ such that $(v_\theta - \delta v_t)(\cdot, t_0) < 1$ and $(v_\theta + \delta v_t)(\cdot, t_0) < 1$. Since $v_\theta \pm \delta v_t - 1$ are also solutions of (9), in the same way as above we get $(v_\theta \pm \delta v_t)(\cdot, t) < 1$ for $t > t_0$. This implies, for $t > t_0$,

$$-\frac{1 - v_\theta}{\delta} < v_t < \frac{1 - v_\theta}{\delta}. \tag{11}$$

Multiplying each side of (11) by $r \in (a, b)$ and integrating by θ from 0 to 2π, we have

$$-\frac{2\pi b}{\delta} < \int_0^{2\pi} r v_t \, d\theta < \frac{2\pi b}{\delta}.$$

Since v satisfies (1) for $t > t_0$, integration by parts yields

$$-2\pi b\, C < \int_0^{2\pi} (rv_r)_r \, d\theta < 2\pi b\, C$$

with $C = (1/\delta) + \|f\|_{C(\overline{\Omega})}$. Integrating each side by r, dividing by r and integrating again, we get

$$-\frac{2\pi b(b-a)^2}{a} C < \int_0^{2\pi} \{v(r,\theta,t) - v(a,\theta,t)\}\, d\theta < \frac{2\pi b(b-a)^2}{a} C.$$

These inequalities and (10) yield

$$-\frac{b(b-a)^2}{a} C - 2\pi < v(r,0,t) - v(a,0,t) < \frac{b(b-a)^2}{a} C + 2\pi.$$

Therefore, again by (10), we obtain

$$-\frac{b(b-a)^2}{a} C - 4\pi < v(r,\theta,t) - v(a,0,t) < \frac{b(b-a)^2}{a} C + 4\pi.$$

Combining these inequalities and the fact that the set $\{v(\cdot,t) \mid 0 \le t \le t_0\}$ is a compact subset of $C(\overline{\Omega})$, we obtain the conclusion. ∎

Lemma 5
Let $\varphi(x) \in C(\overline{\Omega})$ satisfy $\varphi + 2\pi = \Phi_{T_0}(\varphi)$ for some $T_0 > 0$. Then $\varphi(r, \theta - \omega t) + \omega t$ is a solution of (1), where $\omega = 2\pi/T_0$.

We postpone the proof of Lemma 5 until the end of this section.

Proof of Theorem A Denote by $v(x,t)$ a solution of (1) with initial data $v(\cdot,0) \equiv 0$, in other words $v(\cdot,t) = \Phi_t(0)$. First we show that the orbit $\{v(\cdot,t) \mid t \ge 0\}$ is not bounded in $C(\overline{\Omega})$. Assuming that $\{v(\cdot,t) \mid t \ge 0\}$ is bounded in $C(\overline{\Omega})$, we will lead a contradiction. In this case, since a map Φ_t on $C(\overline{\Omega})$ is compact for each $t > 0$, the omega-limit set of 0 defined by

$$W(0) = \bigcap_{t>0} \overline{\{v(\cdot,s) \mid s > t\}} \subset C(\overline{\Omega})$$

is not empty. As is well-known, $W(0)$ is compact and Φ_t-invariant for each $t > 0$, namely $\Phi_t W(0) = W(0)$ (see for example [8]). Put

$$\alpha_0 = \inf\{\alpha > 0 \mid w_1 \le g_\alpha w_2 \text{ for any } w_1, w_2 \in W(0)\},$$

where $g_\alpha w(x) = g_\alpha w(r,\theta) = w(r, \theta - \alpha) + \alpha$ for $w(x) \in C(\overline{\Omega})$. Note that the map g_α on $C(\overline{\Omega})$ is commutative with Φ_t, namely, $g_\alpha \circ \Phi_t = \Phi_t \circ g_\alpha$.

Clearly $w_1 \le g_{\alpha_0} w_2$ holds for any $w_1, w_2 \in W(0)$. We show that $\alpha_0 = 0$. Assume that $\alpha_0 > 0$. If $w_1 < g_{\alpha_0} w_2$ for any $w_1, w_2 \in W(0)$ then $w_1 \ll g_{\alpha_0} w_2$ for any $w_1, w_2 \in W(0)$, since $W(0)$ is Φ_t-invariant and since Φ_t is strong order-preserving for any $t > 0$. In this case, compactness of $W(0)$ implies that if we choose $\delta > 0$ sufficiently small then $w_1 < g_{\alpha_0 - \delta} w_2$

for any $w_1, w_2 \in W(0)$, which contradicts the definition of α_0. Thus there exist some two elements $w_1, w_2 \in W(0)$ such that $w_1 = g_{\alpha_0} w_2$. Then it holds that

$$w \le w_1 \quad \text{for all } w \in W(0) \qquad \text{and} \qquad w_2 < w_1. \tag{12}$$

Since the latter inequality implies $\Phi_t(w_2) \ll \Phi_t(w_1)$ for any $t > 0$, by the definition of $W(0)$ there exist large $t_1,\, t_2 > 0$ satisfying

$$\Phi_{t_1}(0) \ll \Phi_{t_2}(0).$$

Therefore, if we choose $\varepsilon > 0$ sufficiently small then

$$\Phi_{t_1}(0) \ll g_{-\varepsilon}\Phi_{t_2}(0),$$

and hence

$$\Phi_{t_1+s}(0) \ll g_{-\varepsilon}\Phi_{t_2+s}(0)$$

for $s > 0$. Take a sequence $\{s_j\}_j$ such that $\Phi_{t_1+s_j}(0) \to w_1$ as $j \to \infty$. Replacing $\{\Phi_{t_2+s_j}(0)\}_j$ by its subsequence if necessary, we see that $\{\Phi_{t_2+s_j}(0)\}_j$ also converges to some $w_3 \in W(0)$. Then $w_1 \le g_{-\varepsilon} w_3$ holds. This and the former statement of (12) imply $w_3 \le g_{-\varepsilon} w_3$ and we are lead to a contradiction. Thus we obtain $\alpha_0 = 0$, from which for any $w_1,\, w_2 \in W(0)$ it follows that $w_1 \le w_2$ and $w_1 \ge w_2$, that is, $w_1 = w_2$. Hence $W(0)$ is a singleton. As is easily seen, if an omega-limit set is a singleton; then it consists of some equilibrium solution. This means that (1) possesses a spiral traveling wave solution with growth speed 0, which contradicts Lemma 2.

Thus we see that the orbit $\{v(\cdot,t) | t \ge 0\}$ is not bounded. Hence there exists some sequence $\{t_j\}_j$ such that $\|v(\cdot,t_j)\|_{C(\overline{\Omega})} \to \infty$ as $j \to \infty$. We discuss only the case where

$$\max\{v(x,t_j) \mid x \in \overline{\Omega}\} \to \infty, \qquad j \to \infty \tag{13}$$

and prove the existence of a spiral traveling wave solution with positive speed. The case where $\min\{v(x,t_j) \mid x \in \overline{\Omega}\} \to -\infty$ can be treated similarly. In the latter case there exists a spiral traveling wave solution with negative growth speed, which contradicts Lemma 2.

We show that there exists some $T_0 > 0$ such that $\varphi + 2\pi = \Phi_{T_0}(\varphi)$ for some function $\varphi(x) \in C(\overline{\Omega})$. Then, by Lemma 5, we see that (1) possesses a spiral traveling wave solution with growth speed $2\pi/T_0$. As in Lemma 4, there exists some constant $M > 0$ such that

$$\max\{v(x,t) \mid x \in \overline{\Omega}\} - \min\{v(x,t) \mid x \in \overline{\Omega}\} < M, \quad t > 0. \tag{14}$$

We take $n(j) \in \mathrm{N}$ so that the function v_j defined by $v_j(x) = v(x,t_j) - 2\pi n(j)$ satisfies

$$v_j(x) \in [0, M + 2\pi], \quad x \in \overline{\Omega}.$$

Fix $s > 0$ arbitrarily. Then, replacing $\{\Phi_s(v_j)\}_j$ by its subsequence, we see that $\{\Phi_s(v_j)\}_j$ converges to some $\varphi \in C(\overline{\Omega})$.

Note that (13) and (14) imply $2\pi < v(\cdot, T)$ for some $T > 0$. Therefore $\Phi_{s+t}(2\pi) < \Phi_{s+t+T}(0)$ holds for all $t > 0$. Putting $t = t_j$ we have $\Phi_{s+t_j}(0) + 2\pi < \Phi_{s+t_j+T}(0)$ and hence $\Phi_{s+t_j}(0) - 2\pi n(j) + 2\pi < \Phi_{s+t_j+T}(0) - 2\pi n(j) = \Phi_T(\Phi_{s+t_j}(0) - 2\pi n(j))$, since $\Phi_t(u_0) + 2\pi m = \Phi_t(u_0 + 2\pi m)$ holds for any $t > 0$, $m \in \mathrm{N}$ and $u_0 \in C(\overline{\Omega})$. Letting $j \to \infty$, we get $\varphi + 2\pi \leq \Phi_T(\varphi)$. Now set

$$T_0 = \inf\{t \geq 0 \mid \varphi + 2\pi \leq \Phi_t(\varphi)\}.$$

Clearly $0 < T_0 \leq T$ and $\varphi + 2\pi \leq \Phi_{T_0}(\varphi)$. Suppose that $\varphi + 2\pi < \Phi_{T_0}(\varphi)$. Then, for any $\delta > 0$, $\Phi_\delta(\varphi + 2\pi) = \Phi_\delta(\varphi) + 2\pi \ll \Phi_{T_0+\delta}(\varphi)$. From this, for a sufficiently large j_0, it follows that

$$\Phi_{\delta+s}(v_{j_0}) + 2\pi \ll \Phi_{T_0+\delta+s}(v_{j_0}).$$

Therefore, there exists some $\varepsilon \in (0, T_0)$ such that

$$\Phi_{\delta+s}(v_{j_0}) + 2\pi \ll \Phi_{T_0-\varepsilon+\delta+s}(v_{j_0}),$$

and hence

$$\Phi_{\delta+s+t}(v_{j_0}) + 2\pi \ll \Phi_{T_0-\varepsilon+\delta+s+t}(v_{j_0}), \quad t > 0.$$

Adding $2\pi n(j_0) - 2\pi n(j)$ to both sides and putting $t = t_j - t_{j_0} - \delta$, we get

$$\Phi_s(v_j) + 2\pi \ll \Phi_{T_0-\varepsilon}(\Phi_s(v_j)).$$

Hence letting $j \to \infty$ implies

$$\varphi + 2\pi \leq \Phi_{T_0-\varepsilon}(\varphi),$$

which contradicts the definition of T_0. Therefore $\varphi + 2\pi = \Phi_{T_0}(\varphi)$ holds and the proof is completed. ∎

Lemma 6
Let $u_1, u_2 \in C(\overline{\Omega})$ satisfy $u_1 + 2\pi = \Phi_{T_1}(u_1)$ and $u_2 + 2\pi = \Phi_{T_2}(u_2)$ for some $T_1, T_2 > 0$. Then $T_1 = T_2$.

Proof Suppose that the conclusion of the lemma does not hold. Without loss of generality, we may assume that $T_1 < T_2$. Take $n_0 \in \mathrm{N}$ satisfying $u_1 - 2n_0\pi \leq u_2$. Then $\Phi_{nT_2}(u_1) - 2n_0\pi \leq u_2 + 2n\pi$ for all $n \in \mathrm{N}$, and hence $\Phi_{n(T_2-T_1)}(u_1) - 2n_0\pi \leq u_2$. This contradicts $\|\Phi_{l_nT_1+s_n}(u_1)\|_{C(\overline{\Omega})} = \|\Phi_{s_n}(u_1) + 2l_n\pi\|_{C(\overline{\Omega})} \to \infty$ as $n \to \infty$, where $n(T_2 - T_1) = l_nT_1 + s_n$ with $l_n \in \mathrm{N}$, $s_n \in [0, T_1)$. ∎

Proof of Theorem B (i) First, by applying Proposition B1 in [16] (which will be mentioned in Appendix of the present paper), we prove the uniqueness and monotonicity of a spiral traveling wave solution. Set an ordered metric space $X = C(\overline{\Omega})$ with order relation induced by (6) and put

$$X_1 = Y = \{u_0 \in C(\overline{\Omega}) \mid u_0 + 2\pi = \Phi_T(u_0) \text{ for some } T > 0\}.$$

Clearly each spiral traveling wave solution $\overline{u}$ of (1) satisfies $\overline{u}(\cdot, 0) \in Y$. By Lemma 6

$$Y = \{u_0 \in C(\overline{\Omega}) \mid u_0 + 2\pi = \Phi_{T_0}(u_0)\}$$

holds for some $T_0 > 0$. The semiflow $\{\Phi_t\}_{t\in[0,\infty)}$ generated by (1) can be defined on Y for all $t \in \mathbf{R}$. Thus $\{\Phi_t\}_{t\in[0,\infty)}$ is extended to a one-parameter group acting on Y. Denote this group by G. Then condition (G2) in Appendix is satisfied. Further (G1) is fulfilled. Indeed the map Φ_t on Y is also order-preserving for $t < 0$. Fix a spiral traveling wave solution $\overline{u}$ arbitrarily. Then a pair Y and $\overline{\varphi} = \overline{u}(\cdot, 0)$ satisfies (H1) and (H2). Further (H3) holds since by the strong maximum principle $\psi < h\overline{\varphi}$ implies $\psi \ll h\overline{\varphi}$ for any $\psi \in Y$, $h \in G$. Applying Proposition B1 in [16], we see that $Y = G\overline{\varphi}$ and that Y is homeomorphic and order-isomorphic to $\mathbf{R}$. By $Y = G\overline{\varphi}$ we obtain the uniqueness of a spiral traveling wave solution up to translation to the t-direction. Moreover Lemma 2 and monotonicity of $Y = G\overline{\varphi}$ yield that $\overline{u}_t(x,t) \geq 0$ and $\overline{u}_t(x,t) \not\equiv 0$ for $x \in \overline{\Omega}$, $t > 0$. Therefore, from the strong maximum principle it follows that $\overline{u}_t(x,t) > 0$ for $x \in \overline{\Omega}$, $t > 0$.

Next we show that a spiral traveling wave solution $\overline{u}$ is stable. By the positivity of $\overline{u}_t$ if $t_1 < t_2$ then $\overline{u}(\cdot, t_1) \ll \overline{u}(\cdot, t_2)$. Further by the maximum principle we have, for any $\delta_0 > 0$,

$$\overline{u}(\cdot, -\delta_0) \leq u(\cdot, 0) \leq \overline{u}(\cdot, \delta_0) \quad \text{implies} \quad \overline{u}(\cdot, t-\delta_0) \leq u(\cdot, t) \leq \overline{u}(\cdot, t+\delta_0), \ \ t > 0.$$

This proves the stability of a spiral traveling wave solution. Indeed, for any $\varepsilon > 0$, take a $\delta_0 > 0$ satisfying $\|\overline{u}(\cdot, \delta_0) - \overline{u}(\cdot, -\delta_0)\|_{C(\overline{\Omega})} < \varepsilon$ and set

$$\delta = \min\{\overline{u}(x, \delta_0) - \overline{u}(x, 0) \mid x \in \overline{\Omega}\} = \min\{\overline{u}(x, 0) - \overline{u}(x, -\delta_0) \mid x \in \overline{\Omega}\} > 0.$$

Then, for any solution u of (1) satisfying $\|u(\cdot, 0) - \overline{u}(\cdot, 0)\|_{C(\overline{\Omega})} < \delta$, we have

$$\overline{u}(\cdot, -\delta_0) < u(\cdot, 0) < \overline{u}(\cdot, \delta_0).$$

Therefore, from the inequalities

$$\begin{aligned} \overline{u}(\cdot, t-\delta_0) < u(\cdot, t) < \overline{u}(\cdot, t+\delta_0), \\ \overline{u}(\cdot, t-\delta_0) < \overline{u}(\cdot, t) < \overline{u}(\cdot, t+\delta_0) \end{aligned}$$

it follows that

$$\|u(\cdot, t) - \overline{u}(\cdot, t)\|_{C(\overline{\Omega})} < \|\overline{u}(\cdot, t+\delta_0) - \overline{u}(\cdot, t-\delta_0)\|_{C(\overline{\Omega})} = \|\overline{u}(\cdot, \delta_0) - \overline{u}(\cdot, -\delta_0)\|_{C(\overline{\Omega})} < \varepsilon$$

for all $t > 0$. ∎

Proof of Theorem B (ii) As we have shown above, (1) possesses a unique (up to translation to the t-direction) spiral traveling wave solution $\overline{u}$. We denote by ω the growth speed of $\overline{u}$.

Define a map F on $X = C(\overline{\Omega})$ by

$$F(u_0)(r, \theta) = \Phi_{T_0}(u_0)(r, \theta) - 2\pi,$$

where $T_0 = 2\pi/\omega$. Then, $\overline{\varphi} = \overline{u}(\cdot, 0)$ is a fixed point of F and further $\overline{\varphi} - 2m\pi$, $\overline{\varphi} + 2m\pi$ are also fixed points for all $m \in \mathrm{N}$. For any $u_0 \in X$ a sequence $\{F^n(u_0)\}_n$ is bounded in X, since $\overline{\varphi} - 2m\pi \leq u_0 \leq \overline{\varphi} + 2m\pi$ implies $\overline{\varphi} - 2m\pi \leq F^n(u_0) \leq \overline{\varphi} + 2m\pi$ for $m, n \in \mathrm{N}$. Hence the set $K(u_0) = \bigcap_{n \in \mathrm{N}} \overline{\{F^m(u_0) \mid m > n\}} \subset X$ is not empty. Set

$$Y = \{K(u_0) \mid u_0 \in X\}$$

and an acting group G being as in the proof of Theorem B (i). Clearly (G1) and (G2) in Appendix are fulfilled. A pair Y and $\{\overline{\varphi}\}$ satisfies (H4) and (H5). Further the strong maximum principle verifies (H6). Hence applying Proposition B2 in [16] (which will be mentioned in Appendix of the present paper), we see that for any $u_0 \in C(\overline{\Omega})$ there exists some τ_0 satisfying

$$\lim_{n \to \infty} \|F^n u_0 - \overline{u}(\cdot, \tau_0)\|_{C(\overline{\Omega})} = 0.$$

By the definition of F we obtain the conclusion. ∎

Proof of Lemma 5 As we have shown in the proof of Theorem B (i), a function satisfying

$$w(x) + 2\pi = \Phi_{T_0}(w)(x), \qquad x \in \overline{\Omega} \tag{15}$$

is unique up to action of one-parameter group $\{\Phi_t\}_{t \in \mathrm{R}}$. Since $\varphi(r, \theta - (2\pi/m)) + (2\pi/m)$ also satisfies (15) for any $m \in \mathrm{N}$, there exists some $s \in \mathrm{R}$ such that

$$\varphi\left(r, \theta - \frac{2\pi}{m}\right) + \frac{2\pi}{m} = \Phi_s(\varphi)(r, \theta), \qquad a \leq r \leq b,\ 0 \leq \theta \leq 2\pi.$$

It follows from this that

$$\varphi\left(r, \theta - \frac{2 \cdot 2\pi}{m}\right) + \frac{2 \cdot 2\pi}{m} = \Phi_s(\Phi_s(\varphi))(r, \theta) = \Phi_{2s}(\varphi)(r, \theta), \qquad a \leq r \leq b,\ 0 \leq \theta \leq 2\pi.$$

Repeating this calculation, we obtain $\Phi_{ms}(\varphi) = \varphi + 2\pi$. If $ms \neq T_0$ then $\{\Phi_t(\varphi) \mid t \geq |T_0 - ms|\}$ is a periodic orbit with period $|T_0 - ms|$, which contradicts

$$\|\Phi_{nT_0}(\varphi)\|_{C(\overline{\Omega})} = \|\varphi + 2\pi n\|_{C(\overline{\Omega})} \to \infty, \qquad n \to \infty.$$

Hence we get $ms = T_0$, namely $s = T_0/m$. Thus we have, for any $k \in \mathrm{N}$,

$$\varphi\left(r, \theta - \frac{k \cdot 2\pi}{m}\right) + \frac{k \cdot 2\pi}{m} = \Phi_{\frac{kT_0}{m}}(\varphi)(r, \theta), \qquad a \leq r \leq b,\ 0 \leq \theta \leq 2\pi$$

and further, for any rational number $p > 0$,

$$\varphi(r, \theta - 2\pi p) + 2\pi p = \Phi_{pT_0}(\varphi)(r, \theta), \qquad a \leq r \leq b,\ 0 \leq \theta \leq 2\pi.$$

Since the set of positive rational numbers is dense in $(0, \infty)$, if we set $\omega = 2\pi/T_0$ then

$$\varphi(r, \theta - \omega t) + \omega t = \Phi_t(\varphi)(r, \theta), \qquad a \leq r \leq b,\ 0 \leq \theta \leq 2\pi$$

holds for any $t > 0$. The proof is completed. ∎

4 A formal derivation of the interface equation

In this section, we consider equation (2):

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f(u-\theta\,;\varepsilon), & x \in \Omega,\ t>0 \\ u_r = 0, & x \in \partial\Omega,\ t>0. \end{cases}$$

We assume that $f(v\,;\varepsilon) = -\frac{\partial W}{\partial v}(v\,;\varepsilon)$ is a smooth function derived from a multi-well potential $W(v\,;\varepsilon)$ whose local minima lie at $v = 2m\pi$ $(m \in \mathbf{Z})$ for all $\varepsilon \geq 0$. More precisely, we assume that $f(v\,;\varepsilon)$ satisfies the following conditions:

(F1) $f(v\,;\varepsilon)$ is 2π-periodic in v for each $\varepsilon \geq 0$,

(F2) $f(\cdot\,;\varepsilon)$ has exactly three zeroes $0 < \zeta(\varepsilon) < 2\pi$ in $[0, 2\pi]$ for each $\varepsilon \geq 0$,

(F3) $\dfrac{\partial f}{\partial v}(0\,;\varepsilon) < 0$ for each $\varepsilon \geq 0$,

(F4) $\displaystyle\int_0^{2\pi} f(v\,;0)dv = 0,\ \int_0^{2\pi} \frac{\partial f}{\partial \varepsilon}(v\,;0)dv > 0.$

By Theorems A and B, under the conditions (F1)–(F4), there exists a unique spiral traveling wave solution for each $\varepsilon > 0$. Roughly speaking, condition (F4) means that the difference of well-depth $W(2\pi\,;\varepsilon) - W(0\,;\varepsilon)$ is negative and that $W(2\pi\,;\varepsilon) - W(0\,;\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. It follows from (F1)–(F4) that there exists a unique solution $(\psi_\varepsilon(z), c(\varepsilon))$ of

$$\begin{cases} \psi_{zz} + \varepsilon c(\varepsilon)\psi_z + f(\psi\,;0) = 0, \quad z \in \mathbf{R}, \\ \psi(-\infty) = 2\pi,\ \ \psi(0) = \zeta(\varepsilon),\ \ \psi(+\infty) = 0, \end{cases} \tag{16}$$

for each $\varepsilon \geq 0$ ([5]). Note that $c(\varepsilon) > 0$ for $\varepsilon > 0$ and

$$c = \lim_{\varepsilon \to 0} c(\varepsilon) = \frac{\displaystyle\int_0^{2\pi} \frac{\partial f}{\partial \varepsilon}(v\,;0)dv}{\displaystyle\int_{\mathbf{R}} \{\psi_0'(z)\}^2 dz}. \tag{17}$$

Let u^ε be a solution of (2). Since the reaction term is very large and the potential W is multi-well type, u^ε approaches $\theta + 2m\pi$ for some $m \in \mathbf{Z}$ if $\theta + \zeta(\varepsilon) + 2(m-1)\pi < u^\varepsilon(x,0) < \theta + \zeta(\varepsilon) + 2m\pi$. Accordingly, a sharp interface appears between the regions $\{u^\varepsilon \approx \theta + 2m\pi\}$ and $\{u^\varepsilon \approx \theta + 2(m+1)\pi\}$ for each $m \in \mathbf{Z}$. By virtue of (F1), $u^\varepsilon(x,t) = u^\varepsilon(r,\theta,t)$ can be extended to a function (also denoted by u^ε) defined for all $\theta \in \mathbf{R}$ satisfying the following equation:

$$\begin{cases} u_t = u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} + \dfrac{1}{\varepsilon^2} f(u-\theta\,;\varepsilon), & (r,\theta) \in (a,b)\times\mathbf{R},\ t>0, \\ u_r(a,\theta,t) = 0 = u_r(b,\theta,t), & \theta \in \mathbf{R},\ t>0. \end{cases}$$

We fix $T>0$ and define

$$\widetilde{\Gamma}_t^{\varepsilon,m}=\{(r,\theta)\in(a,b)\times\mathbf{R}\mid u^\varepsilon(r,\theta,t)=\theta+\zeta(\varepsilon)+2m\pi\}$$

for $t\in[0,T]$. Since u^ε is 2π-periodic in θ, we have $\widetilde{\Gamma}_t^{\varepsilon,m}=\sigma_{-2m\pi}\widetilde{\Gamma}_t^{\varepsilon,0}$ where σ_s is the translation $\sigma_s:(r,\theta)\mapsto(r,\theta+s)$. For simplicity, we assume that $\widetilde{\Gamma}_t^{\varepsilon,0}$ is a smooth embedded curve in $(a,b)\times\mathbf{R}$ with two boundary points on both $\{a\}\times\mathbf{R}$ and $\{b\}\times\mathbf{R}$ for each $t\in[0,T]$. It follows from the homogeneous Neumann boundary conditions that the closure of $\widetilde{\Gamma}_t^{\varepsilon,0}$ intersects with the lines $r=a$ and $r=b$ perpendicularly at the boundary points. We denote by $\widetilde{D}_t^m$ the domain in $(a,b)\times\mathbf{R}$ between the two curves $\widetilde{\Gamma}_t^{\varepsilon,m}$ and $\widetilde{\Gamma}_t^{\varepsilon,m-1}$. Let Π be the covering map from $(a,b)\times\mathbf{R}$ to Ω defined by $\Pi(r,\theta)=(r\cos\theta,r\sin\theta)$. We take a neighborhood $\widetilde{N}_t$ of $\widetilde{\Gamma}_t^{\varepsilon,0}$ in $(a,b)\times\mathbf{R}$ so that the map $\Pi|_{\widetilde{N}_t}$ is injective. We put $N_t=\Pi(\widetilde{N}_t)$, $D_t^j=\Pi(\widetilde{D}_t^j\cap\widetilde{N}_t)$ $(j=0,1)$ and

$$N=\bigcup_{t\in[0,T]}(N_t\times\{t\}).$$

In what follows we write $\theta=\theta(x)$ for $x\in N_t$ if $x=\Pi(r,\theta)$.

We call the set

$$\Gamma^\varepsilon=\bigcup_{t\in[0,T]}(\Gamma_t^\varepsilon\times\{t\})$$

the interface, where

$$\Gamma_t^\varepsilon=\{\Pi(r,\theta)\in\Omega\mid(r,\theta)\in\widetilde{\Gamma}_t^{\varepsilon,0}\}.$$

We also call Γ_t^ε the interface at time t. We remark that if $x\in\Gamma_t^\varepsilon$ then $u^\varepsilon(x,t)=\theta(x)+\zeta(\varepsilon)$ and that $\overline{\Gamma_t^\varepsilon}$ perpendicularly intersects with $\partial\Omega$.

Let $d^\varepsilon(x,t)$ be the signed distance function to Γ^ε defined in N by

$$d^\varepsilon(x,t)=\begin{cases}\operatorname{dist}(x,\Gamma_t^\varepsilon), & \text{if } x\in D_t^0,\\ -\operatorname{dist}(x,\Gamma_t^\varepsilon), & \text{if } x\in D_t^1,\end{cases}$$

where $\operatorname{dist}(x,\Gamma_t^\varepsilon)$ is the distance from $x\in N_t$ to the curve Γ_t^ε in $\mathbf{R}^2$. We remark that $d^\varepsilon(x,t)=0$ if $x\in\Gamma_t^\varepsilon$ and $|\nabla d^\varepsilon|=1$. We assume that d^ε has the expansion

$$d^\varepsilon(x,t)=d_0(x,t)+\varepsilon d_1(x,t)+\varepsilon^2 d_2(x,t)+\cdots$$

and define

$$\begin{aligned}\Gamma_t &= \{x\in N_t\mid d_0(x,t)=0\},\\ \Omega_t^0 &= \{x\in N_t\mid d_0(x,t)>0\},\\ \Omega_t^1 &= \{x\in N_t\mid d_0(x,t)<0\},\\ \Gamma &= \bigcup_{t\in[0,T]}(\Gamma_t\times\{t\}),\\ Q_0 &= \bigcup_{t\in[0,T]}(\Omega_t^0\times\{t\}),\\ Q_1 &= \bigcup_{t\in[0,T]}(\Omega_t^1\times\{t\}).\end{aligned}$$

Roughly speaking, Γ_t represents the position of the interface at time t in the limit as $\varepsilon \to 0$, while d_0 represents the signed distance function to Γ. In what follows we derive the equation of motion of the interface Γ_t by using matched asymptotic expansions. See [1], [4], [14] and [18] for details.

We assume that the solution u^ε has the expansions

$$u^\varepsilon(x,t) = u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \cdots \tag{18}$$

away from Γ^ε (the outer expansion) and

$$u^\varepsilon(x,t) = U_0(\xi,x,t) + \varepsilon U_1(\xi,x,t) + \varepsilon^2 U_2(\xi,x,t) + \cdots \tag{19}$$

near Γ^ε (the inner expansion), where $\xi = d^\varepsilon(x,t)/\varepsilon$. To make these expansions consistent, we require the matching conditions

$$\begin{array}{ll} U_k(+\infty,x,t) = u_k^1(x,t) & \text{if } x \in \Omega_t^1 \cup \Gamma_t \\ U_k(-\infty,x,t) = u_k^0(x,t) & \text{if } x \in \Omega_t^0 \cup \Gamma_t \end{array} \tag{20}$$

for all $(x,t) \in N$ and $k \geq 0$, where u_k^j $(j=0,1)$ denote the terms of the outer expansion (18) in the region Q_j $(j=0,1)$. Since $u^\varepsilon(x,t) = \theta(x) + \zeta(\varepsilon)$ on Γ^ε, we also require the normalization conditions $U_0(0,x,t) = \theta(x) + \zeta_0$, $U_k(0,x,t) = \zeta_k$ $(k \geq 1)$, where ζ_j denote the terms of the expansion $\zeta(\varepsilon) = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \cdots$.

Substituting the outer expansion (18) into (2) and the collecting the ε^{-2} and ε^{-1} terms respectively, we have

$$\begin{aligned} & f(u_0(x,t) - \theta(x)\,;0) = 0, \\ & \frac{\partial f}{\partial v}(u_0(x,t) - \theta(x)\,;0)u_1(x,t) + \frac{\partial f}{\partial \varepsilon}(u_0(x,t) - \theta(x)\,;0) = 0, \end{aligned}$$

in $Q_0 \cup Q_1$. The first equation implies that

$$u_0(x,t) = \begin{cases} \theta(x) & \text{in } Q_0, \\ \theta(x) + 2\pi & \text{in } Q_1. \end{cases}$$

Hence from the second equation, we get $u_1(x,t) = 0$ in $Q_0 \cup Q_1$.

Next, substituting the inner expansion (19) into (2) and the collecting the ε^{-2} and ε^{-1} terms, we have

$$U_{0\xi\xi} + f(U_0 - \theta(x)\,;0) \;=\; 0, \tag{21}$$

$$\begin{aligned} U_{1\xi\xi} + \frac{\partial f}{\partial v}(U_0 - \theta(x)\,;0)U_1 \;=\; & U_{0\xi}(d_{0t} - \Delta d_0) - 2\nabla(U_{0\xi}) \cdot \nabla d_0 \\ & -\frac{\partial f}{\partial \varepsilon}(U_0 - \theta(x)\,;0). \end{aligned} \tag{22}$$

In both equations we regard x and t as parameters. From (21) together with the matching conditions and the normalization conditions, we obtain

$$U_0(\xi,x,t) = \psi_0(\xi) + \theta(x) \tag{23}$$

where ψ_0 is the unique solution to (16) for $\varepsilon = 0$.

Substituting (23) into (22) and recalling the normalization conditions, we get

$$\begin{cases} U_{1\xi\xi} + \dfrac{\partial f}{\partial v}(\psi_0(\xi)\,;0)U_1 = (d_{0t} - \Delta d_0)\psi_0'(\xi) - \dfrac{\partial f}{\partial \varepsilon}(\psi_0(\xi)\,;0), \\ U_1(0,x,t) = \zeta_1. \end{cases} \tag{24}$$

By Lemma 4.1 in [1], (24) has a bounded solution if and only if

$$(d_{0t} - \Delta d_0)\int_{\mathbf{R}} \{\psi_0'(\xi)\}^2 d\xi - \int_{\mathbf{R}} \frac{\partial f}{\partial \varepsilon}(\psi_0(\xi)\,;0)\psi_0'(\xi)d\xi = 0. \tag{25}$$

Under the solvability condition (25), the solution U_1 of (24) incidentally satisfies the matching conditions (20), since the right-hand side of the first equation of (24) tends to 0 exponentially as $\xi \to \pm\infty$. By (25), we get

$$d_{0t} = \Delta d_0 - c, \tag{26}$$

where c is the positive constant defined in (17). It is known that $-d_{0t} = V$ and $\Delta d_0 = \kappa$, where V and κ are the normal velocity and the curvature of the interface Γ_t, respectively. Thus (26) is equivalent to (4):

$$V = c - \kappa \quad \text{on } \Gamma_t.$$

Moreover $\overline{\Gamma_t}$ intersects with $\partial\Omega$ perpendicularly.

5 Existence of a spiral for the interface equation

In this section we consider the interface equation

$$\begin{cases} V = c - \kappa & \text{on } \Gamma_t, \\ \langle \nu(x), \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \cap \overline{\Gamma_t}, \end{cases} \tag{27}$$

where $\mathbf{n} = \mathbf{n}(x,t)$ and $\nu(x)$ is the outward unit normal at each point of Γ_t and $\partial\Omega$, respectively. We seek for a solution of (27) which is written in the form

$$\overline{\Gamma}(t) = \{(r\cos(\theta(r) + \omega t), r\sin(\theta(r) + \omega t) \mid a \le r \le b,\ t \ge 0\}$$

for some function $\theta(r)$ and some constant ω. We call such $\overline{\Gamma}(t)$ a spiral with angular speed ω. One can easily see that $\overline{\Gamma}(t)$ is a solution of (27) if and only if $q(r) = r\theta'(r)$ satisfies

$$\begin{cases} \dfrac{dq}{dr} = h(r, q\,;\omega), & r > a, \\ q(a) = q(b) = 0, \end{cases}$$

where $h(r,q\,;\omega) = (1+q^2)\left(-c\sqrt{1+q^2} - \frac{q}{r} + \omega r\right)$.

Theorem C

Fix $a > 0$ arbitrarily.

(i) *For any $b > a$, there exists a spiral with angular speed $\omega(b) > 0$. In addition, the spiral is unique up to rotation.*

(ii) *The angular speed $\omega(b)$ is strictly monotone decreasing in b and there exists $\omega_\infty > 0$ such that $\lim_{b\to\infty} \omega(b) = \omega_\infty$.*

(iii) *In the case where $\Omega = \{x \in \mathbf{R} \mid |x| > a\}$, there exists a spiral with speed ω_∞ such that $\lim_{r\to\infty} \theta'(r) = -\dfrac{\omega_\infty}{c}$.*

Remark 3

The statement (iii) of Theorem C shows that the shape of the spiral for (27) looks like Archimedean spiral as $r \to \infty$ in the case where $b = +\infty$.

In what follows we denote by $q(r\,;\omega)$ the solution of the initial value problem

$$\begin{cases} \dfrac{dq}{dr} = h(r, q\,;\omega), & r > a, \\ q(a) = 0, \end{cases} \tag{28}$$

and let (a, R_ω) be the maximal interval of the existence of $q(r\,;\omega)$.

Lemma 7

(i) *If $\omega_1 < \omega_2$ then $q(r\,;\omega_1) < q(r\,;\omega_2)$ for $a < r < \min\{R_{\omega_1}, R_{\omega_2}\}$.*

(ii) *If $\omega > c/a$ then $R_\omega = +\infty$ and $q(r\,;\omega) > 0$ for $r > a$.*

(iii) *R_ω is nondecreasing in $\omega \in \mathbf{R}$.*

(iv) *If ω_n converges to ω_0 then $\liminf_{n\to\infty} R_{\omega_n} \geq R_{\omega_0}$. If, in addition, $\omega_n \leq \omega_0$ for large n then $\lim_{n\to\infty} R_{\omega_n} = R_{\omega_0}$.*

Proof (i) The statement immediately follows from the fact that $h(r, q\,;\omega)$ is strictly increasing in ω for $r > a$.
(ii) If $\omega > c/a$ then $h(r, 0\,;\omega) = -c + \omega r > 0$ for $r > a$. Therefore $q(r\,;\omega) > 0$ for $a < r < R_\omega$. Since $h(r, q\,;\omega) < 0$ if $q \geq \omega r^2$, we have $0 < q(r\,;\omega) \leq \omega r^2$ for any $r \in (a, R_\omega)$. This implies $R_\omega = +\infty$.
(iii) If $R_\omega < +\infty$ then $\lim_{r\nearrow R_\omega} q(r\,;\omega) = -\infty$, since $h(r, q\,;\omega) < 0$ for $q \geq \max\{\omega r^2, 0\}$. Therefore by virtue of (i), R_ω is nondecreasing in ω.
(iv) Put $p_n(r) = q(r\,;\omega_n) - q(r\,;\omega_0)$. Then p_n satisfies

$$\begin{cases} \dfrac{dp_n}{dr} = H_n(r, p_n), & r > a, \\ p_n(a) = 0, \end{cases} \tag{29}$$

where $H_n(r, p) = h(r, q_0(r) + p\,;\omega_n) - h(r, q_0(r)\,;\omega_0)$ and $q_0(r) = q(r\,;\omega_0)$. For any $R < R_{\omega_0}$ and $\delta > 0$ there exists $L > 0$ such that

$$|H_0(r, p) - H_0(r, \widetilde{p})| \leq L|p - \widetilde{p}|, \qquad |p|, |\widetilde{p}| \leq \delta,\ a \leq r \leq R$$

and that

$$\gamma_n = \sup_{\substack{|p|\le\delta \\ a\le r\le R}} |H_n(r,p) - H_0(r,p)| \to 0, \qquad n\to\infty.$$

We define $R_n = \sup\{a < r < R \mid |p_n(r)| \le \delta\}$. Then by (29) we have

$$|p_n(r)| \le \gamma_n(R-a) + L\int_a^r |p_n(s)|ds$$

for $a \le r \le R_n$. Therefore by Gronwall's inequality, we have

$$|p_n(r)| \le \gamma_n(R-a)e^{L(r-a)} \le \gamma_n(R-a)e^{L(R-a)}$$

for $a \le r \le R_n$. This implies $R_n = R$ for sufficiently large n. Thus we get $R_{\omega_n} > R$ for large n, hence

$$\liminf_{n\to\infty} R_{\omega_n} \ge R_{\omega_0}. \tag{30}$$

Combining (ii) and (30), we obtain $\lim_{n\to\infty} R_{\omega_n} = R_{\omega_0}$ if $\omega_n \le \omega_0$ for large n. ∎

Lemma 8

There exists $\widetilde{\omega} \le c/a$ such that $R_{\widetilde{\omega}} > b$ and $q(b\,;\widetilde{\omega}) \le 0$.

Proof Suppose that the statement of the lemma does not hold. Then for any $\omega \le c/a$, either of the following holds:

$$\text{(a) } R_\omega \le b, \qquad \text{(b) } R_\omega > b \text{ and } q(b\,;\omega) > 0.$$

By Lemma 7 (ii), the statement (b) holds for $\omega > c/a$. We define $\omega_0 = \sup\{\omega \in \mathbf{R} \mid R_\omega \le b\}$. Then we have $\omega_0 \ge c/b$, since $h(r,0\,;\omega) < 0$ for $a \le r \le b$ if $\omega < c/b$. Clearly $\omega \le c/a$. By virtue of Lemma 7 (iii), we obtain $R_{\omega_0} \le b$, hence

$$\lim_{r\nearrow R_{\omega_0}} q(r\,;\omega_0) = -\infty.$$

On the other hand, $R_\omega > b$ and $q(b\,;\omega) > 0$ for any $\omega > \omega_0$. Let $r_0 \in (a, R_{\omega_0})$ be such that $q(r_0\,;\omega_0) < -(\omega_0+1)b^2$ and that $h(r_0, q(r_0\,;\omega_0)\,;\omega_0) < 0$. Then $q(r_0\,;\omega_1) < -(\omega_0+1)b^2$ and $h(r_0, q(r_0\,;\omega_1)\,;\omega_1) < 0$ for some ω_1 sufficiently close to ω_0. Since $h(r_1, q\,;\omega) > h(r_2, q\,;\omega)$ for $a \le r_1 < r_2 \le b$, $q \le -(\omega_0+1)b^2$ and $\omega_0 \le \omega \le \omega_0 + 1$, we have

$$\frac{dq(r\,;\omega_1)}{dr} = h(r, q(r\,;\omega_1)\,;\omega_1) < h(r_0, q(r_0\,;\omega_1)\,;\omega_1) < 0$$

for all $r \ge r_0$ satisfying $q(r\,;\omega_1) = q(r_0\,;\omega_1)$. Hence $q(r\,;\omega_1) < q(r_0\,;\omega_1)$ for $r > r_0$, contradicting the fact that $q(b\,;\omega_1) > 0$. ∎

Proof of Theorem C (i) Let $\widetilde{\omega} \in [c/b, c/a]$ be such that $R_{\widetilde{\omega}} > b$ and $q(b\,;\widetilde{\omega}) < 0$. Then $q(b\,;\omega)$ is well-defined for $\omega \ge \widetilde{\omega}$ and is continuous in $\omega \ge \widetilde{\omega}$. Since $q(b\,;\omega) > 0$ for

$\omega > c/a$, there exists $\omega(b) \in (\widetilde{\omega}, c/a]$ satisfying $q(b\,;\omega(b)) = 0$. The uniqueness of $\omega(b)$ is an immediate consequence of Lemma 7 (i). ∎

Proof of Theorem C (ii) By Lemma 7 (i), the rotation speed $\omega(b)$ is strictly monotone decreasing in b. Therefore $\omega(b)$ converges to some $\omega_\infty \geq 0$, since $\omega(b) \geq c/b$. Note that $q(r\,;0) < 0$ for $r \in (a, R_0)$ and hence $q(r\,;0)$ satisfies

$$\frac{dq}{dr} \leq \frac{c}{2} q(1+q^2), \qquad r \geq \frac{2}{c}.$$

This implies $R_0 < +\infty$, from which and the following lemma we obtain $\omega_\infty > 0$. ∎

Lemma 9

$R_{\omega_\infty} = +\infty$.

Proof Suppose that R_{ω_∞} is finite. We fix $b_0 > a$, $k > \omega(b_0)/c$ and take

$$\widetilde{R} = \max\left\{\frac{2k}{ck - \omega(b_0)}, R_{\omega_\infty}\right\}.$$

Then, if $\omega \leq \omega(b_0)$, $-kr$ is a supersolution of (28) for $r \geq \widetilde{R}$ since $h(r, -kr\,;\omega) \leq -k$. We take $\widetilde{q} = \min\{-k\widetilde{R}, -\omega(b_0)\widetilde{R}^2\}$. Then we get $h(r_1, q\,;\omega) \geq h(r_2, q\,;\omega)$ for $a \leq r_1 < r_2 \leq \widetilde{R}$, $q \leq \widetilde{q}$ and $\omega \leq \omega(b_0)$. By the similar argument in the proof of Lemma 8, there exists $b > b_0$ and $r_0 < R_{\omega_\infty}$ such that $q(r_0\,;\omega(b)) < \widetilde{q}$ and $h(r_0, q(r_0\,;\omega(b))\,;\omega(b)) < 0$. Again by the argument in the proof of Lemma 8, we obtain $q(r\,;\omega(b)) < q(r_0\,;\omega(b)) < \widetilde{q}$ for $r_0 < r < \widetilde{R}$ and $q(r\,;\omega(b)) < -kr$ for $r \geq \widetilde{R}$, contradicting the fact that $q(b\,;\omega(b)) = 0$. This contradiction proves the lemma. ∎

Proof of Theorem C (iii) By Lemma 9, $q(r\,;\omega_\infty)$ exists for all $r > a$. Furthermore $q(r\,;\omega_\infty)$ is negative since $q(r\,;\omega(b)) < 0$ for $a < r < b$. This corresponds to a spiral with angular speed ω_∞ for $\Omega = \{x \in \mathbf{R} \mid |x| > a\}$. To complete the proof, we show that

$$\lim_{r\to+\infty} \frac{q(r\,;\omega_\infty)}{r} = -\frac{\omega_\infty}{c}.$$

Fix $k > \omega_\infty/c$. We take b_0 so that $k > \omega(b_0)/c$ and put $r_0 = 2k/(ck - \omega(b_0))$. Since $-kr$ is a supersolution of (28) for $r \geq r_0$ if $\omega \leq \omega(b_0)$, we have $q(r\,;\omega(b)) > -kr$ for $r \geq r_0$ and $b \geq b_0$. This implies $q(r\,;\omega_\infty) \geq -kr$ for $r \geq r_0$, since $q(r\,;\omega(b))$ uniformly converges to $q(r\,;\omega_\infty)$ on any compact subset of $(a, +\infty)$. Hence we have

$$\liminf_{r\to+\infty} \frac{q(r\,;\omega_\infty)}{r} \geq -\frac{\omega_\infty}{c}.$$

We define

$$K_\infty = \left\{(r, q) \,\middle|\, r > \frac{c}{\omega_\infty}\sqrt{1+q^2},\ q < 0\right\}.$$

Let

$$l = \limsup_{n\to\infty} \frac{q(r\,;\omega_\infty)}{r}$$

and suppose that $l > -\omega_\infty/c$. Then there exists $r_0 > a$ such that $(r_0, q(r_0\,;\omega_\infty)) \in K_\infty$. Since $h(r,q\,;\omega_\infty) > 0$ for $(r,q) \in K_\infty$, we have $(r, q(r\,;\omega_\infty)) \in K_\infty$ for all $r \geq r_0$. Therefore by (28) we obtain

$$\frac{d}{dr}\left(\frac{q(r\,;\omega_\infty)}{r}\right) > \frac{-c\sqrt{1+q(r\,;\omega_\infty)^2}+\omega_\infty r}{r} > 0 \tag{31}$$

for $r \geq r_0$, hence

$$-\frac{\omega_\infty}{c} < l = \lim_{r\to+\infty} \frac{q(r\,;\omega_\infty)}{r} \leq 0.$$

On the other hand, by (31) we have

$$\liminf_{r\to+\infty} \frac{d}{dr}\left(\frac{q(r\,;\omega_\infty)}{r}\right) \geq cl + \omega_\infty > 0.$$

This contradiction proves that

$$\limsup_{r\to+\infty} \frac{q(r\,;\omega_\infty)}{r} \leq -\frac{\omega_\infty}{c}.$$

The theorem is proved. ∎

Appendix

In this appendix we present two propositions in [16]. Proposition B1 is concerned with the structure of a subset of an ordered metric space under a group action. Proposition B2 is, in a sense, a set-valued version of the former half of Proposition B1.

Let X be an ordered metric space. In other words, X is a metric space on which a closed partial order relation is defined. We will denote by $\leq$ the order relation in X. Here, we say that a partial order relation in X is *closed* if $\varphi_n \leq \psi_n$ $(n = 1,2,3,\cdots)$ implies $\lim_{n\to\infty} \varphi_n \leq \lim_{n\to\infty} \psi_n$ provided that both limits exist. We write $\varphi < \psi$ if $\varphi \leq \psi$ and $\varphi \neq \psi$. For a subset $V \subset X$, the expression $\varphi \leq V$, $V \leq \varphi$ means $\varphi \leq \psi$, $\psi \leq \varphi$ for all points $\psi \in V$, respectively.

Let G be a metrizable topological group acting on some subset X_1 of X. We say G acts on X_1 if there exists a continuous mapping $\gamma: G \times X_1 \to X_1$ such that $g \mapsto \gamma(g,\cdot)$ is a group homomorphism of G into $Hom(X_1)$, the group of homeomorphisms of X_1 onto itself. For brevity, we write $\gamma(g,\varphi) = g\varphi$ and identify the element $g \in G$ with its action $\gamma(g,\cdot)$. We assume that

(G1) γ is order-preserving (that is, $\varphi \preceq \psi$ implies $g\varphi \preceq g\psi$ for any $g \in G$);

(G2) G is connected.

Let Y be a subset of X and $\overline{\varphi}$ be an element of $Y \cap X_1$ such that

(H1) $g\overline{\varphi} \in Y$ for any $g \in G$;

(H2) for any $\psi \in Y$, there exist some $g_1, g_2 \in G$ satisfying $g_1\overline{\varphi} < \psi < g_2\overline{\varphi}$;

(H3) for any $\psi \in Y$ with $\psi < h\overline{\varphi}$ for some $h \in G$, there exists some neighborhood B of the unit element of G such that $\psi < gh\overline{\varphi}$ for any $g \in B$.

Proposition ([16, Proposition B1])
Let G satisfy (G1), (G2) and Y, $\overline{\varphi}$ satisfy (H1), (H2), (H3). Then Y is a totally-ordered connected set and $Y = G\overline{\varphi}$. Furthermore, if Y is locally precompact, then Y is homeomorphic and order-isomorphic to $\mathbf{R}$.

A similar result holds for the case where the set Y consists of subsets of X. To be more precise, let Y be a set of subsets of X containing $\{\overline{\varphi}\}$ such that

(H4) $\{g\overline{\varphi}\} \in Y$ for any $g \in G$;

(H5) for any $V \in Y$, there exist some $g_1, g_2 \in G$ satisfying $g_1\overline{\varphi} \leq V \leq g_2\overline{\varphi}$ and $V \neq \{g_1\overline{\varphi}\}$, $\{g_2\overline{\varphi}\}$;

(H6) for any $V \in Y$ with $V \leq h\overline{\varphi}$ and $V \neq \{h\overline{\varphi}\}$ for some $h \in G$, there exists some neighborhood B of the unit element of G such that $V \leq gh\overline{\varphi}$ and $V \neq \{gh\overline{\varphi}\}$ for any $g \in B$.

Proposition ([16, Proposition B2])
Let G satisfy (G1), (G2) and Y, $\{\overline{\varphi}\}$ satisfy (H4), (H5), (H6). Then $Y = G\{\overline{\varphi}\} = \{\{g\overline{\varphi}\} \mid g \in G\}$.

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