有理関数族の複素力学系に関する研究

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The Study of Complex Dynamical Systems of Rational Functions

GASR Grand No. 09640217

RESEARCH PROJECT,

Grand–in–Aid for Scientific Research (C)(2)
1997 – 1998

February, 1999

HEAD INVESTIGATOR: Kiyoko NISHIZAWA

ASSOCIATE PROFESSOR
JOSAI UNIVERSITY,
FAC. SCIENCE,
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まえがき

これは１９９７年度、１９９８年度の科学研究費補助金による研究成果報告書である。２年間の研究テーマは有理関数族を対象とする複素力学系の研究である。

複素力学系の立場から２次多項式族のモヂュライ空間及び双曲成分、成分の中心に関しての研究は繰り込み理論、擬等角写像論および、それによる手術理論などを用いて、Doaddy, Hubbard, Milnor, Sullivan, Shishikura, Thurston, Yoccoz らによってほぼ解析されて、現在の研究の対象は２個の特異点を持つ３次多項式族、２次有理関数族の力学系の解析である。すなわちこれらの関数族のモヂュライ空間に於ける双曲成分の配置、Thurston map の吸引領域の解析の研究である。このために新たに angled Hubbard tree, Thurston covering 等の新しい方法が導入された。

一方 1977 年に Milnor-Thurston により実力学系での基本的な位相不変量である位相的エントロピーが、1-パラメター実２次多項式族の場合に、パラメターに関して単調連続関数となる事が、Sullivan による Teichmüller 理論により示された。すなわち非線形性が強くなれば、より複雑な力学系が生じるということである。高次の多項式族に関して、このエントロピー単調性の問題は新しい定式化を必要としていた。1992 年に Milnor は、実３次多項式族の位相的エントロピーが実モヂュライ空間上で単調に作用するか?:すなわち等エントロピー集合は連結で、極大極小値を持たないか?という形に定式化した。また 1994 年には generic な双曲性に関する FATOU 予想の弱い形の仮定の下で肯定的に解決した。(しかし２次多項式族に対する FATOU 予想でさえいまだ未解決である。)

また 1998 年 9 月の彼の preprint では stunted sawtooth maps に対しての単調性の問題を取り上げている。この族は、繰り込み理論を基礎にした組み合わせ的構造を最もよく反映する簡単な族である。Milnor の結果は３次多項式族の力学系のこの族への解釈である。またこの preprint では分岐の単調性の問題も取り上げている。

２次有理関数族、３次多項式族に対するこの単調性の問題に対して、1992 年より西沢は、４タイプの双曲成分の中心に着目してモヂュライ空間に中心曲線という概念を導入し、この曲線に沿ってのエントロピーの単調性を予想した。この予想は大阪市立大の小森氏により kneading Theory を用いての kneading sequence の単調性の問題に還元し、Thurston の有理関数の Rigidity Theorem 及び Teichmüller metric による Distortion Theorem

により示された。

また1994年からは、実2次有理関数族、3次多項式族に対する分岐の単調性の問題を取り上げた。手法はやはりモヂュライ空間にある種の代数曲線を定義し、この曲線に沿っての分岐の様子を見ることである。

研究テーマ『有理関数族を対象とする複素力学系の研究』における我々の手法の特色はモヂュライ空間に定義された代数曲線、特に中心曲線の代数幾何学的な性質を明らかにし、その性質が力学的になにを意味しているかを調べることである。

研究組織

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特別講演

無限反復法と Controlled Topology

山崎 正之　　(城西大学・理学部)

0. 導入

上で言う無限反復法とは次の類のトリック (Eilenberg swindle とよばれる – swindle とは辞書によると「詐取·ぺてん·いんちき」という意味らしい) をいう。

- 任意の射影的加群 P に対して $P \oplus F$ が自由となるような自由加群 F が存在する。実際 $P \oplus Q = F'$ となる射影加群 Q と自由加群 F' をとり、$F = F' \oplus F' \oplus \dots$ とすると

$$P \oplus F = P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \dots$$
$$= (P \oplus Q) \oplus (P \oplus Q) \oplus \dots = F$$

- K を CW 複体とし、半無限区間 $[0,\infty)$ を整数の点に 0 次元セルを置いて胞体分割を与えて置く。するとそれらの局所有限ホモロジー群は自明である：

$$H_*^{lf}(K \times [0,\infty)) = 0$$

- (低次元でない) h-コボルディズム $(W; M_0, M_1)$ は、$\tau(W; M_0) \neq 0 \in Wh(\pi_1)$ のとき直積と同型にならないが、片一方の境界を取り去れば、同型

$$W - M_1 \cong M_0 \times [0,1)$$

が成り立つ。

もちろん、このようなトリックを用いるとき、必然的に “無限” な (無限生成、非コンパクト、…な) ものを扱うことになる。しかし場合によっては、“無限” を用いて “有限” なものに関する情報を得ることもできることがある。例えば、

上の2番目の例に関連して次のような例を考えてみる。K を有限 CW 複体とし、$K\times\mathrm{R}$ を $K\times\{0\}$ で二つに分けてマイヤー・ビートリス完全系列を書いてみる：

$$\ldots\to H^{lf}_{i+1}(K\times(-\infty,0])\oplus H^{lf}_{i+1}(K\times[0,\infty))\to H^{lf}_{i+1}(K\times\mathrm{R})\to H_i(K\times\{0\})$$
$$\to H^{lf}_{i}(K\times(-\infty,0])\oplus H^{lf}_{i}(K\times[0,\infty))\to\ldots$$

半無限区間に関する項は消えているので同型

$$H^{lf}_{*+1}(K\times\mathrm{R})\ \cong\ H_*(K)$$

を得る。つまり有限複体 K の有限なチェインのデータを無限複体の無限チェインのデータで得ることができるのである。ただし次数がずれてしまっている。

題の controlled topology とはこのようなテクニックを用いる topology のことだというと、きっとあまり正確ではないかもしれないが、それほど遠くもないであろう。なお、controlled surgery theory に関しては「数学」に論説が載る (載った？) はずなので細かいことやホモロジー多様体に関する話題については、そちらを見ていただきたい。

1. イプシロン・コントロールと有界コントロール

まず controlled topology で用いる代数的道具を解説する。ただし、話題としては代数的 K 理論を取り上げ、簡単のため係数が一定である場合のみを考える。M を距離空間、R を環とする。$\mathcal{C}_M(R)$ は次のようなカテゴリーとする：

- object は有限生成 R-加群の族 $A=\{A_x|\ x\in M\}$ で、局所有限である (つまり M の任意の有界集合 C に対して A_x ($x\in C$) たちの中で非零なものは有限個のみであるという条件を満たす) もの
- morphism $\varphi:A\to B$ は準同型写像の族 $\{\varphi^x_y: A_x\to B_y\}$ で有界である (つまりある $k>0$ が存在して、$d(x,y)>k$ なら $\varphi^x_y=0$ をみたす) もの

ただし $\varphi : A \to B$ と $\psi : B \to C$ の合成 $\psi \circ \varphi : A \to C$ は $(\psi \circ \varphi)^x_z = \sum_{y \in M} \psi^y_z \circ \varphi^x_y$ で定めるものとする。

今、有限な多面体 P がを十分高い次元の球面 S^{n-1} に埋め込まれており、$O(P)$ でその開錐、つまり $\{ tx \in \mathrm{R}^n \mid x \in P,\ t \geq 0 \}$ を表すとする。P に対してその "boundedly controlled K-groups" $K^b_*(P; R)$ を $K_*(\mathcal{C}_{O(P_+)}(R)\hat{\ })$ で定義する。ここで K_* は Quillen による additive category の代数的 K 群、P_+ は $P \sqcup \{*\}$、$\hat{\ }$ は idempotent completion を表すものとする。このとき次が知られている：

定理 (Pedersen-Weibel, 1986) 次の同型がある：

$$H_{*-1}(P; \mathbb{K}(R)) \cong K^b_*(P; R)\ .$$

ただし左辺は環 R の代数的 K スペクトル $\mathbb{K}(R)$ の定める一般ホモロジー群である。

[注意：idempotent completion は K_i (i は 0 または負) のみに影響を与えるので $* \geq 1$ では $H_{*-1}(P; \mathbb{K}(R)) \cong K_*(\mathcal{C}_{O(P_+)}(R))$ となる。]

一方、Quinn は $\mathcal{C}_P(R)$ の中の morphism たちのうち $\epsilon > 0$ で bound されるもののみを考えることにより ϵ-controlled K_1-group $K_1(P, \epsilon)$ を考え、次を得ていた ($R = \mathrm{Z}$ の場合だが、一般でも同様)：

定理 ["Squeezing"](Quinn, 70 年代後半) 多面体 P に対しある数 ϵ_0 が存在し、任意の $0 < \delta < \epsilon < \epsilon_0$ に対し、自然な写像 $K_1(P, \delta; R) \to K_1(P, \epsilon; R)$ は同型であり、これらを $\epsilon \to 0$ の極限 $K^c_1(P; R)$ とみなしてよい。

Ferry は 1990 年頃、bounded topology を用いて Quinn の定理を導き、$K^b_2(P; R)_- \cong K^c_1(P; R)$ であることを証明した。講演では、そのアイデアを紹介するが、例の Eilenberg swindle を繰り返し用いることがキーである。

代数的K理論と同様なことは代数的L理論でも行うことができる(ただし、一般的な理論を作る際にはLではなくRanickiの$L^{-\infty}$を用いる必要がある)。筆者はQuinn流のやり方でcontrolled $L^{-\infty}$群を定義し、それが一般ホモロジー群になることを証明した(1982)。またある条件の下でのsqueezingも証明できる(Ranicki-Yamasaki, 準備中)。またFerry-Pedersenのbounded topologyによる理論もある。彼らもsqueezingを主張しているが、彼らの論文も我々のものと同様いまだ準備中である。

代数的K-理論におけるsqueezingの応用として、例えばコンパクトANRの有限性やWhitehead torsionの位相不変性などがある。またL-理論におけるsqueezingの応用としては、有理ポントリャーギン類の位相不変性などがある。

2. 連続なコントロール

前節のイプシロン・コントロールや有界コントロールは、対象となる空間の「距離」を用いていることに注意する。Anderson-Connolly-Ferry-Pedersenらの導入した連続コントロールの理論(90年代初頭)では、距離を使わない。

空間対$(\bar{E}, \Sigma)$を考え、$E = \bar{E} - \Sigma$は$\bar{E}$で稠密あるとする(ΣがEの無限遠点の空間と思えばよい)。カテゴリー$\mathcal{B}(\bar{E}, \Sigma; R)$を次のように定める:

- objectは有限生成R-加群の族$A = \{A_x | x \in E\}$で、局所有限である(つまり$\bar{E}$の任意のコンパクト集合Cに対してA_x $(x \in C)$たちの中で非零なものは有限個のみであるという条件を満たす)もの、
- morphism $\varphi : A \to B$は準同型写像の族$\{\varphi_y^x : A_x \to B_y\}$で無限遠で連続であるもの、つまり2条件
  1. 任意の$x \in E$に対し、$\{y | \varphi_y^x \neq 0\}$は有限、
  2. 任意の$z \in \sum$と$\bar{E}$におけるzの任意の近傍Uに対し、zの近傍Vで、$x \in V$かつ$y \notin U$なら$\varphi_y^x = 0$となるものが存在する

  が成り立つもの。

例えば S^{n-1} に埋め込まれた有限多面体 P があるとき、閉錐 $c(P)$ を原点からの錐、つまり $\{ tx \mid 0 \leq t \leq 1,\ x \in P \}$ とし、小開錐 $o(P)$ を $\{ tx \mid 0 \leq t < 1,\ x \in P \}$ で定める。P の "continuously controlled K-groups" $K_*^{cc}(P;R)$ を

$$K_*^{cc}(P;R) = \pi_* \{ K(\mathcal{B}(c\,\Sigma^n P_+, \Sigma^n P_+; R)^\wedge) \}_{n=1}^{\infty}$$

で定める。これも bounded な場合と同様に、ホモロジーになることがわかる。$O(P)$ を $o(P)$ と同一視するとき、有界な morphism は無限遠で連続になり、写像 $K_*^b(P;R) \to K_*^{cc}(P;R)$ ができるが、共にホモロジーであることと係数の比較により、これは同型となる。したがって、前節の controlled K-groups は距離には無関係に位相のみによってきまることがわかる。

3. Novikov 予想

最後に Carlsson, Pedersen, Vogell, Higson, Roe らによる Novikov 予想 (またはその類似) の部分的解決について紹介しておく。証明は同変版の連続コントロールを用いる。

定理 群 Γ の分類空間 $B\Gamma$ が有限複体で、その普遍被覆 $E\Gamma$ が、Čech 可縮で同変でハウスドルフなコンパクト化 $\overline{E\Gamma}$(必ずしも距離づけ可能ではない) を持つとし、さらに次のいずれかが成り立つとする：

1. Γ の作用は <u>無限遠で小さい</u>：K を $E\Gamma$ の任意のコンパクト集合、y を $E\Gamma$ の任意の無限遠点、U を $\overline{E\Gamma}$ における y の任意の近傍とする。このとき、y の近傍 V で $\gamma K \cap V \neq \emptyset$ ならば $\gamma K \subset U$ が成り立つようなものが存在する。
2. $\partial E\Gamma = \overline{E\Gamma} - E\Gamma$ の "boundedly saturated"(説明略) な部分集合による被覆の Γ-不変な族 $\mathcal{F}$ で、$\mathcal{F}$ に関する Čech homotopy type が通常のČech homotopy type と一致するようなものが存在する。

このとき、次のおのおのの assembly map は split injective である。

a) 任意の環 R に対する assembly map $H_*(B\Gamma; \mathbb{K}^{-\infty}(R)) \to K_*^{-\infty}(R\Gamma)$,

b) 十分大きい i に対して $K_{-i}(R) = 0$ が成り立つような、involution をもつ環 R に対する assembly map $H_*(B\Gamma; \mathbb{L}^{-\infty}(R)) \to L_*^{-\infty}(R\Gamma)$,

c) 十分大きい i に対して $K_{-i}(\pi_1(X)) = 0$ が成り立つような空間 X に対する assembly map $H_*(B\Gamma; \mathbb{A}_{-\infty}(X)) \to \pi_*^s(\mathbb{A}_{-\infty}(B\Gamma \times X))$,

c) C^*-algebra R に対する assembly map $H_*(B\Gamma; \mathbb{K}^{top}(R)) \to \pi_*^s(\mathbb{K}^{top}(C_R^*\Gamma))$.

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山口 博

- On the product of Riese sets in dual objects of compact groups

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第７回 関数空間セミナー 報告集

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On the product of Riesz sets in dual objects of compact groups

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Abstract. Let E_i be a Riesz set in the dual object of a compact group $K_i (i = 1, 2)$. We show that the product set $E_1 \times E_2$ is a Riesz set in the dual object of $K_1 \oplus K_2$. We also give a result on compact groups related to a result of Glicksberg and Graham concerned with "small p set".

§1　　T 上の F.and M. Riesz の定理の $\mathbf{T}^2$ への拡張として次の Bochner の結果がある。

<u>定理 1.1.</u>　　$\mu \in M(\mathbf{T}^2)$, $\hat{\mu}(n, m) = 0$ for $(n, m) \notin \mathbf{Z}^+ \times \mathbf{Z}^+ \Longrightarrow \mu \ll m_{\mathbf{T}^2}$.

定理1.1 は $\hat{\mathbf{T}} \cong \mathbf{Z}$ の Riesz 集合 $\mathbf{Z}^+$ の積 $\mathbf{Z}^+ \times \mathbf{Z}^+$ は $\hat{\mathbf{T}}^2 \cong \mathbf{Z} \oplus \mathbf{Z}$ の Riesz 集合になることを示している。この種のことについては、局所コンパクト可換群についても成り立つ。

<u>定義 1.1.</u>　　G を局所コンパクト可換群とし、p を自然数とする。$\hat{G}$ の閉集合 E が次を満たすとき、small p set と呼ばれる。

(1. 1)　$\forall \mu \in M_E(G) \Longrightarrow \mu^p = \overbrace{\mu * \cdots * \mu}^{p} \in L^1(G)$.

但し、$M_E(G) = \{\mu \in M(G) : \hat{\mu} = 0 \text{ on } E^c\}$. 特に、small 1 set は Riesz 集合と呼ばれる。

<u>定理 1.2</u> (cf.[6]).　　G_1, G_2 を局所コンパクト可換群とし、p を自然数とする。E_1, E_2 をそれぞれ $\hat{G}_1, \hat{G}_2$ の small p set とする。すると、$E_1 \times E_2$ は $G_1 \hat{\oplus} G_2$ の small p set である。

small 2 set になるための条件としては、Graham によって与えられた次の結果がある。

<u>定理 1.3</u> (cf.[3]).　　G を局所コンパクト可換群とし、S を次を満たす $\hat{G}$ の Borel 集合とする。

(1. 2)　$\{\gamma \in \hat{G} : m_{\hat{G}}(S \cap (\gamma - S)) < \infty\}$ は $\hat{G}$ で稠密。

すると、$\mu, \nu \in M_S(G) \Longrightarrow |\mu| * |\nu| \in L^1(G)$.

ここでは、定理 1.1、定理 1.3 に関連したことを（非可換）コンパクト群にたいして考えてみる。

§2　K をコンパクト群とし、Σ_K を K の dual object とする。$M(K)$ を K 上の bounded regular measures の空間とし、m_K を K 上の Haar measure とする。$\sigma \in \Sigma_K$ に対して、$U^{(\sigma)}$ を σ に属し、H_σ を表現空間として持つ K の continuous irreducible unitary representation とする。$\mu \in M(K)$ に対して、μ の Fourier 変換 $\hat{\mu}$ を次のように定義する：$\sigma \in \Sigma_K; \xi, \eta \in H_\sigma$ に対して、

$$< \hat{\mu}(\sigma)\xi, \eta >= \int_K < \bar{U}_x^{(\sigma)}\xi, \eta > d\mu(x).$$

但し、$\bar{U}_x^{(\sigma)} = D_\sigma U_x^{(\sigma)} D_\sigma$。又、$D_\sigma$ は H_σ 上の conjugation。そして、$spec(\mu) = \{\sigma \in \Sigma_K : \hat{\mu}(\sigma) \neq 0\}$ とおく。Σ_K には、conjugation "$^-$" と積 "$\times$" の２つの operations が定義される。

<u>定義 2.1</u>.　p を自然数とする。Σ_K の部分集合 E が次を満たすとき s-small p set と言うことにする。

(2. 1)　$\forall \mu_1, \cdots, \mu_p \in M_E(K) \Longrightarrow \mu_1 * \cdots * \mu_p \in L^1(K)$.

但し、$M_E(K) = \{\mu \in M(K) : spec(\mu) \subset E\}$.　特に、s-small 1 set は Riesz 集合と呼ばれる。

<u>注意</u>.　K が compact abelian group の時は、"s-small p set" と "small p set" は同じ概念である。

<u>定理 2.1</u>.　p を自然数とし、K_1, K_2 を compact groups とする。E_1, E_2 をそれぞれ $\Sigma_{K_1}, \Sigma_{K_2}$ のs-small p sets とする。すると、$E_1 \times E_2$ は　$\Sigma_{K_1 \oplus K_2} \cong \Sigma_{K_1} \times \Sigma_{K_2}$ における s-small p set である。

<u>系</u>.　E_1, E_2 をそれぞれ $\Sigma_{K_1}, \Sigma_{K_2}$ の Riesz 集合とする。すると、$E_1 \times E_2$ は$\Sigma_{K_1 \oplus K_2} \cong \Sigma_{K_1} \times \Sigma_{K_2}$ における Riesz 集合である。

次に、定理 1.3 に対応したことを compact group の場合について考えてみる。G が compact abelian group の場合は、定理 1.3 の条件 (1. 2) は次の条件 (1. 2)' となる。

(1. 2)'　$\forall \gamma \in \hat{G}$ にたいして、$S \cap (\gamma - S)$ は有限集合。

又、条件 (1. 2)' は次の条件 (1. 2)" と同値である。

(1. 2)"　$\forall \gamma_1, \gamma_2 \in \hat{G}$ に対して、$(\gamma_1 + S) \cap (\gamma_2 - S)$ は有限集合。

<u>定理 2.2</u>.　K を compact group とし、Δ を次を満たす Σ_K の部分集合とする。

(2. 2)　$\forall \sigma, \tau \in \Sigma_K$ に対して、$(\sigma \times \Delta) \cap (\tau \times \bar{\Delta})$ は有限集合。

すると、$\forall \mu, \nu \in M_\Delta(K) \Longrightarrow |\mu| * |\nu| \in L^1(K)$。

但し、$\bar{\Delta} = \{\bar{\omega} : \omega \in \Delta\}, \sigma \times \Delta = \{\sigma \times \eta : \eta \in \Delta\}$。特に、$\Delta$ は s-small 2 set である。

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- Moduli spaces of maps with two critical points

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Moduli spaces of maps with two critical points

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Abstract

We give directly a defining equation of the symmetry locus, a singular part of the moduli space of the quadratic rational maps. We show a characterization of this locus. We can expand analogous discussion for the cubic polynomials and give a "chart" making a comparison between properties of these moduli spaces in Appendix A. Moreover, we apply these method to the polynomials of degree n, and give some conjectures.

1 Quadratic rational maps

1.1 Moduli space of quadratic rational maps

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\mathrm{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\mathrm{Rat}_2(\mathbf{C})$ by conjugation,

$$g \circ f \circ g^{-1} \in \mathrm{Rat}_2(\mathbf{C}) \quad \text{for} \quad g \in \mathrm{PSL}_2(\mathbf{C}),\ f \in \mathrm{Rat}_2(\mathbf{C}).$$

Two maps $f_1, f_2 \in \mathrm{Rat}_2(\mathbf{C})$ are **holomorphically conjugate**, denoted by $f_1 \sim f_2$, if and only if there exists $g \in \mathrm{PSL}_2(\mathbf{C})$ with $g \circ f_1 \circ g^{-1} = f_2$. The quotient space of $\mathrm{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the **moduli space** of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f.

Milnor introduced coordinates in $\mathcal{M}_2(\mathbf{C})$ as follows; for each $f \in \mathrm{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be the fixed points of f and μ_i the multipliers of z_i; $\mu_i =$

$f'(z_i)$ $(1 \leq i \leq 3)$. Consider the elementary symmetric functions of the three multipliers,

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3, \quad \sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad \sigma_3 = \mu_1\mu_2\mu_3.$$

These three multipliers determine f up to holomorphic conjugacy, and are subject only to the restriction that

$$\sigma_3 = \sigma_1 - 2.$$

Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ with coordinates σ_1 and σ_2 (Lemma 3.1 in [Mil93]).

For each $\mu \in \mathbf{C}$ let $\mathrm{Per}_n(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps f which having a periodic point of period n and multiplier μ.

Each of $\mathrm{Per}_1(\mu)$ and $\mathrm{Per}_2(\mu)$ forms a straight lines as follows:

$$\begin{aligned} \mathrm{Per}_1(\mu) &= \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1}) \right\} \\ \mathrm{Per}_2(\mu) &= \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = -2\sigma_1 + \mu \right\}, \end{aligned}$$

(Lemmas 3.4 and 3.6 in [Mil93]).

Remark $\mathrm{Per}_1(-1) \subseteq \mathrm{Per}_2(1)$ by definition. But, in the case of $\mathcal{M}_2(\mathbf{C})$, it is clear that two families coincide.

1.2 Symmetry locus

By an automorphism of a quadratic rational map f, we will mean $g \in \mathrm{PSL}_2(\mathbf{C})$ which commutes with f. The collection $\mathrm{Aut}(f)$ of all automorphisms of f forms a finite group. It is clear that $\mathrm{Aut}(\tilde{f})$ is isomorphic to $\mathrm{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$.

The set

$$\mathcal{S} = \{\langle f \rangle \, ; \ \mathrm{Aut}(f) \text{ is non-trivial}\} \subset \mathcal{M}_2(\mathbf{C})$$

is called the **symmetry locus**.

Corollary 1 *The symmetry locus $\mathcal{S}$ of quadratic rational maps forms an irreducible algebraic curve as follows;*

$$S(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \quad (1)$$

Proof of Corollary 1.

Aut(f) coincides with the group consisting of all permutations of the fixed points which preserve the multipliers. In the case of f has the three distinct fixed points, Aut(f) has order 1, 2, or 6 according as three multipliers are distinct, two are equal, or all the three are equal, respectively, while, if f has multiple fixed points then Aut(f) is non-trivial if and only if f has a triple fixed point. The multipliers μ_i are the roots of the equation:

$$\mu^3 - \sigma_1\mu^2 + \sigma_2\mu - \sigma_1 + 2 = 0. \tag{2}$$

The equation (2) has multiple roots if and only if its discriminant is equal to zero. Hence we have

$$(\sigma_2 - 2\sigma_1 + 3)(2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36) = 0.$$

The first factor corresponds with $\mathrm{Per}_1(1)$. Considering the line of the first factor ($\mathrm{Per}_1(1)$) tangent to the curve of the second factor (S) with tangency of degree three, the second factor is the required equation.

∎

The following result is obtained immediately by the definition of the envelope of the family of curves.

Corollary 2 *The envelope of* $\{Per_1(\mu)\}_\mu$ *coincides with the symmetry locus.*

Remark (Theorem 5.1.of [Mil93]) A quadratic rational map has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form $f(z) = k(z + \frac{1}{z})$ with $k \in \mathbf{C} \setminus \{0\}$.

1.3 Real moduli space

Let $\mathrm{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. Then the parameters σ_i $(1 \leq i \leq 3)$ are all real, because the three fixed points and the corresponding multipliers are either all real or one real and a pair of complex conjugate numbers. According to J. Milnor, we define the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\mathrm{Rat}_2(\mathbf{R})$ to be simply the real (σ_1, σ_2)-plane. This notation needs some care when used: if we put $\mathcal{S}_{\mathbf{R}} = \mathcal{S} \cap \mathcal{M}_2(\mathbf{R})$, and denote by $\langle\ \rangle_{\mathbf{R}}$ the real conjugacy class, then $(\mathrm{Rat}_2(\mathbf{R})/\mathrm{PGL}_2(\mathbf{R})) \setminus \{\left\langle a(x + \frac{1}{x})\right\rangle_{\mathbf{R}}, \left\langle a(x - \frac{1}{x})\right\rangle_{\mathbf{R}}\}_{a \in \mathbf{R}^\times}$ is canonically isomorphic to $\mathbf{R}^2 \setminus \mathcal{S}_{\mathbf{R}}$, whereas there is a canonical two-to-one correspondence between $\{\left\langle a(x \pm \frac{1}{x})\right\rangle\}_{a \in \mathbf{R}^\times}$ and $\mathcal{S}_{\mathbf{R}}$.

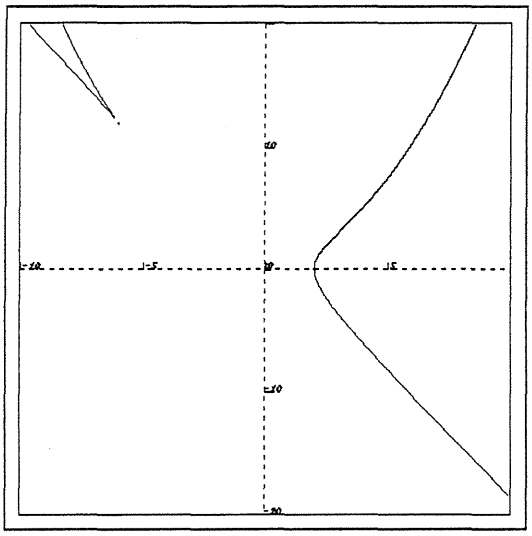

Figure 1: $\mathcal{M}_2(\mathbf{R})$ with the real cut of $\mathcal{S}$.

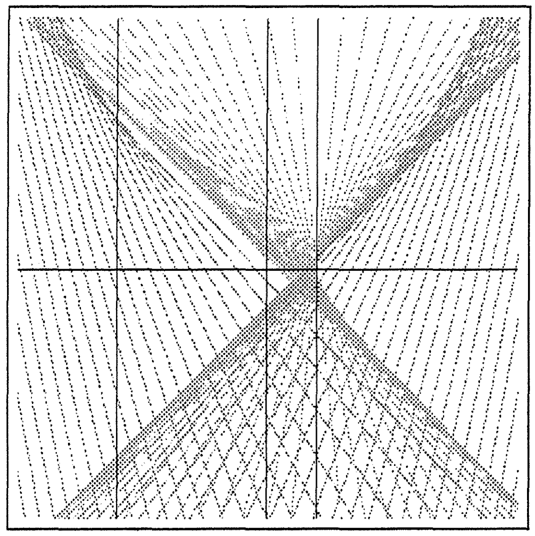

Figure 2: Lines $\{\mathrm{Per}_1(\mu)\}$ in the real cut of the moduli space $\mathcal{M}_2(\mathbf{C})$

2 Cubic polynomials

2.1 Moduli space of cubic polynomials

Let $\mathrm{Poly}_3(\mathbf{C})$ be the space of all cubic polynomials from $\mathbf{C}$ to itself. The group $\mathrm{Poly}_1(\mathbf{C})$ of affine transformations acts on the space $\mathrm{Poly}_3(\mathbf{C})$ by conjugation,

$$g \circ p \circ g^{-1} \in \mathrm{Poly}_3(\mathbf{C}) \quad \text{for} \quad g \in \mathrm{Poly}_1(\mathbf{C}),\ p \in \mathrm{Poly}_3(\mathbf{C}).$$

Two maps $p_1, p_2 \in \mathrm{Poly}_3(\mathbf{C})$ are **holomorphically conjugate**, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \mathrm{Poly}_1(\mathbf{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\mathrm{Poly}_3(\mathbf{C})$ under this action will be denoted by $\mathrm{M}_3(\mathbf{C})$, and called the **moduli space** of holomorphic conjugacy classes $\langle p \rangle$ of cubic polynomials p.

Doing the same as the case of quadratic rational maps, we introduce coordinates in $\mathrm{M}_3(\mathbf{C})$ as follows; for each $p \in \mathrm{Poly}_3(\mathbf{C})$, let $z_1,\ z_2,\ z_3,\ z_4(=\infty)$ be the fixed points of p and μ_i the multipliers of z_i; $\mu_i = p'(z_i)$ $(1 \le i \le 3)$, and $\mu_4 = 0$. Consider the elementary symmetric functions of the four multipliers,

$$\begin{aligned}
&\sigma_1 = \mu_1 + \mu_2 + \mu_3 + \mu_4 = \mu_1 + \mu_2 + \mu_3 \\
&\sigma_2 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\
&\sigma_3 = \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 = \mu_1\mu_2\mu_3 \\
&\sigma_4 = \mu_1\mu_2\mu_3\mu_4 = 0.
\end{aligned}$$

These multipliers determine uniquely p up to holomorphic conjugacy, and are subject only to the restriction that

$$3 - 2\sigma_1 + \sigma_2 = 0.$$

Hence the moduli space $\mathrm{M}_3(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ with coordinates σ_1 and σ_3.

Proposition 1 *The locus $Per_1(\mu)$ forms a straight lines as follows:*

$$Per_1(\mu) = \left\{ \langle f \rangle \in M_3(\mathbf{C}); \sigma_3 = (-\mu^2 + 2\mu)\sigma_1 + \mu^3 - 3\mu \right\}.$$

The locus $Per_2(\mu)$ forms an algebraic curve of degree three as follows:

$$\begin{aligned} Per_2(\mu) \quad = \quad & \{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_3^2 + (4\sigma_1^2 - (\mu + 57)\sigma_1 + 252)\sigma_3 - (4\mu - 16)\sigma_1^3 \\ & +(61\mu - 252)\sigma_1^2 - (4\mu^2 + 246\mu - 1134)\sigma_1 - \mu^3 + 51\mu^2 \\ & -99\mu - 459 = 0 \}. \end{aligned}$$

Note that this curve is irreducible if and only if $\mu \neq 1$. In the case of $\mu = 1$,

$$Per_2(1) = Per_1(-1) \bigcup \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_3 + 4\sigma_1^2 - 61\sigma_1 + 254 = 0 \right\}.$$

2.2 Symmetry locus

Using conjugation described in above, we can define symmetry locus of this moduli space as one in $\mathcal{M}_2(\mathbf{C})$, and we obtain next results.

Theorem 1 *The symmetry locus $\mathcal{S}$ of cubic polynomials forms an irreducible algebraic curve:*

$$S(\sigma_1, \sigma_3) = 27\sigma_3 + (\sigma_1 - 6)(2\sigma_1 - 3)^2 = 0. \tag{3}$$

The following result is obtained immediately by the definition of the envelope of the family of curves.

Corollary 3 *The envelope of $\{Per_1(\mu)\}_\mu$ coincides with the symmetry locus.*

Remark A cubic polynomial has non-trivial automorphism if and only if it is conjugate to a map in the unique normal form $p(z) = z^3 + az$.

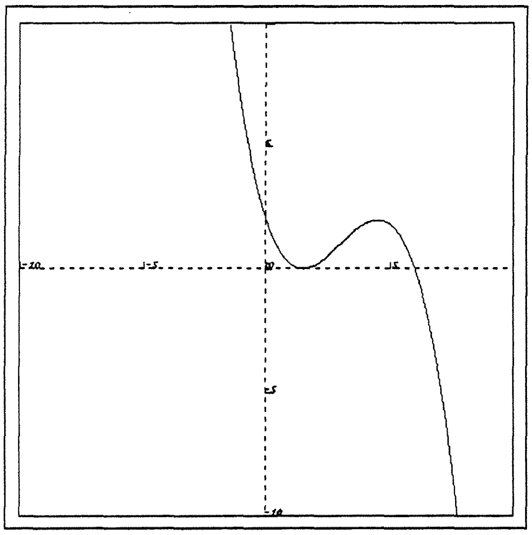

Figure 3: $\mathrm{M}_3(\mathbf{R})$ with the real cut of $\mathcal{S}$.

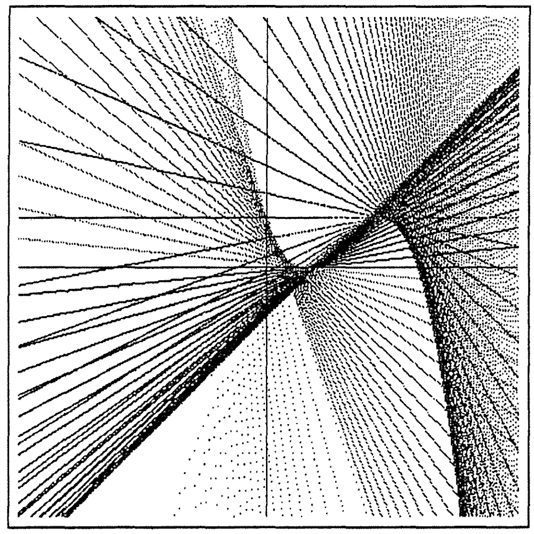

Figure 4: Lines $\{\mathrm{Per}_1(\mu)\}$ in the real cut of the moduli space $\mathrm{M}_3(\mathbf{C})$.

2.3 Real moduli space

Let $\mathrm{Poly}_3(\mathbf{R})$ be the set of real cubic polynomials. By the same reason for the case of $\mathcal{M}_2$, we define the real moduli space $\mathrm{M}_3(\mathbf{R})$ for $\mathrm{Poly}_3(\mathbf{R})$ to be simply the real (σ_1, σ_3)-plane. This notation needs some care when used: if we put $\mathcal{S}_{\mathbf{R}} = \mathcal{S} \cap \mathrm{M}_3(\mathbf{R})$, and denote by $\langle\ \rangle_{\mathbf{R}}$ the real conjugacy class, then $(\mathrm{Poly}_3(\mathbf{R})/\mathrm{Poly}_1(\mathbf{R})) \setminus \{\langle x^3 + ax\rangle_{\mathbf{R}},\ \langle -x^3 + ax\rangle_{\mathbf{R}}\}_{a \in \mathbf{R}^\times}$ is canonically isomorphic to $\mathbf{R}^2 \setminus \mathcal{S}_{\mathbf{R}}$, whereas there is a canonical two-to-one correspondence between $\{\langle \pm x^3 + ax\rangle\}_{a \in \mathbf{R}^\times}$ and $\mathcal{S}_{\mathbf{R}}$.

3 Polynomials of degree n

3.1 Moduli space of polynomials of degree n

Now we discuss about the moduli space $\mathrm{M}_n(\mathbf{C})$ for the space, $\mathrm{Poly}_n(\mathbf{C})$, of polynomials of degree n.

Doing the same as the case of cubic polynomials, we try introducing coordinates in $\mathrm{M}_n(\mathbf{C})$ as follows; for each $p(z) \in \mathrm{Poly}_n(\mathbf{C})$, let $z_1, \cdots, z_n, z_{n+1}(= \infty)$ be the fixed points of p and μ_i the multipliers of z_i; $\mu_i = p'(z_i)$ $(1 \leq i \leq n)$, and $\mu_{n+1} = 0$. Consider the elementary symmetric functions of the n

multipliers,

$$\begin{aligned}
&\sigma_{n,1} = \mu_1 + \cdots + \mu_n, \\
&\sigma_{n,2} = \mu_1\mu_2 + \cdots + \mu_{n-1}\mu_n = \textstyle\sum_{i=1}^{n-1} \mu_i \sum_{j>i}^{n} \mu_j, \\
&\quad \cdots \\
&\sigma_{n,n} = \mu_1\mu_2 \cdots \mu_n, \\
&\sigma_{n,n+1} = 0.
\end{aligned}$$

Example 1 For example, we assume $p(z) \in \mathrm{Poly}_4(\mathbf{C})$;

- fixed points: $z_1, z_2, z_3, z_4, \infty$
- multiplier: $\mu_1, \mu_2, \mu_3, \mu_4, 0$
- elementary symmetric functions:

$$\begin{cases}
\sigma_{4,1} = \mu_1 + \mu_2 + \mu_3 + \mu_4 \\
\sigma_{4,2} = \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 \\
\sigma_{4,3} = \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 \\
\sigma_{4,4} = \mu_1\mu_2\mu_3\mu_4 \\
\sigma_{4,5} = 0
\end{cases}$$

Applying Fatou-index theorem to these fixed points;

$$\frac{1}{1-\mu_1} + \frac{1}{1-\mu_2} + \frac{1}{1-\mu_3} + \frac{1}{1-\mu_4} + \frac{1}{1-0} = 1, \tag{4}$$

where $\mu_i \neq 1$ $(1 < i < n)$. Arranging this equation for the form of elementary symmetric functions;

$$\begin{aligned}
4 - 3(\mu_1 + \mu_2 + \mu_3 + \mu_4) + 2(\mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4) \\
-(\mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4) = 0.
\end{aligned}$$

Hence we have

$$4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0. \tag{5}$$

For the equation (5), the cases $\mu_i = 1$ are also allowable.

Now we consider a polynomial $p(z) = a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0 \in \mathrm{Poly}_4(\mathbf{C})$ that has at least two fixed points. After affine conjugation, we can assume they are 0 and 1. Then, we will solve the following question: *"Do the four multipliers*

$$\mu_0 = p'(0),\ \mu_1 = p'(1),\ \mu_2 = p'(z_2),\ \mu_3 = p'(z_3),$$

where z_1, z_2 are fixed points of $p(z)$, determine the five coefficients a_4, a_3, a_2, a_1, a_0 of $p(z)$?"

In fact, the following equations hold;

$$\begin{array}{ll} a_0 = 0 & \text{because of } f(0) = 0, \\ a_1 = \mu_0 & \text{because of } f'(0) = \mu_0, \\ a_2 = a_4 + 3 - 2\mu_0 - \mu_1 & \text{because of } f'(1) = \mu_1, \\ a_3 = 1 - a_4 - a_2 - \mu_0 & \text{because of } f(1) = 1, \end{array}$$

and a_4 is a common root of the following two equations;

$$\begin{aligned} A_1 = {} & (\mu_2^2 - 2\mu_3\mu_2 + \mu_3^2 - \mu_0^2 + 2\mu_1\mu_0 - \mu_1^2)a_4^4 + (-4\mu_0^3 + (4\mu_1 + \\ & 8)\mu_0^2 + (-4\mu_1^2 - 8)\mu_0 + 4\mu_1^3 - 8\mu_1^2 + 8\mu_1)a_4^3 + (-6\mu_0^4 + (-4\mu_1 + \\ & 28)\mu_0^3 + (4\mu_1^2 + 4\mu_1 - 44)\mu_0^2 + (-4\mu_1^3 + 4\mu_1^2 - 8\mu_1 + 32)\mu_0 - \\ & 6\mu_1^4 + 28\mu_1^3 - 44\mu_1^2 + 32\mu_1 - 16)a_4^2 + (-4\mu_0^5 + (-12\mu_1 + 32)\mu_0^4 + \\ & (-8\mu_1^2 + 64\mu_1 - 96)\mu_0^3 + (8\mu_1^3 - 96\mu_1 + 128)\mu_0^2 + (12\mu_1^4 - 64\mu_1^3 + \\ & 96\mu_1^2 - 64)\mu_0 + 4\mu_1^5 - 32\mu_1^4 + 96\mu_1^3 - 128\mu_1^2 + 64\mu_1)a_4 - \mu_0^6 + \\ & (-6\mu_1 + 12)\mu_0^5 + (-15\mu_1^2 + 60\mu_1 - 60)\mu_0^4 + (-20\mu_1^3 + 120\mu_1^2 - \\ & 240\mu_1 + 160)\mu_0^3 + (-15\mu_1^4 + 120\mu_1^3 - 360\mu_1^2 + 480\mu_1 - 240)\mu_0^2 + \\ & (-6\mu_1^5 + 60\mu_1^4 - 240\mu_1^3 + 480\mu_1^2 - 480\mu_1 + 192)\mu_0 - \mu_1^6 + 12\mu_1^5 - \\ & 60\mu_1^4 + 160\mu_1^3 - 240\mu_1^2 + 192\mu_1 - 64 = 0, \\ A_2 = {} & (\mu_2 + \mu_3 + \mu_0 + \mu_1 - 4)a_4^2 + (2\mu_0^2 - 4\mu_0 - 2\mu_1^2 + 4\mu_1)a_4 + \mu_0^3 + \\ & (3\mu_1 - 6)\mu_0^2 + (3\mu_1^2 - 12\mu_1 + 12)\mu_0 + \mu_1^3 - 6\mu_1^2 + 12\mu_1 - 8 = 0. \end{aligned}$$

Above two equations have common roots if and only if $\mu_0, \mu_1, \mu_2, \mu_3$ satisfy the equation (5). Since $\mu_0, \mu_1, \mu_2, \mu_3$ are the four multipliers of $p(z)$ and they should satisfy the equation (5), the two equations always have common roots. Hence five coefficients of $p(z)$ are calculated by its four multipliers, however, this calculation is not decisive when they have distinct two common roots.

For the case of $\mathrm{Poly}_n(\mathbf{C})$, it is clear from (4) that the equation corresponds to (5) cannot have the term of $\sigma_{n,n}$. Hence we can put

$$c_0 + c_1\sigma_{n,1} + c_2\sigma_{n,2} + \cdots + c_{n-1}\sigma_{n,n-1} = 0$$

where c_k $(0 \le k \le n-1)$ are functions of n variable.

Paying attention to the form of elementary symmetric functions, we obtain the following equation;

$$c_k = (-1)^k \frac{\begin{pmatrix} n-1 \\ k \end{pmatrix} n}{\begin{pmatrix} n \\ k \end{pmatrix}} = n - k.$$

where $\left(\begin{array}{c} n \\ k \end{array}\right)$ means binomial coefficient. For convenience, put $\sigma_{n,0} = 1$. we have

$$\sum_{k=0}^{n-1}(-1)^k(n-k)\sigma_{n,k} = 0. \tag{6}$$

Question Is the moduli space $\mathrm{M}_n(\mathbf{C})$ for polynomials of degree n canonically isomorphic to $\mathbf{C}^{n-1}$ with coordinates $\sigma_1, \sigma_2, \cdots, \sigma_{n-2}$, and σ_n?

3.2 Symmetry locus

Proposition 2 *A polynomial of degree four has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form*

$$\{z^4 + az\}, \quad a \in \mathbf{C}.$$

For a map $p(z)$ in this normal form, $Aut(p)$ is a cyclic group of order three.

Outline of proof. Let $p(z) \in \mathrm{Poly}_4(\mathbf{C})$.

1. In the case of a map $p(z)$ with multiple fixed points.
   (a) The case of $p(z)$ with a fixed point of order four: $\mathrm{Aut}(p)$ is non-trivial.
   (b) The case of $p(z)$ with a fixed point of order three: $\mathrm{Aut}(p)$ is trivial.
   (c) The case of $p(z)$ with two fixed points of order two: there is not such $p(z)$.
   (d) The case of $p(z)$ with a fixed point of order two: $\mathrm{Aut}(p)$ is trivial.
2. In the case of a map $p(z)$ with four distinct fixed points.
   (a) The case of four distinct multipliers: $\mathrm{Aut}(p)$ is trivial.
   (b) The case that only two of multipliers are coincide: $\mathrm{Aut}(p)$ is trivial.
   (c) The case of two pair of same multipliers: there is not such $p(z)$.
   (d) The case of three same multipliers: By an affine conjugation, if three fixed points (whose multipliers are same) are mapped on the vertices of a regular triangle whose barycenter is the origin and the other fixed point on the origin, then $\mathrm{Aut}(p)$ is non-trivial. Otherwise $\mathrm{Aut}(p)$ is trivial.

(e) The case of four same multipliers: there is not such $p(z)$.

Therefore a map $p(z)$ has non-trivial automorphisms if and only if $p(z)$ is in the case 1-(a) and the first part of 2-(d). We can check easily that these maps coincide with the normal form $\{z^4 + az\}$. ∎

Conjecture A polynomial of degree n has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form

$$\left\{ z^n + \sum_{k|(n-1), k \neq n-1} A(k) z^k \right\}$$

where $A(k)$ are parameters in $\mathbf{C}$.

A Comparison between the quadratic rational maps and cubic polynomials

	Quadratic rational maps	Cubic polynomials
Moduli Space	$\mathcal{M}_2(\mathbf{C}) \simeq \mathbf{C}^2$	$\mathrm{M}_3(\mathbf{C}) \simeq \mathbf{C}^2$
Real Moduli Space	$\mathcal{M}_2(\mathbf{R}) \simeq \mathbf{R}^2$ *excepts on the symm. locus*	$\mathrm{M}_3(\mathbf{R}) \simeq \mathbf{R}^2$ *excepts on the symm. locus*
Coordinates	(σ_1, σ_2), $\sigma_3 = \sigma_1 - 2$	(σ_1, σ_3), $3 - 2\sigma_1 + \sigma_2 = 0$
Normal Forms	*Fixed Pint Normal Form, etc.*	$\{f(z) = z^3 + az + b\}_{(a,b)}$
Periodic Orbits	$\mathrm{Per}_1(\mu)$: $\sigma_2 = (\mu + \frac{1}{\mu})\sigma_1 - (\mu^2 + \frac{2}{\mu})$ $\mathrm{Per}_2(\mu)$: $2\sigma_1 + \sigma_2 = \mu$ $\mathrm{Per}_1(-1) = \mathrm{Per}_2(1)$	$\mathrm{Per}_1(\mu)$: $\sigma_3 = (-\mu^2 + 2\mu)\sigma_1 + \mu^3 - 3\mu$ $\mathrm{Per}_2(\mu)$: *cubic algebraic curve* $\mathrm{Per}_1(-1) \subset \mathrm{Per}_2(1)$
Symmetry Locus	*the envelope of* $\{\mathrm{Per}_1(\mu)\}$ *normal form* : $\{k(z + \frac{1}{z})\}$	*the envelope of* $\{\mathrm{Per}_1(\mu)\}$ *normal form* : $\{z^3 + az\}$
Topological Partition	*degree*± 2, *monotone, unimodal, bimodal*	$\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$
Hyp. Components	B, C, D, E	A, B, C, D

A.1 Topological Partition

For map $f \in \mathcal{M}_2(\mathbf{R})$, the two critical points of f are two real numbers or a pair of complex conjugate numbers. If f has a pair of complex conjugate critical points, this map is two-to-one covering map on $S^1 = \mathbf{R} \cup \{\infty\}$. In this case, if $f' > 0$ then f is called the map of degree $+2$, else $f' < 0$ then the map of degree -2.

While a map f with real critical points is called monotone (resp. unimodal, bimodal) if the interval $I = int(f(S))$ contains no (resp. one, two) critical points [Mil93].

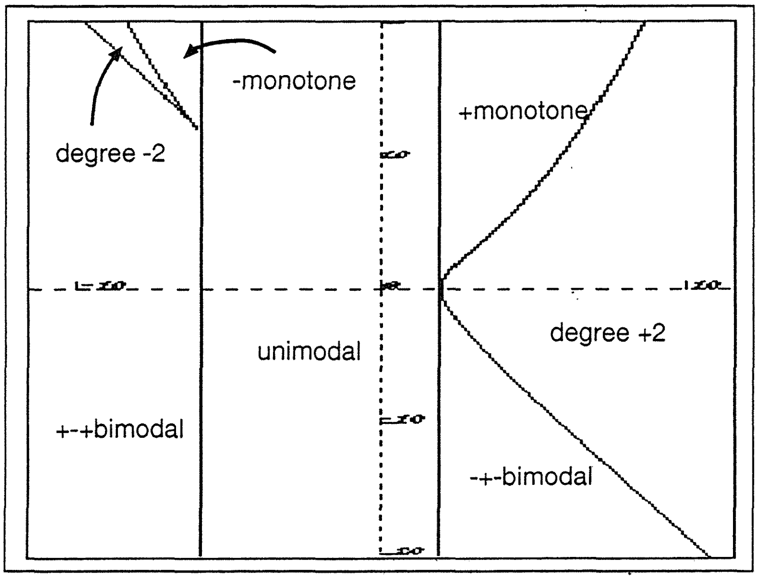

Figure 5: The topological partition of the $\mathcal{M}_2(\mathbf{R})$.

Boundary curves of Figure 5

$$\begin{aligned} CD_1 &: \sigma_1 = 2 \\ BC_1 &: \sigma_1 = 6 \\ \text{Symmetry locus} &: S(\sigma_1, \sigma_2) = 0 \end{aligned}$$

where the curves CD_1 ($\mathrm{Per}_1(0)$) and BC_1 are "center curve" defined in [NN93].

Remark Two curves BC_1 and CD_1 are boundary courves of the "unimodal" region.

For map $p \in M_3(\mathbf{R})$, if the real filled-in Julia set of p is a single point then it is said that p in the class $\mathcal{R}_0$. Let J be the smallest closed interval which contains the real filled-in Julia set of p. For $p \notin \mathcal{R}_0$, it is said that p belongs to the class $\mathcal{R}_n$ if the graph of p intersected with $J \times J$ has n distinct components [Mil90].

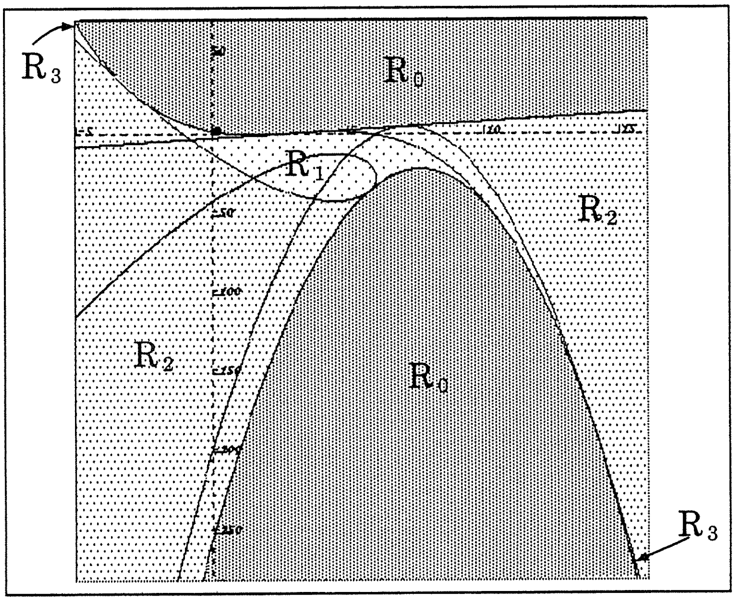

Figure 6: The topological partition of the $M_3(\mathbf{R})$.

Boundary curves of Figure 6

$$
\begin{aligned}
\mathrm{Per}_1(1) \quad &: \quad \sigma_1 - \sigma_3 - 2 = 0 \\
\mathrm{Preper}_{(1)}1 \quad &: \quad -4\sigma_1^2 + 57\sigma_1 - \sigma_3 - 198 = 0 \\
\text{Symmetry locus} \quad &: \quad S(\sigma_1, \sigma_3) = 0 \\
\mathrm{Per}_2(1) \quad &: \quad -8\sigma_1^3 + 180\sigma_1^2 - 1809\sigma_1 - 27\sigma_3 + 6966 = 0 \\
\mathrm{Preper}_{(1)}2 \quad &: \quad 64\sigma_1^6 - 1152\sigma_1^5 + 7776\sigma_1^4 + (432\sigma_3 - 25056)\sigma_1^3 \\
&\quad\ +(-3888\sigma_3 + 41796)\sigma_1^2 + (8748\sigma_3 - 34992)\sigma_1 + 729\sigma_3^2 \\
&\quad\ -45198\sigma_3 + 543105 = 0
\end{aligned}
$$

A.2 Hyperbolic components

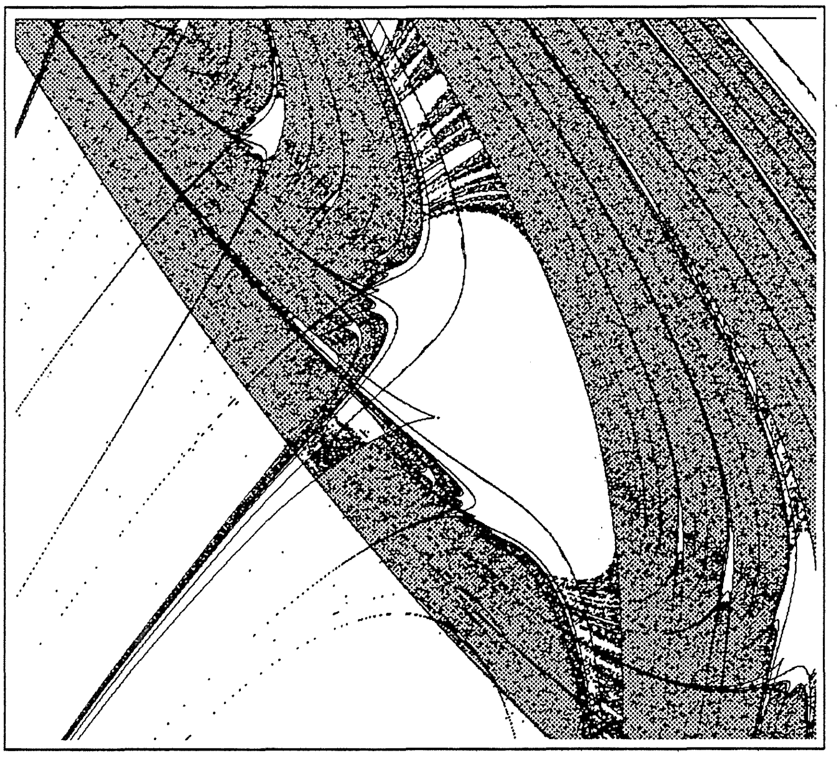

Figure 7: "Swallow configuration" in the moduli space $\mathbf{M}_3(\mathbf{C})$ of the cubic polynomials. $0.8 < \sigma_1 < 3.7$, $-37.6 < \sigma_3 < -20$.

A rational map is hyperbolic if and only if the orbit of every critical point converges to some attracting periodic orbit. The hyperbolic maps form an open subset of moduli space, and the connected components of this open set are called hyperbolic components.

The hyperbolic components of $\mathcal{M}_2(\mathbf{C})$ can be devided into the following four classes; Bitransitive-, Capture-, Disjoint-, and Escape-types [Ree90].

The hyperbolic components of $\mathbf{M}_3(\mathbf{C})$ can be devided into the following four classes; Adjacent-, Bitransitive-, Capture-, and Disjoint-types [Mil90].

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1 Quadratic rational maps

1.1 Moduli space of quadratic rational maps

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\mathrm{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\mathrm{Rat}_2(\mathbf{C})$ by conjugation,

$$g \circ f \circ g^{-1} \in \mathrm{Rat}_2(\mathbf{C}) \quad \text{for} \quad g \in \mathrm{PSL}_2(\mathbf{C}),\ f \in \mathrm{Rat}_2(\mathbf{C}).$$

Two maps $f_1, f_2 \in \mathrm{Rat}_2(\mathbf{C})$ are **holomorphically conjugate**, denoted by $f_1 \sim f_2$, if and only if there exists $g \in \mathrm{PSL}_2(\mathbf{C})$ with $g \circ f_1 \circ g^{-1} = f_2$. The quotient space of $\mathrm{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the **moduli space** of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f.

Milnor introduced in [Mil92] coordinates in $\mathcal{M}_2(\mathbf{C})$ as follows; for each $f \in \mathrm{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be the fixed points of f and μ_i the multipliers of z_i; $\mu_i = f'(z_i)$ $(1 \le i \le 3)$. Consider the elementary symmetric functions of the three multipliers,

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3, \quad \sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad \sigma_3 = \mu_1\mu_2\mu_3.$$

These three multipliers determine f up to holomorphic conjugacy, and are subject only to the restriction that

$$\sigma_3 = \sigma_1 - 2.$$

Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ with coordinates σ_1 and σ_2 (Lemma 3.1 in [Mil92]).

By an automorphism of a quadratic rational map f, we will mean $g \in \mathrm{PSL}_2(\mathbf{C})$ which commutes with f. The collection $\mathrm{Aut}(f)$ of all automorphisms of f forms a finite group. It is clear that $\mathrm{Aut}(\tilde{f})$ is isomorphic to $\mathrm{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$.

The set

$$\mathcal{S} = \{\langle f \rangle\, ;\ \mathrm{Aut}(f) \text{ is non-trivial}\} \subset \mathcal{M}_2(\mathbf{C})$$

is called the **symmetry locus**.

For each $\mu \in \mathbf{C}$ let $\mathrm{Per}_n(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps f which having a periodic point of period n and multiplier μ.

Each of $\mathrm{Per}_1(\mu)$ and $\mathrm{Per}_2(\mu)$ forms a straight lines as follows:

$$\begin{aligned} \mathrm{Per}_1(\mu) &= \left\{\langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1})\right\} \\ \mathrm{Per}_2(\mu) &= \{\langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = -2\sigma_1 + \mu\}, \end{aligned}$$

(Lemmas 3.4 and 3.6 in [Mil92]).

Proposition 1 *The symmetry locus $\mathcal{S}$ is defined by an irreducible algebraic curve in $\mathcal{M}_2(\mathbf{C})$ as follows;*

$$S(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \qquad (1)$$

We give an proof in [FN], [FN2].

Corollary 1 *T he symmetry locus $\mathcal{S}$ is the envelope of the family of the lines $\mathrm{Per}_1(\mu)$.*

Milnor describes the curve (1) implicitly (compare Figure 15 in [Mil92]). Here we can give a defining equation (1) of this cubic curve.

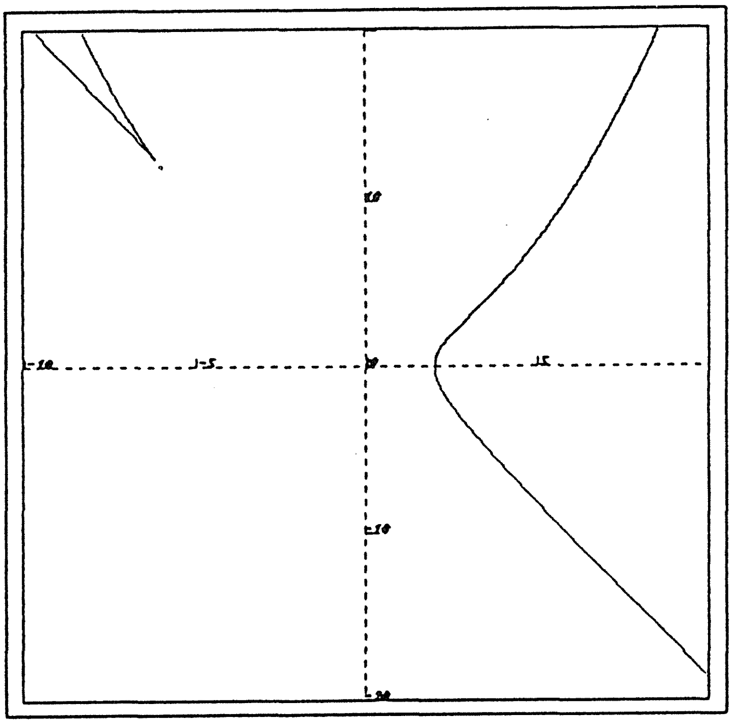

Figure 1: The real cut of the Symmetry locus.

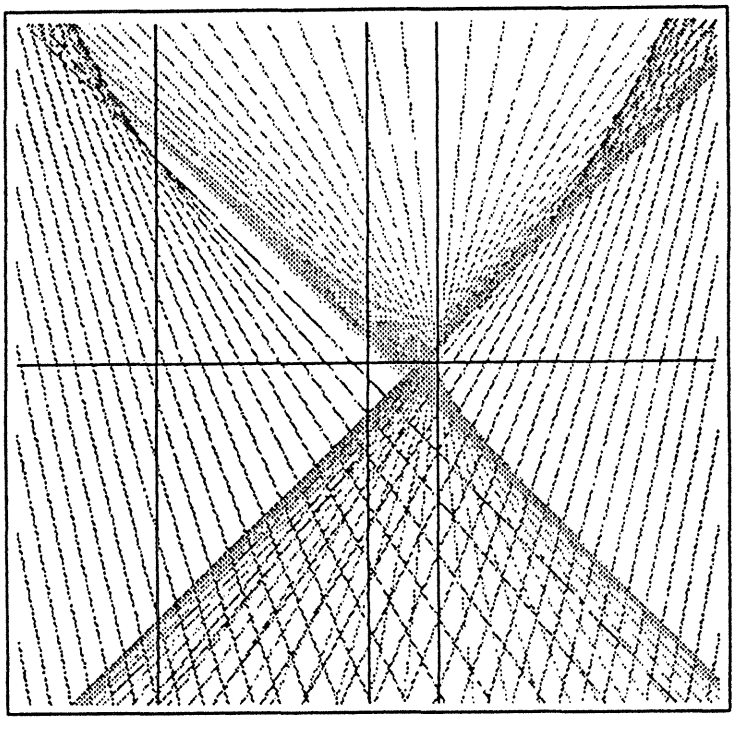

Figure 2: The lines $\mathrm{Per}_1(\mu)$.

1.2 Real moduli space

Let $\mathrm{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. Then the parameters σ_i $(1 \le i \le 3)$ are all real, because the three fixed points and the corresponding multipliers are either all real or one real and a pair of complex conjugate numbers. According to J. Milnor, we define the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\mathrm{Rat}_2(\mathbf{R})$ to be simply the real (σ_1, σ_2)-plane. This notation needs some care when used: if we put $\mathcal{S}_{\mathbf{R}} = \mathcal{S} \cap \mathcal{M}_2(\mathbf{R})$, and denote by $\langle\ \rangle_{\mathbf{R}}$ the real conjugacy class, then $(\mathrm{Rat}_2(\mathbf{R})/\mathrm{PSL}_2(\mathbf{R})) \setminus \{\langle a(x+\frac{1}{x})\rangle_{\mathbf{R}}, \langle a(x-\frac{1}{x})\rangle_{\mathbf{R}}\}_{a \in \mathbf{R}^\times}$ is canonically isomorphic to $\mathbf{R}^2 \backslash \mathcal{S}_{\mathbf{R}}$, whereas there is a canonical two-to-one correspondence between $\{\langle a(x \pm \frac{1}{x})\rangle\}_{a \in \mathbf{R}^\times}$ and $\mathcal{S}_{\mathbf{R}}$.

2 A quadratic rational family with non-monotone bifurcations

let $\{f_\lambda\}_\Lambda$ be a one-parameter family of discrete dynamical systems on $\mathbf{R}$ where Λ is an interval of $\mathbf{R}$. As the parameter increased, a parameter value λ_0 is called **orbit creating** if, at λ_0, new periodic orbits are created and no periodic orbits are annihilated; λ_0 is called **orbit annihilating** if periodic orbits are annihilated and no new periodic orbits are created; λ_0 is called **neutral** if no periodic orbits are annihilated and no periodic orbits are created.

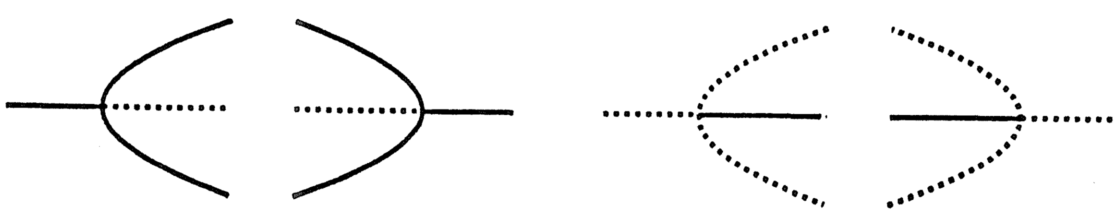

Figure 3: Regular period-doubling (-halving) bifurcations and irregular period-doubling (-halving) bifurcations.

A family $\{f_\lambda\}_\Lambda$ is said to be **monotone increasing** (resp. **decreasing**) if every parameter value in Λ is neutral or orbit creating (resp. annihilating). A family $\{f_\lambda\}_\Lambda$

is called **non-monotone** if Λ contains both orbit creating and orbit annihilating parameter values.

Note that the sign of Schwarzian derivative $Sf = f'''(x)/f'(x) - \frac{3}{2}\left(f''(x)/f'(x)\right)^2$ determines the type of local bifurcation: For a family of maps with negative Schwarzian derivative, a period-doubling bifurcation necessarily involves only an attracting (regular) orbit of period two, and not the reverse one which involves a repelling (irregular) orbit of period two ([?]). See Figure 3.

Now, we investigate the dynamics of a certain real 2-parameter family given by M. Bier and T. C. Bountis [BB84] and rewritten by H. E. Nusse and J. A. Yorke ([NY88]):

$$\left\{f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}.$$

We note that quadratic rational maps have negative Schwarzian derivatives. Hence, only regular period-doubling (or -halving) bifurcations may occur in this family.

Since the maps $f_{m,r}$ and $f_{m,-r}$ are conjugate to each other for any r, it suffices to consider the case $r \geq 0$.

Since $\mathcal{M}_2(\mathbf{C})$ is isomorphic to $\mathbf{C}^2$ with coordinate σ_1 and σ_2, there is a natural compactification $\hat{\mathcal{M}}_2(\mathbf{C}) \cong \mathbf{CP^2}$, consisting of $\mathcal{M}_2(\mathbf{C})$ together with a 2-sphere of **ideal points** at infinity. Elements of this 2-sphere can be thought as limits of quadratic rational maps which degenerate towards a fractional linear or constant map ([Mil92]). Therefore for the case $m = 0$ of this family $f_{m,r}$, it makes sense that we should consider it as a degenerated limit.

Theorem 1 *In $\mathcal{M}_2(\mathbf{R})_{\mathbf{R}}$, the one parameter family $\{f_{m,r}(x)\}_m$ for each fixed r ($r \geq 0$) lies exactly on an irreducible algebraic curve:*

For $r \neq \frac{1}{2}, 0$, this curve is of degree 4 defied by the equation

$$\begin{aligned} H_r(\sigma_1,\sigma_2) &= -r^2\sigma_1^4 + (8r^2-2)\sigma_1^3 + ((8r^2-1)\sigma_2 - 128r^4 + 8r^2 + 1)\sigma_1^2 \\ &\quad + ((-32r^2+8)\sigma_2 + 512r^4 - 96r^2 - 12)\sigma_1 + (-16r^2+4)\sigma_2^2 \\ &\quad + (512r^4 - 96r^2 - 12)\sigma_2 - 4096r^6 + 1536r^4 - 144r^2 + 36 = 0. \quad (2) \end{aligned}$$

For $r = \frac{1}{2}$, the corresponding curve is of degree 3, i.e.,

$$H_{\frac{1}{2}}(\sigma_1,\sigma_2) = -\sigma_1^3 - 2\sigma_1^2 + (4\sigma_2 - 24)\sigma_1 + 8\sigma_2 - 64 = 0. \quad (3)$$

For $r = 0$, the corresponding curve is also of degree 3, i.e.,

$$H_0(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \qquad (4)$$

Proof. The three fixed points z_1, z_2, z_3 of $f_{m,r}$ are the roots of the equation

$$z^3 - mrz^2 + (1-m)z - mr = 0,$$

i.e.,

$$\begin{cases} z_1 + z_2 + z_3 = mr, \\ z_1z_2 + z_2z_3 + z_3z_1 = 1 - m, \\ z_1z_2z_3 = mr. \end{cases}$$

The multiplier μ_i of each fixed point z_i is given by

$$f'(z_i) = \mu_i = m\frac{z_i^2 - 1}{(z_i^2+1)^2} \qquad (i = 1, 2, 3).$$

By using "**Gröbner basis** of **Risa/Asir**, Symbolic and algebraic computation system by FUJITSU, we can obtain the coordinates $\sigma_1(= \mu_1 + \mu_2 + \mu_3)$ and $\sigma_2(= \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1)$ as functions of m and r:

$$\begin{cases} 4m^2r^2 - m^2 + (\sigma_1 + 2)m - 4 = 0 \\ -4m^4r^4 + (m^4 - 12m^3 - 8m^2)r^2 + 2m^3 + (\sigma_2 - 5)m^2 + 4m - 4 = 0. \end{cases} \qquad (5)$$

Using "**Gröbner basis** again, we can remove m from (5) for each fixed r,and get the defining equation (2). We can check easily that (2)is irreducible if and only if $r \neq \frac{1}{2}$, from which follows the first and the last cases. In the case of $r = \frac{1}{2}$, substituting $r = \frac{1}{2}$ in (5) directory, then we obtain (3), which is clearly irreducible.

Conversely, to see any point on the curve $H_r(\sigma_1, \sigma_2) = 0$ comes from an $f_{m,r}$ for some m, observe carefully the process that m is removed from (5). Thus we can see that, except for finite number of points which annihilates the leading coefficients of some polynomial in m appearing in the course of the procedure, every point on the curve corresponds to an $f_{m,r}$ for some m. Then so does any point on the whole curve due to the continuity of the solution of (5), when regarded as equation of m. ∎

Remark 1 The equation of σ_1 in (5) is obtained by the following Program 2, which is suggested us by Takeshi Shimoyama, advaced researcher of ISIS, FUJITSU LABORATORIES LTD.

Program 2

```
if (vtype(gr)!=3) load("gr")$$
extern Ord$

def moduliS1()
{
        S1=nm(m*((z1^2-1)/(z1^2+1)^2
                +(z2^2-1)/(z2^2+1)^2+(z3^2-1)/(z3^2+1)^2)-s1);
        X=z1+z2+z3-m*r;
        Y=z1*z2+z2*z3+z3*z1-1+m;
        Z=z1*z2*z3-m*r;
        Ord=2;
        G=gr([S1,X,Y,Z],[z1,z2,z3,m,r,s1]);
        for (I=length(G)-1;I>=0;I--){
                E=G[I];
                if (vars(E)==[r,m,s1])
                        break;
        }
        return E;
}
end$
```

To say superfluously, the required equation (2) is obtained from following command of Risa/Asir.

Command of Risa/Asir

```
gr([4*m^2*r^2-m^2+(s1+2)*m-4-4*m^4*r^4,
+(m^4-12*m^3-8*m^2)*r^2+2*m^3+(s2-5)*m^2+4*m-4],[m,r]);
```

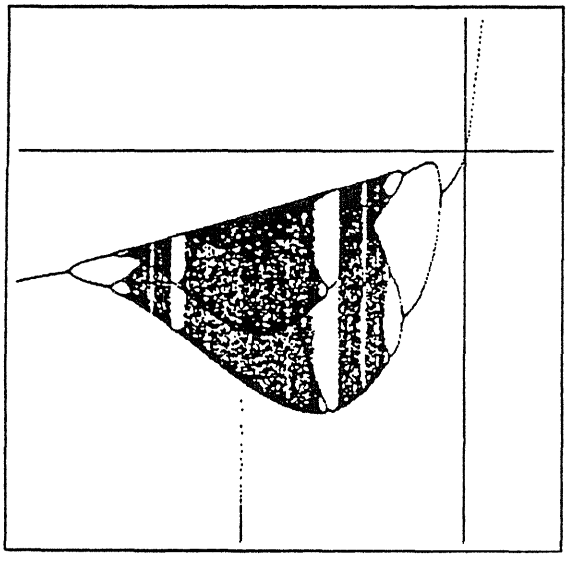

Figure 4: Non-monotone bifurcation; $-25.0 \leq m \leq 5.0$, $-3.0 \leq x \leq 1.0$, $r = 0.54$.

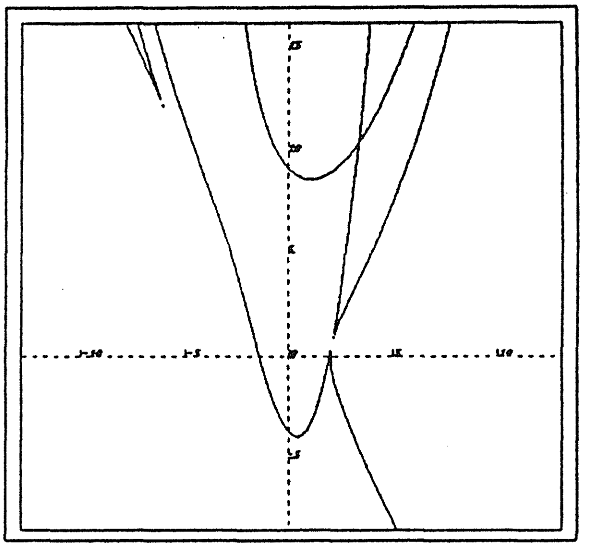

Figure 5: Algebraic curve of degree 4 and cubic curve in the moduli space. In the case of $r = 0.54$.

o

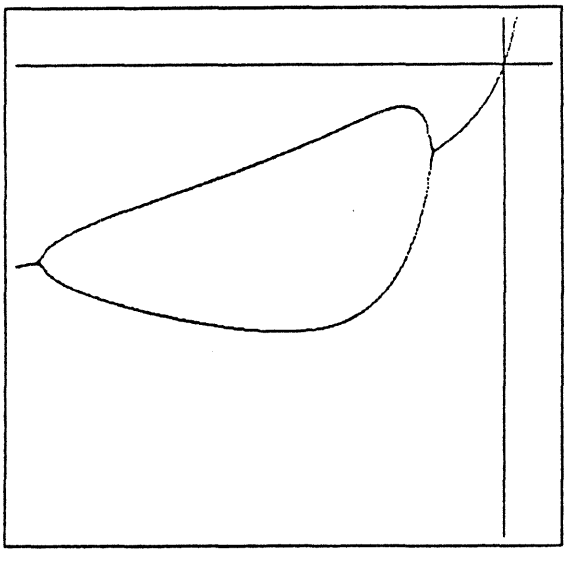

Figure 6: Period-bubbling bifurcation: $-10 \leq m \leq 1$, $-2 \leq x \leq 0.2$, Parameter $r = 0.58$.

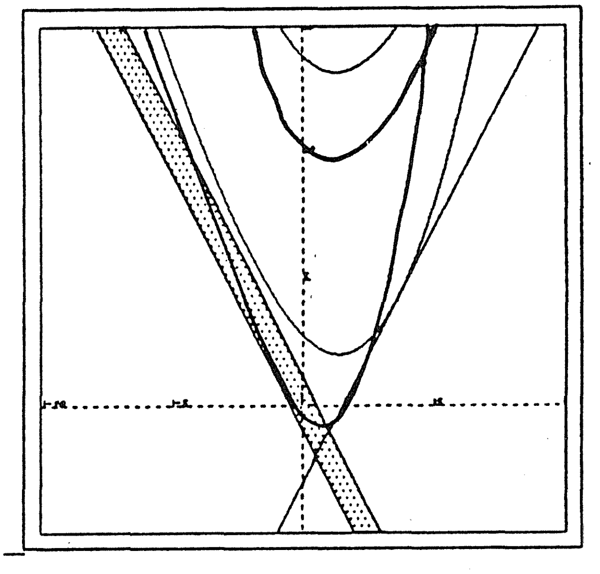

Figure 7: Algebraic curves of degree 4 in the "classified" moduli space. Thick curve corresponds with $r = 0.58$, thin curve corresponds with $r = 0.7$.

Example 1 Non-monotone bifurcation can occur at $r = 0.54$, See Figure 4. And its charasteristic curve is Figure 5.

We can analyze the non-monotone bifurcation by overwriting the algebraic curve of degree 4 on the $\mathcal{M}_2(\mathbf{R})$.

Example 2 One parameter family $\{f_{m,0.58}\}$ has non-monotone (period-bubbling) bifurcation. See Figure 6.

In Figure 7, the thick line indicates this family, and the gray belt is the region on which each map has attracting period 2 cycle. When algebraic curve of degree 4 through this gray belt, period-doubling bifurcation occurs. In this case, the curve intersects the gray belt (period-doubling occurs) and intersects again the period 1 region (period-halving occurs). Hence period-bubbling bifurcation occurs, as in Figure 6.

Theorem 2 *For a fixed parameter r, there are following three possibilities;*

1. *various bifurcations occur if $0 < r \leq \frac{1}{2}$,*
2. *non-monotone bifurcations occur if $\frac{1}{2} < r < \frac{3\sqrt{3}}{8}$, or*
3. *any bifurcation can't occur if $\frac{3\sqrt{3}}{8} \leq r$.*

A`oof is given in ([FN] and [FN2]).

Acknowledgements: Takeshi Shimoyama (Fujitsu Laboratories) taught us about Gröbner basis of Risa/Asir, computer algebra system, developed by Fujitsu Laboratories. We thank him for his advice.

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Two affine structures imposed in the polynomials with degree four

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Abstract

In the paper of [NF97b] we studied the geometrical and topological properties of the moduli space of polynomial maps of degree 3 from a viewpoint of complex dynamical systems. Making use of the discussion of [FN97] and [NF97a], we decide the branch locus and give the "topological partition" of the real moduli space of polynomial maps of degree 4.

1 Polynomials of degree 4

1.1 Coefficient coordinate on polynomials of degree 4

Let $\mathrm{Poly}_4(\mathbf{C})$ be the space of all polynomial maps of the form

$$\begin{aligned}&p : \mathbf{C} \to \mathbf{C},\\ &p(z) = a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0 \quad (a_4 \neq 0).\end{aligned}$$

The group $\mathfrak{A}(\mathbf{C})$ of all affine transformations acts on $\mathrm{Poly}_4(\mathbf{C})$ by conjugation:

$$g \circ p \circ g^{-1} \in \mathrm{Poly}_4(\mathbf{C}) \quad \text{for} \quad g \in \mathfrak{A}(\mathbf{C}),\ p \in \mathrm{Poly}_4(\mathbf{C}).$$

Two maps $p_1, p_2 \in \mathrm{Poly}_4(\mathbf{C})$ are *holomorphically conjugate* if and only if there exists $g \in \mathfrak{A}(\mathbf{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\mathrm{Poly}_4(\mathbf{C})$ under this action will be denoted by $\mathrm{M}_4(\mathbf{C})$, and called the *moduli space* of holomorphic conjugacy classes $\langle p \rangle$ of polynomial maps p of degree 4.

Under the conjugacy of the action of $\mathfrak{A}(\mathbf{C})$, it can be assumed that any map in $\mathrm{Poly}_4(\mathbf{C})$ is "monic" and "centered", i.e.,

$$p(z) = z^4 + c_2 z^2 + c_1 z + c_0.$$

This p is determined up to the action of the group $G(3)$ of 3-rd roots of unity, where each $\eta \in G(3)$ acts on $p \in \mathrm{Poly}_4(\mathbf{C})$ by the transformation $p(z) \mapsto p(\eta z)/\eta$.

Let $\mathcal{P}_1(4)$ be the affine space of all monic and centered polynomials of degree 4 with coordinate (c_0, c_1, c_2). Then we have an three-to-one canonical projection

$$\Phi : \mathcal{P}_1(4) \to \mathrm{M}_4(\mathbf{C})$$

from $\mathcal{P}_1(4)$ onto $\mathrm{M}_4(\mathbf{C})$. Thus we can use $\mathcal{P}_1(4)$ as coordinate space for $\mathrm{M}_4(\mathbf{C})$ though there remains the ambiguity up to the group $G(3)$.

Now we introduce "multipliers' coordinates" in $\mathrm{M}_4(\mathbf{C})$ (see [Mil93]):

for each $p(z) \in \mathrm{Poly}_4(\mathbf{C})$, let $z_1, \cdots, z_4, z_5 (= \infty)$ be the fixed points of p and μ_i the multipliers of z_i; $\mu_i = p'(z_i)$ $(1 \leq i \leq 4)$, and $\mu_5 = 0$. Consider the elementary symmetric functions of the four multipliers,

$$\begin{aligned}
\sigma_{4,1} &= \mu_1 + \mu_2 + \mu_3 + \mu_4, \\
\sigma_{4,2} &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 \\
\sigma_{4,3} &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4, \\
\sigma_{4,4} &= \mu_1\mu_2\mu_3\mu_4 \\
\sigma_{4,5} &= 0.
\end{aligned}$$

Note that these are well-defined on the moduli space $\mathrm{M}_n(\mathbf{C})$, since μ_i's are invariant by affine conjugacy. Applying the Fatou index theorem, we have a linear relation ([NF97b]):

$$4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0. \tag{1}$$

Let $\Sigma(4)$ be an affine space with coordinates $(\sigma_{4,1}, \sigma_{4,2}, \sigma_{4,4})$, so-called multipliers' coordinates.

We have a natural projection:

$$\Psi : \mathrm{M}_4(\mathbf{C}) \to \Sigma(4).$$

Definition 1 $\mathrm{Per}_1(\mu) \subset \mathrm{M}_4(\mathbf{C})$ is the locus of all classes having a fixed point with multiplier μ. Similarly, $\mathrm{Preper}_{(n)}1$ is the locus of all classes having a pre-fixed critical orbit with tale-length $n \neq 0$.

2 Comparison between $\mathrm{Poly}_3(\mathbf{C})$ and $\mathrm{Poly}_4(\mathbf{C})$

Now we summarize the properties of the $\mathrm{Poly}_3(\mathbf{C})$ and $\mathrm{Poly}_4(\mathbf{C})$ given by [Mil92], [NF97b] and [FN97].

2.1 $\mathrm{Poly}_3(\mathbf{C})$ case

Moduli space:

- The moduli space $\mathrm{M}_3(\mathbf{C})$ is isomorphic to the space $\Sigma(3)$, hence it is isomorphic to $\mathbf{C}^2$.
- $\mathcal{P}_1(3)$ is a two-sheeted ramified covering of $\mathbf{C}^2$

Real moduli space: The real moduli space $\mathrm{M}_3(\mathbf{R})$ has one-to-one correspondence with $\mathbf{R}^2$, excepting on the symmetry locus. While on the symmetry locus, there is two-to-one correspondence.

Multiplier's Coordinates: $(\sigma_{3,1}, \sigma_{3,3})$ with the linear relation $3 - 2\sigma_{3,1} + \sigma_{3,2} = 0$.

Normal Forms ($\mathcal{P}_1(3)$): $\{f(z) = z^3 + c_1 z + c_0\}$

Transformation formula:

$$\begin{aligned}\sigma_{3,1} &= -3c_1 + 6,\\ \sigma_{3,3} &= 27c_0^2 + a(2c_1 - 3)^2\end{aligned}$$

Dynamical curves:

- $\mathrm{Per}_1(\mu)$: $\sigma_{3,3} = (-\mu^2 + 2\mu)\sigma_{3,1} + \mu^3 - 3\mu$
- $\mathrm{Per}_2(\mu)$: cubic algebraic curve

- $\mathrm{Per}_1(-1) \subset \mathrm{Per}_2(1)$

Symmetry locus: The symmetry locus coincides with the envelope of the lines family $\{\mathrm{Per}_1(\mu)\}$. And it forms an irreducible algebraic curve in $\mathrm{M}_3(\mathbf{C})$:

$$S_3(\sigma_{3,1}, \sigma_{3,3}) = 4\sigma_{3,1}^3 - 36\sigma_{3,1}^2 + 81\sigma_{3,1} + 27\sigma_{3,3} - 54 = 0.$$

And its normal form is given by a one parameter family $\{z^3 + az\}$.

Topological partition: The real moduli space is divided into the following four parts $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$. And its boundary curves are the following dynamical curves:

$$\mathrm{Per}_1(1),\ \mathrm{Per}_2(1),\ \mathrm{Preper}_{(1)}1,\ \mathrm{Preper}_{(1)}2,\ \text{Symmetry locus}$$

2.2 $\mathrm{Poly}_4(\mathbf{C})$ case

Moduli space: The number of the inverse images of the space $\Sigma(4)$ under the map Ψ is 0, 1, 2, or ∞. The space $\mathcal{P}_1(4)$ is a three-sheeted ramified covering of $\mathbf{C}^3$

Multiplier's Coordinates: $(\sigma_{4,1}, \sigma_{4,2}, \sigma_{4,4})$ with linear relation $4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0$

Normal Forms ($\mathcal{P}_1(4)$): $\{f(z) = z^4 + c_2z^2 + c_1z + c_0\}$

Transformation formula:

$$\begin{aligned}
\sigma_{4,1} &= -8c_1 + 12 && (2)\\
\sigma_{4,2} &= 4c_2^3 - 16c_0c_2 + 18c_1^2 - 60c + 1 + 48 && (3)\\
\sigma_{4,4} &= 16c_0c_2^4 + (-4c_1^2 + 8c_1)c_2^3 - 128c_0^2c_2^2 + (144c_0c_1^2 - 288c_0c_1\\
&\quad +128c_0)c_2 - 27c_1^4 + 108c_1^3 - 144c_1^2 + 64c_1 + 256c_0^3 && (4)
\end{aligned}$$

Dynamical curves:

$$\Psi\left(\mathrm{Per}_1(\mu)\right) : \mu^4 - \sigma_{4,1}\mu^3 + \sigma_{4,2}\mu^2 + (3\sigma_{4,1} - 2\sigma_{4,2} - 4)\mu + \sigma_{4,4} = 0$$

Symmetry locus: The symmetry locus is a proper subspace of the envelope of the plane family $\{\mathrm{Per}_1(\mu)\}$. The symmetry locus $\mathcal{S}_4$ in $\mathrm{M}_4(\mathbf{C})$ forms the following algebraic curve:

$$\begin{cases} \sigma_{4,1} = s \\ \sigma_{4,2} = 3(3s-4)(s+4)/32 \\ \sigma_{4,4} = -(3s-4)^3(s-12)/4096. \end{cases}$$

And its normal form is given by a one parameter family $\{z^4 + az\}$.

Remark A. F. Beardon [Bea90] studies symmetries of Julia sets. He gave a sufficient and necessary condition for the Julia set of two polynomials P and Q are same. There are significant relations between symmetries of Julia sets and the symmetry locus ([FN]).

3 Branch locus

In the case of cubic polynomials, the envelope of the line family $\{\mathrm{Per}_1(\mu)\}_\mu$ coincides with the symmetry locus ([NF97b]). But, in the case of polynomials of degree 4, the symmetry locus is the proper subspace of the envelope ([NF97a]).

In fact, the images of the surfaces $\mathrm{Per}_1(\mu)$ are easily obtained by using the linear relation (1):

$$\Psi(\mathrm{Per}_1(\mu)) \ : \ \mu^4 - \sigma_{4,1}\mu^3 + \sigma_{4,2}\mu^2 + (3\sigma_{4,1} - 2\sigma_{4,2} - 4)\mu + \sigma_{4,4} = 0.$$

And a defining equation of the envelope of $\{\Psi(\mathrm{Per}_1(\mu))\}_\mu$ is

ENV :

$$54\sigma_{4,1}^5 + (-81\sigma_{4,2} - 27\sigma_{4,4} - 135)\sigma_{4,1}^4 + (36\sigma_{4,2}^2 - 144\sigma_{4,2} - 1008)\sigma_{4,1}^3 + (-4\sigma_{4,2}^3 + 360\sigma_{4,2}^2 + (144\sigma_{4,4} + 2976)\sigma_{4,2} + 576\sigma_{4,4} + 4192)\sigma_{4,1}^2 + (-160\sigma_{4,2}^3 - 2176\sigma_{4,2}^2 + (-384\sigma_{4,4} - 6400)\sigma_{4,2} - 1280\sigma_{4,4} - 5376)\sigma_{4,1} + 16\sigma_{4,2}^4 + 448\sigma_{4,2}^3 + (-128\sigma_{4,4} + 2176)\sigma_{4,2}^2 + (256\sigma_{4,4} + 3840)\sigma_{4,2} + 256\sigma_{4,4}^2 + 768\sigma_{4,4} + 2304 = 0.$$

This defining equation is obtained by seeking the common factor of $\Psi(\mathrm{Per}_1(\mu))$ and $\frac{\partial}{\partial\mu}\Psi(\mathrm{Per}_1(\mu))$ where the singular factor $\Psi(\mathrm{Per}_1(1))$ is removed.

A defining equation of the symmetry locus satisfies a defining equation of ENV.

To say more intuitively, the symmetry locus corresponds with the condition that the equation $\mathrm{Per}_1(\mu)$ has triple root, while the envelope corresponds with the condition of double root.

In the case of polynomials of degree 4, the envelope deeply concerns the branch locus.

In this paper, branch locus is defined the locus where the number of inverse images is not two.

Theorem 1 *The branch locus is characterized as follows;*
branch locus $= \{\sigma_{4,1} - 4 = 0\} \cup ENV$.

Before proving this theorem, we need "inverse problem" described in [NF97a] (Proposition 2): *for any* $(\sigma_{4,1}, \sigma_{4,2}, \sigma_{4,4})$ *given, there exists* (c_0, c_1, c_2) *satisfying the transformation formula or not.*

> **Proposition 2** in [NF97a] *The composition* $\Psi \circ \Phi : \mathcal{P}_1(4) \to \Sigma(4)$ *is not surjective: this map has no inverse image for any point on the "punctured" curve* $\mathcal{E}$*:*
>
> $$(\sigma_{4,1}, \sigma_{4,2}, \sigma_{4,4}) = (4, s, s^2/4 - 2s + 4), s \neq 6.$$

Proof of outline of "inverse problem" Fix a point $(\sigma_{4,1}, \sigma_{4,2}, \sigma_{4,4}) \in \Sigma(4)$. The following equation is obtained by substituting the equation (2) to (3) of transformation formula:

$$4c_2^3 - 16c_0c_2 = -\sigma_{4,2} - \frac{9}{32}\sigma_{4,1}^2 - \frac{3}{4}\sigma_{4,1} + \frac{3}{2} \tag{5}$$

Let V be the value of the right hand of the relation (5):

$$V = \frac{1}{32}(-32\sigma_{4,2} + 9\sigma_{4,1}^2 + 24\sigma_{4,1} - 48) \tag{6}$$

First we start the case of $V = 0$. We put $c_1 = \frac{12-\sigma_{4,1}}{8}$ and $c_2 = 0$. Then c_0 is a one of the solutions of the equation given by (4):

$$1048576c_0^3 - 4096\sigma_{4,4} - 27\sigma_{4,1}^4 + 432\sigma_{4,1}^3 - 1440\sigma_{4,1}^2 + 1792\sigma_{4,1} - 768 = 0.$$

It is important that the coefficient of the c_0^3 term does not vanish.

Second, we assume that $V \neq 0$. From the relation (5), if there exists inverse images then we have $c_2 \neq 0$. Therefore dividing (3) by c_2, and substituting it into (4) we obtain the following equation:

$$Ac_2^6 + Bc_2^3 + C = 0 \tag{7}$$

where

$$\begin{aligned}
A &= 262144(\sigma_{4,1} - 4)^2, \\
B &= 1024(128\sigma_{4,2} + (-144\sigma_{4,1}^2 + 384\sigma_{4,1} - 256)\sigma_{4,2} - 512\sigma_{4,4} + 27\sigma_{4,1}^4 \\
&\quad -576\sigma_{4,1}^2 + 1280\sigma_{4,1} - 768), \\
C &= -(32\sigma_{4,2} - 9\sigma_{4,1}^2 - 24\sigma_{4,1} + 48)^3.
\end{aligned}$$

Here, we will make sure that the above equation (7) have solution(s) c_2 in the cases of $A \neq 0$ or $B \neq 0$. Now we note that $C = (32V)^3 \neq 0$.

1. If $A \neq 0$ or $B \neq 0$ then the equation (7) has solution(s) c_2. Substituting these c_2 to (3), c_0 is also obtained. The parameter c_1 depends only on $\sigma_{4,1}$.

2. If $A = 0$ and $B = 0$, then we have $\sigma_{4,1} = 4$ and $\sigma_{4,4} = (\sigma_{4,2}^2 - 8\sigma_{4,2} + 16)/4$. Now, suppose the equation (7) has solution(s) c_2. Substituting above two conditions into the transformation formula, we have a relation $4c_0 - c_2^2 = 0$. As this relation is a factor of the left hand of the equation (5), it contradicts to the condition $C \neq 0$.

   Therefore there is not a solution c_2 satisfying the equation (7).

   We remark that if C is also 0 (that is $(\sigma_{4,1}, \sigma_{4,2}, \sigma_{4,4}) = (4, 6, 1)$) then there are infinitely many inverse images $(c_0, c_1, c_2) = (c_2^2/4, c_1, c_2)$. However, in this case, we mention again $V = 0$.

Therefore the equation (7) always has solution(s) c_2, except for $(\sigma_{4,1}, \sigma_{4,2}, \sigma_{4,4}) = (4, s, s^2/4 - 2s + 4),\ s \neq 6$. If there is solution(s) c_2, substituting these c_2 to (3), c_0 is also obtained. The parameter c_1 depends only on $\sigma_{4,1}$. ∎

Making use of this proof, we prove Theorem 1 as below.

Proof of Theorem 1 If $V = 0$, then $c_2 = 0$ or $4c_0 - c_2^2 = 0$.

- In the case of $c_2 = 0$ and $4c_0 - c_2^2 = 0$:

  The points $(0, c_1, 0)$ correspond with the symmetry locus on $\Sigma(4)$ and the number of the inverse image is one. Hence these points (symmetry locus) belong to the branch locus and it is already known that the symmetry locus is a proper subspace of ENV.

- In the case that one of c_2 or $4c_0 - c_2^2$ is equal to zero:

  1. In the case of $c_2 = 0$ and $4c_0 - c_2^2 \neq 0$:
     We have $c_1 = (12 - \sigma_{4,1})/8$ and c_0 is a root of the equation

     $$1048576c_0^3 - 4096\sigma_{4,4} - 27\sigma_{4,1}^4 + 432\sigma_{4,1}^3 - 1440\sigma_{4,1}^2 + 1792\sigma_{4,1} - 768 = 0.$$

     The above equation have three roots $c_0 = k, k\omega, k\omega^2$, however, these three maps $(c_0, c_1, c_2) \in \mathcal{P}_1(4)$ belong to same conjugacy class.
  2. In the case of $c_2 \neq 0$ and $4c_0 - c_2^2 = 0$:
     The one parameter family $\{(c_2^2/4, 1, c_2)\}_{c_2}$ corresponds to one point $(4, 6, 1) \in \Sigma(4)$. Only on this point, there are infinitely many inverse images.
     For the other points $(c_2^2/4, c_1, c_2)$, we know that there is only one inverse image (conjugacy class) by using the same argument as above case 1.

  Putting together above two cases, there are two inverse images except for the point $(4, 6, 1)$. The point $(4, 6, 1)$ belongs to the symmetry locus (of course it belongs to the ENV). Although this point does not belong to the "branch locus", we treat this point is an element of the branch locus in meaning that the number of inverse images is not two.

On the other hand, if $V \neq 0$ then the equation $Ac_2^6 + Bc_2^3 + C = 0$ is obtained from the inverse problem. This equation has multiple roots if and only if $A = 0$ or discriminant $= 0$. $A = 0$ means $\sigma_{4,1} = 4$ and the discriminant $= 0$ coincides with the defining equation ENV.

At last, we note that the exceptional curve $\mathcal{E}$ is included in the plane $\sigma_{4,1} = 4$. Therefore there are two inverse images except for $\sigma_{4,1} = 0$ or on ENV. ∎

4 Real moduli space

4.1 coordinates of real moduli space

Let $\mathrm{Poly}_4(\mathbf{R})$ be the set of real polynomials of degree 4. Then it is easily shown that the parameters $\sigma_{4,i}$ $(1 \leq i \leq 4)$ are all real. But "real inverse problem" is not so easy.

Now we discuss the following real inverse problem for a while: *for any* $(\sigma_{4,1}, \sigma_{4,2}, \sigma_{4,4}) \in \mathbf{R}^3$ *given, whether there exists* $(c_0, c_1, c_2) \in \mathbf{R}^3$ *satisfying the transformation formula or not.*

Fix any $(\sigma_{4,1}, \sigma_{4,2}, \sigma_{4,4}) \in \mathbf{R}^3$. For the case $V = 0$ it is clear from a proof of inverse problem that there exists suitable $(c_0, c_1, c_2) \in \mathbf{R}^3$.

In the case of $V \neq 0$, put $c_2^3 = t$. If the discriminant $D = B^2 - 4AC$ of the quadratic equation (7) of t variable is negative, then any root is not real number.

Here, the discriminant D is as follows:

$$D = 54\sigma_{4,1}^5 - 27(3\sigma_{4,2} + \sigma_{4,4} + 5)\sigma_{4,1}^4 + 36(\sigma_{4,2}^2 - 4\sigma_{4,2} - 28)\sigma_{4,1}^3 + 4(-\sigma_{4,2}^3 + 90\sigma_{4,2}^2 + (36\sigma_{4,4} + 744)\sigma_{4,2} + 144\sigma_{4,4} + 1048)\sigma_{4,1}^2 + 32(-5\sigma_{4,2}^3 - 68\sigma_{4,2}^2 + (-12\sigma_{4,4} - 200)\sigma_{4,2} - 40\sigma_{4,4} - 168)\sigma_{4,1} + 16(\sigma_{4,2}^4 + 28\sigma_{4,2}^3 + (-8\sigma_{4,4} + 136)\sigma_{4,2}^2 + (16\sigma_{4,4} + 240)\sigma_{4,2} + 16\sigma_{4,4}^2 + 48\sigma_{4,4} + 144).$$

Therefore, for $\sigma_{4,1} \ll -1$ this discriminant is negative and $c_2 \in \mathbf{C} \setminus \mathbf{R}$. Hence we conclude that for suitable $(\sigma_{4,1}, \sigma_{4,2}, \sigma_{4,4}) \in \mathbf{R}^3$, we can not find a real polynomial corresponding to this coordinate. Precise arguments are written in [NF97a].

Under the conjugacy of the action $\mathfrak{A}(\mathbf{R})$, it can be assumed any map in $\mathrm{Poly}_4(\mathbf{R})$ is in the suitable branch of the real part of $\mathcal{P}_1(4)$. Note that this correspondence makes bijective map. Hence $\mathrm{M}_4(\mathbf{R}) \simeq \Re\{\mathcal{P}_1(4)\} \simeq \mathbf{R}^3$.

From now on, to carry out topological partition, we use the real part of $\mathcal{P}_1(4)$, denoted by $\Re\mathcal{P}_1(4)$, and the real (c_0, c_1, c_2)-space.

4.2 Topological Partition

At first, we will divide real (c_0, c_1, c_2)-space into two parts; the maps with three real critical points and the maps with one real critical point and a pair of complex conjugate critical point.

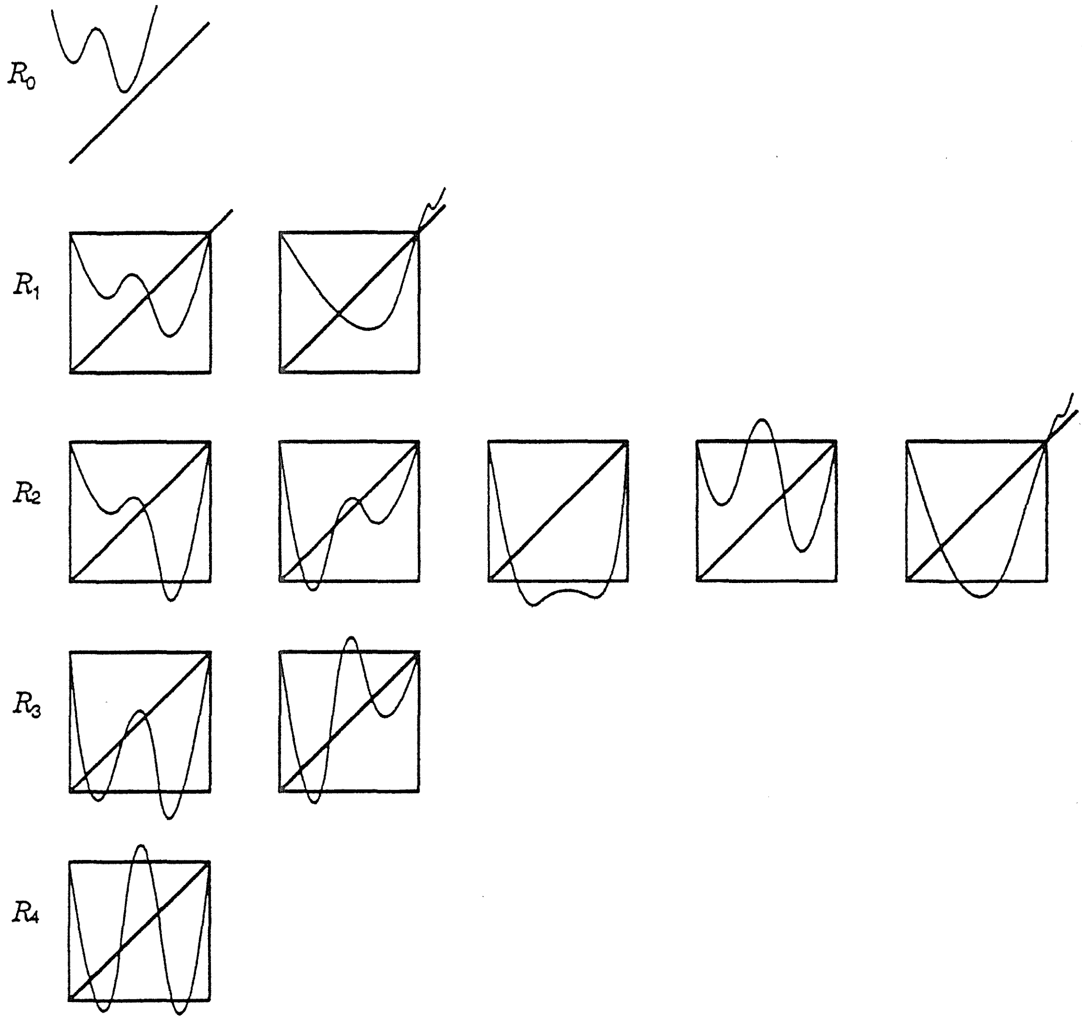

Figure 1: $\mathcal{R}_0$ to $\mathcal{R}_4$, for the case of $\mathrm{Poly}_4(\mathbf{R})$.

Let $p(x)$ be a monic and centered polynomial of degree 4 with real coefficients, i.e., $p(x) = x^4 + c_2x^2 + c_1x + c_0$. The discriminant of the equation $p'(x) = 4x^3 + 2c_2x + c_0 = 0$ is given by $D = -(c_2^3 + \frac{27}{16}c_1^2)$. Hence, a map $p(x)$ have a pair of complex conjugate critical points if and only if $p(x)$ in the region $\{(c_0, c_1, c_2); c_2^3 + \frac{27}{16}c_1^2 > 0\}$.

Next, we give a topological partition on this space. For map $p \in \Re\mathcal{P}_1(4)$, if the real filled-in Julia set of p is a single point then it is said that p in the class $\mathcal{R}_0$. Let J be the smallest closed interval which contains the real filled-in Julia set of p. For $p \notin \mathcal{R}_0$, it is said that p belongs to the class $\mathcal{R}_n$ if the graph of p intersected with $J \times J$ has n distinct components [Mil92]. In this case, $0 \leq n \leq 4$.

The boundary curves which give the above partitions are as follows:

- $Per_1(1)$: $\{-16c_0c_2^4 + (4c_1^2 - 8c_1 + 4)c_2^3 + 128c_0^2c_2^2 + (-144c_0c_1^2 + 288c_0c_1 - 144c_0)c_2 + 27c_1^4 - 108c_1^3 + 162c_1^2 - 108c_1 - 256c_0^3 + 27 = 0\}$
- $Preper_{(1)}1$: $\{-256c_2^9 - 256c_0^2c_2^8 + (128c_0c_1^2 + 256c_0c_1 + 4096c_0)c_2^7 + (-16c_1^4 - 64c_1^3 - 3776c_1^2 - 7168c_1 + 4096c_0^3 - 4096)c_2^6 + (-5632c_0^2c_1^2 - 11264c_0^2c_1 - 28672c_0^2)c_2^5 + (2016c_0c_1^4 + 8064c_0c_1^3 + 38912c_0c_1^2 + 57344c_0c_1 - 24576c_0^4 + 32768c_0)c_2^4 + (-216c_1^6 - 1296c_1^5 - 17856c_1^4 - 59648c_1^3 + (38912c_0^3 - 49152)c_1^2 + (77824c_0^3 - 24576)c_1 + 98304c_0^3 - 16384)c_2^3 + (-27648c_0^2c_1^4 - 110592c_0^2c_1^3 - 175104c_0^2c_1^2 - 139264c_0^2c_1 + 65536c_0^5 - 81920c_0^2)c_2^2 + (7776c_0c_1^6 + 46656c_0c_1^5 + 96768c_0c_1^4 + 73728c_0c_1^3 + (-73728c_0^4 - 147456c_0)c_1^2 + (-147456c_0^4 - 147456c_0)c_1 - 131072c_0^4)c_2 - 729c_1^8 - 5832c_1^7 - 27216c_1^6 - 76032c_1^5 + (13824c_0^3 - 145152)c_1^4 + (55296c_0^3 - 165888)c_1^3 + (-73728c_0^3 - 110592)c_1^2 - 163840c_0^3c_1 - 65536c_0^6 = 0\}$
- $Preper_{(2)}1$: the degree of this defining equation is 33 with respect to c_0, 44 with respect to c_1, and 47 with respect to c_2.

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非線形解析学と凸解析学の研究

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Both ways pitch-fork bifurcation diagrams of families of one-dimensional maps

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1　Introduction

パラメターの変化で非線形関数族の力学系の複雑さを測ろうとするときに、その変化に伴う周期軌道の生成と消滅は最も基本的な分岐過程であって、この系の複雑度を示している。特に吸引的周期軌道の変化の様子はしばしば一方方向の、または両方向の熊手分岐図として描かれる。

特に, メイによる個体生物学で有名な例であるロジスチック関数族 $\{f_\lambda(x) = \lambda x(1-x)\ ;\ \lambda \in [1,4]\}$ は最も簡単な非線形写像でありながら、大変複雑な力学系である。すなわち周期倍分岐カスケイドがカオスにいたる一般的なルートの一つであることを認識させた例であり、物理学、化学、生物学に於いて実区間離散力学系はカオス的挙動を持つモデルとし重要であることは知られている。

しかし、一般には周期軌道の分岐はロジスチック関数族の力学が示す分岐よりははるかに複雑である。すなわちパラメター λ が単調に増加するとき、ロジスチック関数族は、周期軌道が生成されることがあっても、決して消滅はしないという単調性を有するが、非線形力学系の多くはこの性質を持たない。さらにあるパラメターの近傍では周期軌道の生成と消滅が無限回おこりることさえある。

ここでは、こうした１パラメター非線形関数族の非単調な分岐現象を論じる。

はじめに分岐に関する幾つかの定義をする：

１パラメター非線形関数族を $\{f_\lambda\}_\Lambda$ とする。パラメター λ が単調に変化するとき、分岐値 λ_0 が **orbit creating** と呼ばれるのは、λ_0 で新たに周期点が生じ、存在していた周期点は消滅しないときで、**orbit annihilating** と呼ばれるのは、λ_0で存在していた周期点は消滅し、新たに周期点が生じないときとする。さらに **neutral** とは周期点の誕生も消滅も起こらないときとする。族 $\{f_\lambda\}_\Lambda$ が **monotone increasing (resp. decreasing)** であるとは、すべての分岐値が neutral か orbit creating (resp. annihilating) のとき

で、**non-monotone** であるとは orbit creating と orbit annihilating な分岐値を持つときとする。

パラメター λ が単調に変化するとき、分岐値 λ_0 が **anti-monotone** と呼ばれるのは、λ_0 の任意の近傍に無限個の orbit creating と orbit annihilating が存在するときで、族がこのような anti-monotone 分岐値を一つでももてば、この族を **anti-monotone** と呼ぶ。

anti-monotonicity パラメター値 については、dissipative planar diffeomorphisms の 1 パラメター系において研究がなされている。その系では homclinic 点 および heteroclinic 点の挙動が本質的である。すなわち Newhause & Robinson ([New79], [Rob83]) によってある種の非退化性の仮定のもとに、任意の homoclinic-tangency パラメター値の任意の近傍には homoclinic-tangency パラメター値からなる dense な集合を部分集合としてもつパラメター値の区間が存在する事が示されている。この区間を **Newhause interval** と呼ぶ。続いて、Kan,Kocak, and York ([KKY92]) によって homoclinic-tangency パラメター値の近傍では anti-monotone 分岐が生じているという次の定理が得られた：

Antimonotonicity Theorem

In any neighborhood of a nondegenerate, homoclinic-tangency parameter value of a one-parameter family of dissipative C^3 diffeomorphisms of the plane, there must be both infinitely many orbit-creation and infinitely many orbit-annihilation parameter values.

この族の分岐図は反対向きのの熊手分岐図となる。具体的な例としてはヘノン写像族か知られている。

$$H_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda - x^2 + by \\ x \end{pmatrix}$$

1 次元写像族 $\{f_\lambda\}$ に対しての anti-monotone 分岐に関しては Jonassenn ([Jona93]) の方法がある: $\beta-$ lift という 2 次元写像

$$F_{\beta,\lambda} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_\lambda(x) + \beta y \\ x \end{pmatrix} \quad 0 < |\beta| < 1$$

を作る。このとき $\{f_\lambda\}$ の non-degenerate homoclinic orbit は丁度$\beta-$lift $F_{\beta,\lambda}$ の transversal homoclinic point に対応している。$\beta-$ lift は transversal homoclinic point を含む chaotic inveriant set を持つ。

十分に小さな $\beta = \beta_0$ を固定して

$$F_{\beta_0,\lambda} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_\lambda(x) + \beta_0 y \\ x \end{pmatrix}$$

とすると $\{F_{\beta_0,\lambda}\}$ は non-degenerate, homoclinic-tangency parameter を有する。従って、Antimonotonicity Theorem によって、この族は anti-monotone である。また anti-monotone を保証する Newhouse interval が存在する。$\beta \to 0$ とするとき、この区間の幅が 0 にならなければ $\{f_\lambda\}$ は anti-monotone である。しかしこの極限操作で Newhouse interval が いつ生き残るかについての $\{f_\lambda\}$ に対する条件は求まっていない。

Milnor and Thurston , Doudy and Hubbard ([MT88],[DH85]) は Teichmüller 空間の理論を使って、ロジスチック関数族 $\{f_\lambda(x) = \lambda x(1-x)\ ;\ \lambda \in [1,4]\}$ は monotone increasing であることを示した（図 1 を参照）。しかしロジスチック関数族の分岐の単調性は, 1 パラメター非線形関数族の中ではむしろ特異であって、多くの非線形関数族は非単調な分岐過程をもつ。ここでパラメターの入れ方がロジスチック関数族のようにある特定の関数 fを固定して $\{mf(x)\}_{m\in\mathbf{R}}$ と限ったものに対してはどうかという問題が生じてくる。すなわち、固定される関数に対してどのような制限があれば分岐の単調性が保証されるか？である。この問題に関して多くの研究が為されている。現在 fに対しては単峰でありシュワルツ微分が負である条件の基で、単調な分岐をを持つ 1 パラメター関数族 $\{mf(x)\}_m$の特徴付けが研究されている。分岐の単調性はまたこの族の位相的エントロピーの単調性をも示している。

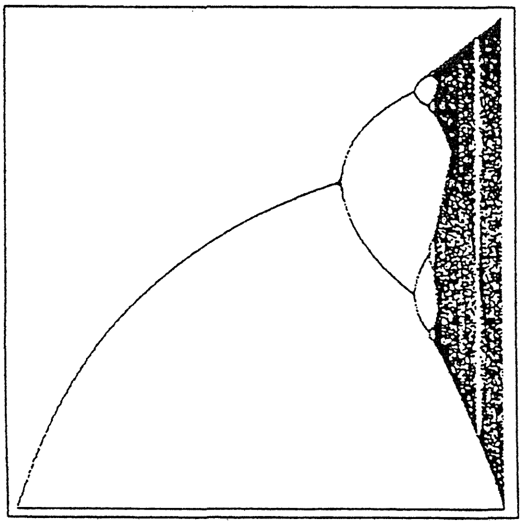

Figure 1: One-way pitchfork in the logistic family. $\{\lambda x(1-x)\}, (1 \le m \le 4)$.

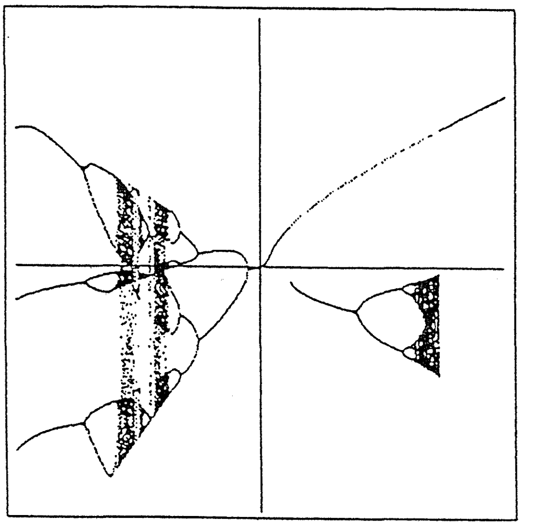

Figure 2: Both-ways pitchforks in quadratic rational maps. $\{m(\frac{1}{5} + \frac{x}{1+x^2})\}$.

ここでは 2 次有理関数 $f(x)$ に対しての 1 パラメター関数族 $\{mf(x)\}_m$ の分岐現象に付いて述べる。すなわち fの選び方によって様々な分岐が生ずる ([FN95]) （図 2 を参照）。この分岐の多様性を説明するためにとる我々の基本的な方法は実 2 次有理関

数族のモヂュライ空間でこの関数族 $\{mf(x)\}_m$ が定義する代数曲線γ_f（定理１）の性質を調べることに帰着させる。具体的には、モヂュライ空間に適当な座標をいれ、関数の力学的挙動によってこの空間を分割し、双曲成分の配置と分岐現象を引き起こす特定の代数曲線群を定義し、つぎに曲線γ_fがどの双曲成分を通過するか、またどの特定の代数曲線群と交差するかを調べることである。

特に anti-monotone 分岐については$\beta-$ lift をつかわずに代数曲線の方法で例のみを示す。

Section 2 で、複素２次有理関数族のモヂュライ空間、双曲成分の配置、分岐現象を引き起こす特定の代数曲線群の定義などを J.MILNOR([Mil92]) から引用し、さらに実２次有理関数族のモヂュライ空間中に目的に会わせた領域と直線族を用意する。

Section 3 で分岐問題に対する議論を具体的な関数族 $\{m(r+\frac{x}{1+x^2})\}_m$ （ r fixed）に対して展開する。これは H. E. Nusse and J. A. Yorke の論文 (p.329 in [NY88]) の次の部分に触発されたためである。

> *" If it is written in our form, i.e.,* $m[b_0+a_0\frac{x}{1+x^2}]$*, by fixing the ratio of a and b, it apparently does not exhibit period-halving bifurcation as the parameter* m *is increased."*

この記述にはかなりの曖昧さがある。すなわち比の取り方によって、この関数族は様々な分岐過程を生じる。たとえば比が 0.58 の１パラメター関数族 $\{m(0.58+\frac{x}{1+x^2})\}$ では非単調な分岐過程、さらに詳しくはバブリング、を持つ（例１）。さらに定理２として

" もし $r \geq \frac{3\sqrt{3}}{8}$ であれば、rを固定したときこの関数族 $\{m(r+\frac{x}{1+x^2})\}$ は　どのような分岐も起こさない、すなわち, この関数族はパラメター m がどのように変化しても常に唯一つの吸引的不動点をもち、他のいかなる周期点も持たない。さらにこの値 $\frac{3\sqrt{3}}{8}$ は次の意味で最良のものである：$r<\frac{3\sqrt{3}}{8}$ であれば 関数族 $\{m(r+\frac{x}{1+x^2})\}$ は必ず周期倍分岐を引き起こす "

ことを示す。さらに $r=0$ のときの関数族 $\{m(\frac{x}{1+x^2})\}$ は、複素有理関数族とみなすことによって、非自明な自己同型群を持つ関数の共役類からなるモヂュライ空間の特異集合 (Symmnetry Locus) の normal form を与えてて、対応する代数曲線がこの特異集合の定義方程式である。

2　2次有理関数族

2.1　複素2次有理関数族のモヂュライ空間

$\overline{\mathbf{C}}$ を Riemann sphere とする。$\mathrm{Rat}_2(\mathbf{C})$ を$\overline{\mathbf{C}}$から$\overline{\mathbf{C}}$への2次有理関数族とする。群 $\mathrm{PSL}_2(\mathbf{C})$ が 空間 $\mathrm{Rat}_2(\mathbf{C})$ 上に次のように作用する：

$$g \circ f \circ g^{-1} \in \mathrm{Rat}_2(\mathbf{C}) \quad \text{for} \quad g \in \mathrm{PSL}_2(\mathbf{C}),\ f \in \mathrm{Rat}_2(\mathbf{C}).$$

2つの関数 $f_1, f_2 \in \mathrm{Rat}_2(\mathbf{C})$ が **holomorphically conjugate** であるとは $g \circ f_1 \circ g^{-1} = f_2$となる $g \in \mathrm{PSL}_2(\mathbf{C})$ があるときとする。これを $f_1 \sim f_2$と書く。この作用による $\mathrm{Rat}_2(\mathbf{C})$ の商空間を $\mathcal{M}_2(\mathbf{C})$ とし、2次有理関数 fの正則共役類 $\langle f \rangle$ の **moduli space** とする。$f \in \mathrm{Rat}_2(\mathbf{C})$ の不動点を z_1, z_2, z_3 とし、その固有値を $\mu_i = f'(z_i)$ とする。また$\sigma_1 = \mu_1 + \mu_2 + \mu_3$,　$\sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1$,　$\sigma_3 = \mu_1\mu_2\mu_3$. とする。このとき $\sigma_3 = \sigma_1 - 2$ と言う関係が得られ、Milnor(Lemma 3.1 in [Mil92]) によって $\mathcal{M}_2(\mathbf{C})$ に座標 $(\sigma_1\ \sigma_2)$ が入り、$\mathcal{M}_2(\mathbf{C})$ と $\mathbf{C}^2$は同一視される。また $\mathcal{M}_2(\mathbf{C})$ の自然なコンパクト化として 射影空間 $\mathbf{CP}^2$を考える。無限遠での ideal point は2次有理関数の極限関数で、1次分数関数または定数に退化したものが対応しているとみなすことが出来る。

fの自己同型群 $\mathrm{Aut}(f) \subset \mathrm{PSL}_2(\mathbf{C})$ とは $g \circ f \circ g^{-1} = f$ をみたす $g \in \mathrm{PSL}_2(\mathbf{C})$ の集合とする。有限群である。$\tilde{f} \in \langle f \rangle$ であるような$\tilde{f}$に対して $\mathrm{Aut}(\tilde{f})$ と $\mathrm{Aut}(f)$ とは同型であることに注意すれば、自明でない自己同型群をもつ共役類 $\langle f \rangle$ が意味を持ち、従ってその全体を　bf symmetry locus と呼び、$\mathcal{S}(\subset \mathcal{M}_2(\mathbf{C}))$ と書く。

各 $\mu \in \mathbf{C}$ にたいして、$\mathrm{Per}_n(\mu) \subset \mathcal{M}_2(\mathbf{C})$ とは固有値がμである周期 n の周期点をもつ関数 fの共役類 $\langle f \rangle$ の全体とする。すなわち

$$\mathrm{Per}_n(\mu) = \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); f^n(z) = z,\ (f^n)'(z) = \mu \right\}.$$

特に周期1又は2のとき $\mathrm{Per}_1(\mu), \mathrm{Per}_2(\mu)$ は直線束である (Lemma 3.4 and Lemma 3.6 in [Mil92]):

$$\begin{aligned}
\mathrm{Per}_1(\mu) &= \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1}) \right\} \\
\mathrm{Per}_2(\mu) &= \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = -2\sigma_1 + \mu \right\}.
\end{aligned}$$

Symmetry locus $\mathcal{S}$ は既約3次代数曲線である。その定義方程式は以下で与えられる ([FN98])：

$$S(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \tag{1}$$

有理関数が双曲的であるとは各 critical orbit が ある吸引周期軌道に収束することである。双曲的な2次有理関数はモヂュライ空間で開集合であって、その連結成分は

双曲的成分 と呼ばれる。M. Rees ([Ree90]) によれば 双曲的成分 は 4 クラスに分けられる：

Type B: Bitransitive. Each of the two critical points belongs to the immediate basin of some attracting periodic point, where these two periodic points are distinct but belong to the some orbit. Evidently the period must be two or more.

Type C: Capture. Only one critical point belongs to the immediate basin on a periodic point, but the orbit of the other critical point eventually falls into this immediate basin. Again the period must be two or more.

Type D: Disjoint attractors. The two critical points belong to the attracting basins for two disjoint attracting periodic orbits.

Type E: Escape. Both critical orbits converge to the same attracting fixed point.

Type E(Escape) 双曲的成分はただ 1 つ存在することが知られている。

2.2 実モヂュライ空間

実係数 2 次有理関数族 $\mathrm{Rat}_2(\mathbf{R})$ に対して 3 個の固定点と対応する固有値は全てが実数か、1 個が実で残りの 2 個は互いに複素共役である。従ってσ_i $(1 \leq i \leq 3)$ はやはり実数が対応する。J.Milnor に従って $\mathrm{Rat}_2(\mathbf{R})$ の実モヂュライ空間 $\mathcal{M}_2(\mathbf{R})$ を単に実 $(\sigma_1,\ \sigma_2)$−平面と定義する。$\mathcal{M}_2(\mathbf{R})$ には次のような "特異点" が存在することに注意しておく：いま $\mathcal{S}_{\mathbf{R}} = \mathcal{S} \cap \mathcal{M}_2(\mathbf{R})$, とし、$< >_{\mathbf{R}}$ で、$\mathrm{Rat}_2(\mathbf{R})$ の関数の $\mathrm{PSL}_2(\mathbf{R})$ での実共役類を表せば $(\mathrm{Rat}_2(\mathbf{R})/\mathrm{PSL}_2(\mathbf{R})) \setminus \{\left\langle a(x+\frac{1}{x})\right\rangle_{\mathbf{R}},\ \left\langle a(x-\frac{1}{x})\right\rangle_{\mathbf{R}}\}_{a\in\mathbf{R}^\times}$ は $\mathbf{R}^2 \setminus \mathcal{S}_{\mathbf{R}}$, と同型であり、一方 $\{\left\langle a(x\pm\frac{1}{x})\right\rangle_{\mathbf{R}}\}_{a\in\mathbf{R}^\times}$ と $\mathcal{S}_{\mathbf{R}}$ のあいだには 2 ：1 の対応がある。

以後 $\mathcal{M}_2(\mathbf{R})$ を $\mathbf{R}^2$ と同一視する。

次に [Mil92] から、$\mathcal{M}_2(\mathbf{R})$ での双曲的成分 **B**, **C**, **D**, **E** の配置を引用し、これらの事実に双曲成分 (escape locus) についての考察を加え、分岐に付いての議論のために幾つかの領域を用意する。

2.2.1 双曲的成分 (Type D) の実部

相異なる 2 個の吸引固定点を持つ関数の共役類からなる双曲的成分 (Type D) $D_{1,1}$ の実部は

$$D_{1,1} = \{(\sigma_1, \sigma_2); -2\sigma_1 + 1 < \sigma_2 < 2\sigma_1 - 3\}.$$

また双曲的 成分 (Type D) $D_{1,2}$の実部は

$$D_{1,2} = \{(\sigma_1, \sigma_2); -2\sigma_1 - 1 < \sigma_2 < -2\sigma_1 + 1, \sigma_2 < 2\sigma_1 - 3\}.$$

その他の disjoint 成分の実部は半平面 $\{\sigma_1 < -6\}$ に存在する (P.42 of [Mil92])。

2.2.2 双曲的成分 (Type E) の実部

$\mathcal{M}_2(\mathbf{C})$ では双曲的成分 (Type E) は連結であるが、その実部は 2 つの部分に分かれる：その上部を E_1、下部を E_2とする。E_1に属する関数は吸引固定点以外にはどんな実周期点をも持ち得ないことが知られている (Caption of Figure 16 in [Mil92])。この事実は後の我々の議論での最も重要なことである。

双曲的成分 (Type E) の実部 $E = E_1 \cup E_2^{(1)} \cup E_2^{(2)}$ の境界は次の曲線で与えられる (

2.2.3 双曲的成分 (Type B) の実部

Per_2-bend として次の領域を定義する：

$$\{(\sigma_1, \sigma_2); -2\sigma_1 - 1 < \sigma_2 < -2\sigma_1 + 1,\ \sigma_1 > -6,\ \sigma_2 > 2\sigma_1 - 3\}.$$

Per_2-bend に属する関数は type B または type C である。従って 2 つの critical orbits は終局的に同じものである。

2.2.4 3 本の分岐実直線

$n = 1,\ 2$ に対して $\mathrm{Per}_n(\mu)$ は直線であり、さらに$\mu = 1$ のときには $\mathrm{Per}_2(1) = \mathrm{Per}_1(-1)$ である。後の議論のために次の 3 本の実直線を分岐実直線と呼ぶ：

$$\mathrm{Per}_1(1) \ :\ \sigma_2 = 2\sigma_1 - 3 \tag{2}$$
$$\mathrm{Per}_1(-1) \ :\ \sigma_2 = -2\sigma_1 + 1 \tag{3}$$
$$\mathrm{Per}_2(-1) \ :\ \sigma_2 = -2\sigma_1 - 1 \tag{4}$$

分岐実直線は双曲成分 E_1, $D_{1,1}$,$D_{1,2}$及び Per_2-bend の境界である。

3 非単調な分岐過程をもつ 2 次有理関数族

ここでは M. Bier and T. C. Bountis [BB84] によって与えられ、H. E. Nusse and J. A. Yorke ([NY88]) によって書き改められた次の 2 パラメター数 2 次有理関数族の分岐について述べる：

$$\left\{ f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}.$$

ここで $m=0$ の場合に $f_{0,r}(x)$ は、コンパクト化されたモヂュライ空間の中での有理関数の極限としての定数関数として意味を持つことに注意する。また $f_{m,r}$ と $f_{m,-r}$ とは共役であることから、以下では $0 \leq r$ の場合のみ考えれば良い。

Schwarzian derivative: $Sf = f'''(x)/f'(x) - \frac{3}{2}\left(f''(x)/f'(x)\right)^2$ の符号は局所的な分岐のタイプを決めることが知られている。$f_{m,r}$ の Schwarzian derivative は常に負なので、この族には regular period-doubling または regular period-halving bifurcations しか生じない。

定理 1 *(see [FN98])* パラメター r を固定して得られる 1 パラメター関数族 $\{f_{m,r}(x)\}_m$ は、モヂュライ空間 $\mathcal{M}_2(\mathbf{R})$ で、既約代数曲線 $\mathcal{H}_r$ 上の全ての点と一致する。$r(\neq \frac{1}{2}, 0)$ に対しては次の方程式が対応する既約 4 次代数曲線 $\mathcal{H}_r$ の定義方程式である：

$$\begin{aligned} H_r(\sigma_1,\sigma_2) &= -r^2\sigma_1^4 + (8r^2-2)\sigma_1^3 + ((8r^2-1)\sigma_2 - 128r^4 + 8r^2 + 1)\sigma_1^2 \\ &\quad +((-32r^2+8)\sigma_2 + 512r^4 - 96r^2 - 12)\sigma_1 + (-16r^2+4)\sigma_2^2 \\ &\quad +(512r^4 - 96r^2 - 12)\sigma_2 - 4096r^6 + 1536r^4 - 144r^2 + 36 = 0. \quad (5) \end{aligned}$$

$r = \frac{1}{2}$のときには 3 次代数曲線 $\mathcal{H}_{1/2}$：

$$H_{\frac{1}{2}}(\sigma_1,\sigma_2) = -\sigma_1^3 - 2\sigma_1^2 + (4\sigma_2 - 24)\sigma_1 + 8\sigma_2 - 64 = 0. \quad (6)$$

$r=0$ のときには $\mathcal{H}_0$:

$$H_0(\sigma_1,\sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \quad (7)$$

がそれぞれ対応する。

$r=0$ のときについて：このとき曲線 $\mathcal{H}_0$は、

$$f_{m,0}(x) = m\left(\frac{x}{1+x^2}\right) = m\left(\frac{1}{x+\frac{1}{x}}\right) \sim \frac{1}{m}\left(x + \frac{1}{x}\right),$$

に注意すれば J. Milnor([Mil92]) による次の結果とリンクする：symmetry locas $\mathcal{S}$ の関数は次のようなパラメター表示を持つ

$$\mathcal{S}=\{\left\langle k\left(z+\frac{1}{z}\right)\right\rangle ; k\in\mathbf{C}\setminus\{0\}\}.$$

我々のモデルの変数 r, m, σ_1, σ_2 を $\mathbf{R}$ から $\mathbf{C}$ に拡張すれば $\mathcal{S}=\{\langle f_{m,0}\rangle ; m\in\mathbf{C}\setminus\{0\}\}$. 従って次の結果を得る。

系 1 *(see [FN98])* 曲線 $\mathcal{H}_0$ は $\mathcal{M}_2(\mathbf{C})$ で *symmetry locus* $\mathcal{S}$ と一致する。

以下では r を固定したときの関数族 $\{f_{m,r}\}_m$ の分岐ダイヤグラムを調べる：パラメター m が単調に変化するとき代数曲線 $\mathcal{H}_r(\sigma_1,\sigma_2)=0$ がモヂュライ空間のどの部分を通過するかを調べることで、関数族

$$\left\{f_{m,r}(x)=m\left(r+\frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}$$

の分岐を決定する。

例 1 (see [FN98]) $r=0.58$ とする。$\{f_{m,0.58}\}$ は period-doubling と period- -halving bifurcations をともにもつ。図 3に 3 分岐直線とともに代数曲線 $H_{0.58}=0$ を、また図 4に対応する関数族の非単調分岐ダイヤグラムを示した。この分岐はさらに詳しく言えばいわゆる primary bubbling bifurcation ([BB84]) である。

定理 2 *(see [FN98])* r を範囲 $\frac{3\sqrt{3}}{8}\leq r$ にとり固定すれば、$\{f_{m,r}\}$ には如何なる分岐も生じない。さらにこの値 $\frac{3\sqrt{3}}{8}$ は次の意味で最良のものである：$r<\frac{3\sqrt{3}}{8}$ であれば関数族 $\{m(r+\frac{x}{1+x^2})\}$ は必ず周期倍分岐を引き起こす。

例 2 最後に anti-monotone 分岐を生じている族 $r=0.54$ の場合の代数曲線（図 5）と、対応する関数族の分岐ダイヤグラム（図 6）を示しておく。

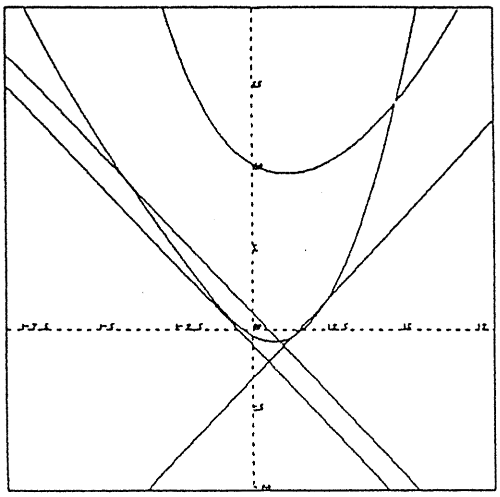

Figure 3: The curve $H_{0.58} = 0$ together with the bifurcation lines. $-8 < \sigma_1 < 8$, $-10 < \sigma_2 < 20$.

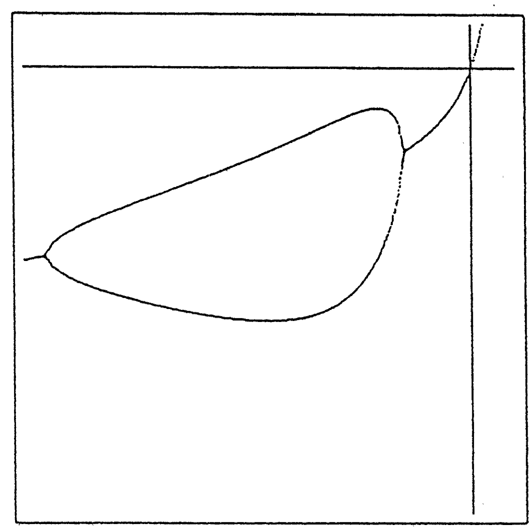

Figure 4: Period-bubbling bifurcation diagra m corresponding to $H_{0.58}$. $-10 \leq m \leq 1$, $-2 \leq x \leq 0.2$.

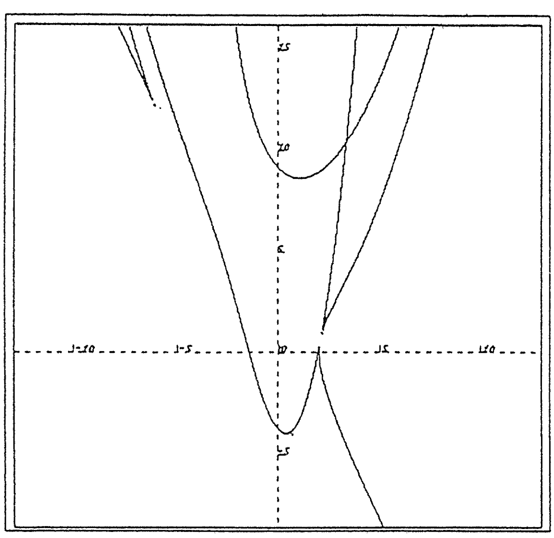

Figure 5: The curve $H_{0.54} = 0$ together with the bifurcation lines. $-8 < \sigma_1 < 8$, $-15 < \sigma_2 < 25$.

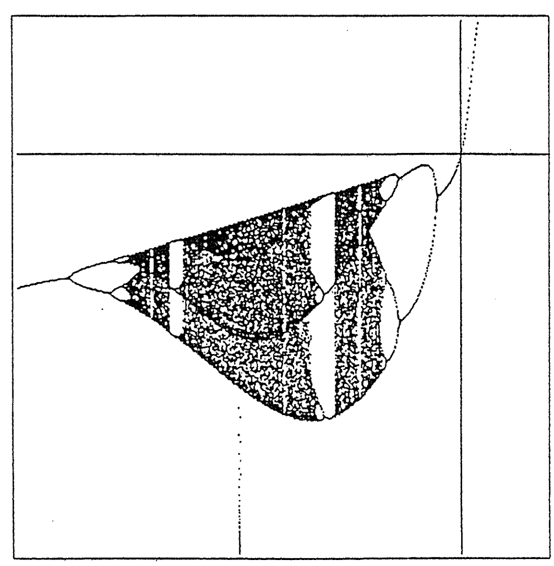

Figure 6: Bifurcation diagram corresponding to $H_{0.54}$. $-10 \leq x \leq 10$.

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調和・解析関数空間と線形作用素

京 都 大 学 数 理 解 析 研 究 所

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非単調な分岐過程をもつ1径数1次元関数族

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1 Introduction

パラメターの変化で非線形関数族の力学系の複雑さを測ろうとするときに、その変化に伴う周期軌道の生成と消滅は最も基本的な分岐過程であって、この系の複雑度を示している。特に吸引的周期軌道の変化の様子はしばしば一方方向の、または両方向の熊手分岐図として描かれる。

ロジスチック関数族 $\{f_\lambda(x) = \lambda x(1-x)\ ;\ \lambda \in [1,4]\}$ は最も簡単な非線形写像でありながら、その力学的挙動は複雑である。すなわち周期倍分岐カスケイドがカオスにいたる一般的なルートの一つであることを認識させた例である。しかし、この族の分岐はパラメター $\lambda \in \mathbb{R}$ が単調に増加するとき、周期軌道が生成されることがあっても、決して消滅はしないという単調性を有する。一般の非線形関数族では、周期軌道の分岐はロジスチック関数族の力学が示す分岐よりははるかに複雑で、この単調性という性質を持たない。さらにあるパラメターの近傍では周期軌道の生成と消滅が無限回生じることさえある。

ここでは、こうした1径数非線形関数族の非単調な分岐現象を論じる。

はじめに分岐に関する幾つかの定義をする。1径数非線形関数族を $\{f_\lambda\}_\Lambda$ とする。パラメター λ が単調に変化するとき、分岐値 λ_0 が **orbit creating** と呼ばれるのは、λ_0 で新たに周期点が生じ、存在していた周期点は消滅しないときで、**orbit annihilating** と呼ばれるのは、λ_0 で存在していた周期点は消滅し、新たに周期点が生じないときとする。さらに **neutral** とは周期点の誕生も消滅も起こらないときとする。族 $\{f_\lambda\}_\Lambda$ が **monotone increasing (resp. decreasing)** であるとは、すべての分岐値が neutral か orbit creating (resp. annihilating) のときで、**non-monotone** であるとは orbit creating と orbit annihilating な分岐値を同時に持つときとする。族 $\{f_\lambda\}_\Lambda$ が nonmonotone のときにはさらに細かくその分岐が区別されている。すなわち、first-bubble, second-bubble と呼ばれる安定な状態から、次に述べる antimonotone と呼ばれる複雑な分岐現象までさまざまな段階が生じている。

パラメター λ が単調に変化するとき、分岐値 λ_0 が **anti-monotone** と呼ばれるのは、λ_0 の任意の近傍に無限個の分岐値で **orbit creating** と **orbit annihilating** であるものが存在するときで、族がこのような **anti-monotone** 分岐値を一つでももてば、この族を **anti-monotone** と呼ぶ。antimonotone な族に対する議論を 3 節で行う。

Milnor and Thurston ([MT88],[DH85]) は Teichmüller 空間の理論を使って、ロジスチック関数族 $\{f_\lambda(x) = \lambda x(1-x)\ ;\ \lambda \in [1,4]\}$ は monotone increasing であることを示した。また区分的線形関数のテント写像族 $T_t(x) = t(1-|2x-1|)$ に対しても単調性が示されている。

しかしロジスチック関数族やテント写像族の分岐の単調性は，1 径数非線形関数族の中ではむしろ特異であって、この二つの関数族に十分近い族でも単調性を持つかどうかは解っていない。多くの非線形関数族は非単調な分岐過程をもつ。

従ってパラメターの入れ方がロジスチック関数族のようにある特定の関数 f を固定して $\{mf(x)\}_m,\ \ m \in \mathbb{R}$ と限ったものに対してはどうかという問題が生じてくる。すなわち、固定される関数に対してどのような制限があれば分岐の単調性が保証されるか？である。この問題に関して多くの研究が為されている。現在 f に対しては単峰でありシュワルツ微分が負である条件の基で、単調な分岐を持つ 1 パラメター関数族 $\{mf(x)\}_m$ の特徴付けが研究されている。分岐の単調性はまたこの族の位相的エントロピーの単調性をも示している。このタイプの族に対する分岐については 2 節で論ずる。

2　非単調な分岐過程をもつ 2 次有理関数族

我々は論文 [FN96], [Nis96] で、この分岐の単調性問題に対する議論を具体的な関数族 $\{m(r + \frac{x}{1+x^2})\}_m$ (r, fixed) に対して展開した。これは H. E. Nusse and J. A. Yorke の論文 (p.329 in [NY88]) の次の部分に触発されたためであった。

> *" If it is written in our form, i.e., $m[b_0 + a_0 \frac{x}{1+x^2}]$, by fixing the ratio of a and b, it apparently does not exhibit period-halving bifurcation as the parameter m is increased."*

この記述にはかなりの曖昧さがある。すなわち比の取り方によって、この関数族は様々な分岐過程を生じる。たとえば比が 0.58 のとき関数族 $\{m(0.58 + \frac{x}{1+x^2})\}$ は非単調な分岐過程、詳しくは first bubble を持つ。

関数族の分岐の多様性を説明するためにとる我々の基本的な方法 ([FN96],[FN97],[Nis96]) は実 2 次有理関数族のモヂュライ空間でこの関数族 $\{mf(x)\}_m$ が定義する代数曲線 γ_f の

性質を調べることに帰着させる。具体的には、モヂュライ空間に適当な座標 (σ_1, σ_2) をいれ、関数の力学的挙動によってこの空間を分割し、双曲成分の配置と分岐現象を引き起こす特定の代数曲線群を定義し、つぎに曲線 γ_f がどの双曲成分を通過するか、またどの特定の代数曲線群と交差するかを調べることである。

定理 1 　パラメター r を固定して得られる 1 パラメター関数族 $\{f_{m,r}(x)\}_m$ は、モヂュライ空間 $\mathcal{M}_2(\mathbb{R})$ で、既約代数曲線 $\mathcal{H}_r$ 上の全ての点と一致する。$r(\neq \frac{1}{2}, 0)$ に対しては次の方程式が対応する既約 4 次代数曲線 $\mathcal{H}_r$ の定義方程式である：

$$\begin{aligned} H_r(\sigma_1, \sigma_2) &= -r^2\sigma_1^4 + (8r^2-2)\sigma_1^3 + ((8r^2-1)\sigma_2 - 128r^4 + 8r^2 + 1)\sigma_1^2 \\ &\quad +((-32r^2+8)\sigma_2 + 512r^4 - 96r^2 - 12)\sigma_1 + (-16r^2+4)\sigma_2^2 \\ &\quad +(512r^4 - 96r^2 - 12)\sigma_2 - 4096r^6 + 1536r^4 - 144r^2 + 36 = 0. \qquad (1) \end{aligned}$$

$r = \frac{1}{2}$ のときには 3 次代数曲線 $\mathcal{H}_{1/2}$：

$$H_{\frac{1}{2}}(\sigma_1, \sigma_2) = -\sigma_1^3 - 2\sigma_1^2 + (4\sigma_2 - 24)\sigma_1 + 8\sigma_2 - 64 = 0. \qquad (2)$$

$r = 0$ のときには $\mathcal{H}_0$:

$$H_0(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \qquad (3)$$

がそれぞれ対応する。

ここで $r = 0$ のときの曲線 $\mathcal{H}_0$ は、

$$f_{m,0}(x) = m\left(\frac{x}{1+x^2}\right) = m\left(\frac{1}{x+\frac{1}{x}}\right) \sim \frac{1}{m}\left(x + \frac{1}{x}\right),$$

に注意すれば J. Milnor ([Mil92]) による次の結果とリンクする：モヂュライ空間の特異部分 $\mathcal{S}$ の関数は次のようなパラメター表示を持つ

$$\mathcal{S} = \{\left\langle k\left(z + \frac{1}{z}\right)\right\rangle; \;\; k \in \mathbb{C} \setminus \{0\}\}.$$

我々のモデルの変数 r, m, σ_1, σ_2 を $\mathbb{R}$ から $\mathbb{C}$ に拡張すれば $\mathcal{S} = \{\langle f_{m,0}\rangle \,; m \in \mathbb{C} \setminus \{0\}\ \}$ であって、曲線 $\mathcal{H}_0$ は $\mathcal{M}_2(\mathbb{C})$ で symmetry locus $\mathcal{S}$ と一致する。

パラメター m が単調に変化するとき代数曲線 $H_r(\sigma_1, \sigma_2) = 0$ がモヂュライ空間のどの部分を通過するかを調べることで、r を固定したときの関数族

$$\left\{f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbb{R}^2}$$

のおおまかな分岐が決まる。すなわち、

定理 2 　r を範囲 $\frac{3\sqrt{3}}{8} \leq r$ にとり固定すれば、$\{f_{m,r}\}$ には如何なる分岐も生じない。さらにこの値 $\frac{3\sqrt{3}}{8}$ は次の意味で最良のものである：$r < \frac{3\sqrt{3}}{8}$ であれば 関数族 $\{m(r+\frac{x}{1+x^2})\}$ は必ず周期倍分岐を引き起こす。

3 Antimonotonicity

S. Dawson, C. Grebogi, J. Yorke, I. Kan and H. Koçak 達は論文 [DGYKK92] で、1 係数の 1 次元及び 2 次元関数族が antimonotone であるための十分条件を与えてる:

> *our main findings are*
> *(A) A smooth one-dimensional map depending on oneparameter has an antimonotone parameter value whenever at least two independent critical points are contained in the interior of a chaotic attractor.*
> *(B) A smooth invertible dissipative two-dimensional map depending on one parameter has an antimonotone parameter value at any nondegenerate homoclinic tangency value.*
>
> *We now present a heuristic argument and numerical evidence to support (A).*

2 次元関数族に対する条件 (B) が正しいことは証明されているが、1 次元関数族に対する条件 (A) に対しては次の 3 次多項式族を引用して正しいであろうと述べている:

$$f_\lambda(x) = -x^3 + 1.2675x - \lambda.$$

この記述 (A) に対して, (A) の条件を満たしながら antimonotone parameter value を持たない 1 係数 3 次多項式族が存在する。

2 節の場合と同様に、3 次多項式族のモヂュライ空間に適当な座標 (σ_1, σ_3) をいれ、関数の力学的挙動によってこの空間を分割し、双曲成分の配置と分岐現象を引き起こす特定の代数曲線群を定義し、つぎに曲線がどの双曲成分を通過するか、またどの特定の代数曲線群と交差するかを調べることである。

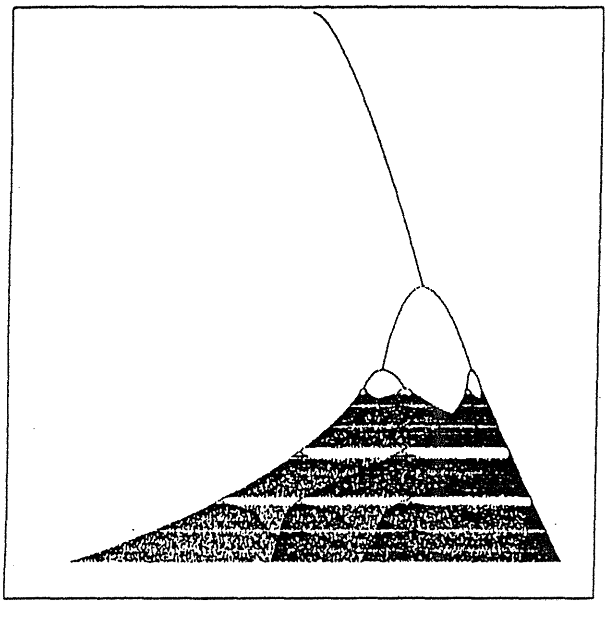
(A) の条件を満たしながら単調である 1 径数 3 次多項式族の例は次式である。

$$\begin{aligned}
f_{\sigma_1}(x) &= -x^3 - 2Ax + \sqrt{|B|}, \\
A &= \frac{\sigma_1 - 6}{9}, \\
B &= \frac{\sigma_3 + 1/27(\sigma_1 - 6)(2\sigma_2 - 3)^2}{27}, \\
& 8{\sigma_1}^2 - 96\sigma_1 + 3\sigma_3 + 288 = 0.
\end{aligned}$$

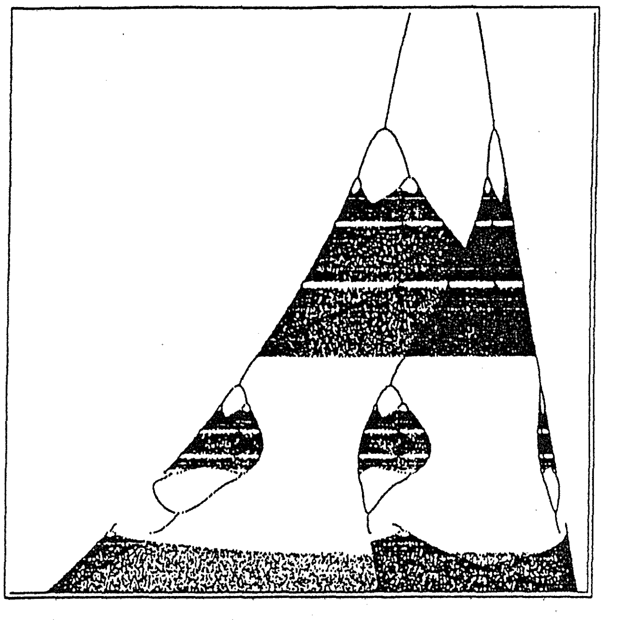
この3次多項式族は、3次多項式族のモヂュライ空間内でBC1と呼ばれる中心曲線([NN93])である。すなわち相異なる2個の特異点に関して、一方が他に写されるような3次多項式の集合である。

Milnor([Mil90])による双曲成分の分類によれば、BC1はB-タイプの双曲成分の1つの中心を通る既約代数曲線であり、Kneading sequence, topological entropyの単調性が示されている。したがってその分岐図は1方向の熊手分岐図を示す。一方antimonotoneの3次多項式族の分岐図は向き合った熊手分岐図を示し、かつカオス帯でぶつかっている。

曲線BC1に沿っての$\{f_{\sigma_1}(x)\}_{\sigma_1}$の分岐図と, $\lambda \in [0.51.1]$での$\{f_\lambda(x)\}_\lambda$の分岐図を示す。防衛大の藤村雅代氏によるものである。



Bifurcation diagram of cubic family (BC1): $-2 < x < 2,\ 0 < a < 2$



Bifurcation diagram of cubic family: $-1.5 < x < 1.7,\ 0.5 < b < 1.1$

2次元の場合には, 関数族がdissipative planar diffeomorphismsのときにはhomoclinic-tangency parameterの存在からantimonotonicityが示されている([KKY92])のと対照的に、antimonotoneな1次元関数族の解析はほとんど手が付けられていない。β-liftという1次元関数族$\{f_\lambda(x)\}_\lambda$から2次元関数族$\{F_{\beta,\lambda}(x,y)\}_{\beta,\lambda}$を作り:

$$F_{\beta,\lambda}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_\lambda(x) + \beta y \\ x \end{pmatrix}, \quad 0 < |\beta| < 1$$

, これにAntimonotonicity Theoremを適用する方法も取られているが([Jona93])、このときNewhouse-intervalが、βを0にちかずけた時に生き残るかの判定が難しい。ロジスチック関数族に対するヘノン写像族のようにNewhouse-intervalが消滅する場合がある。

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Two Yorke's conjectures on chaotic bifurcations

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Abstract

We discuss problems of non-monotone bifurcations for two cases of one-parameter families: real quadratic rational maps and real cubic polynomials. We present counter examples by computer experiments to the monotonicity conjecture and the antimonotonicity conjecture.

1 Introduction

System of iterated maps, viewed as real dynamical systems is considered as an important model for the chaotic behavior in certain parameterized systems. Creation and annihilation of periodic orbits is one of the most fundamental bifurcation processes, often illustrated by the pitchforks oriented either one-way or both-ways.

We discuss in this paper some topics from the bifurcation problems for a one parameter real family of quadratic rational maps or of cubic polynomials. J. Milnor and W. Thurston ([11]) proved by using Teichmüller theory that the logistic family $\{\lambda x(1-x)\ ;\ \lambda \in [1,4]\}$, which is a family of simple maps with extremely complicated dynamics, has only orbit-creation parameter values and no orbit-annihilation values as the parameter increases. Unlike monotonicity of the logistic family, however, there exist many one-parameter families exhibiting a non-monotone orbit-bifurcation structure, namely the pitchforks oriented both-ways.

We discuss monotonicity conjecture (M) indicated in several papers, now reformulated by [18] as follows:

> (M) *Let $f_m(x) = mf(x)$ be a one-parameter family of differential maps from closed interval I_m into itself which satisfies the following properties:* (1) *f_m is concave on I_m,* (2) *the set of periodic points of f_1 consists of two fixed points,* (3) *f_m has a negative schwarzian derivative. As the parameter m is increased, this one-parameter family is monotone.*

We consider a family $\{mf(x)\}$, where $f(x) = r + \frac{x}{1+x^2}$. The bifurcation diagram of this family can be monotone, non-monotone, or antimonotone according to the choice of the function f, namely the choice of r (cf. [6]). To the monotonicity conjecture, we will give a counter example using the defining equation of the lower escape locus, obtained in the section 2.1.

Next, we present an example to the antimonotonicity conjecture (A), enounced in the paper ([2]) with their heuristic argument and numerical evidence:

> (A) *A smooth one-dimensional map depending on one parameter has an antimonotone parameter value whenever at least two independent critical points are contained in the interior of a chaotic attractor.*

Hereafter we call the part *"at least two independent critical points are contained in the interior of a chaotic attractor"*, anti-condition:(Anti). To construct a one-parameter family under (Anti), having no antimonotone parameter value, we use an algebraic curve, so-called center curve defined in our papers ([16], [4]), in the moduli space of the cubic maps with the multiplier-coordinates system.

Our method of approach to a bifurcation problem is to analyze an algebraic curve, defined by one-parameter family in the moduli space associated of a family, e.g., we examine "which hyperbolic locus does the curve lie in?" or "which dynamical curves does the curve intersect with?"

2 Moduli space of quadratic rational maps

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\mathrm{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\mathrm{Rat}_2(\mathbf{C})$ by conjugation, $g \circ f \circ g^{-1} \in \mathrm{Rat}_2(\mathbf{C})$ for $g \in \mathrm{PSL}_2(\mathbf{C})$, $f \in \mathrm{Rat}_2(\mathbf{C})$. The quotient space of $\mathrm{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the moduli space of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f. The multipliers coordinates are introduced in $\mathcal{M}_2(\mathbf{C})$. For each $f \in \mathrm{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be the fixed points of f and μ_i the multipliers of z_i; $\mu_i = f'(z_i)$ $(1 \le i \le 3)$. Consider the elementary symmetric functions of the three multipliers, $\sigma_1 = \mu_1 + \mu_2 + \mu_3$, $\sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1$, $\sigma_3 = \mu_1\mu_2\mu_3$, which are subject only to the restriction that $\sigma_3 = \sigma_1 - 2$. Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ (Lemma 3.1 in [10]). Let $\mathrm{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. We remark that the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\mathrm{Rat}_2(\mathbf{R})$ is the real cut of $\mathcal{M}_2(\mathbf{C})$ (see [6]).

By an automorphism of a quadratic rational map f, we will mean $g \in \mathrm{PSL}_2(\mathbf{C})$ which commutes with f. The collection $\mathrm{Aut}(f)$ of all automorphisms of f forms a finite group. Since $\mathrm{Aut}(\tilde{f})$ is isomorphic to $\mathrm{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$, the set

$$\mathcal{S} = \{\langle f \rangle\, ;\ \mathrm{Aut}(f) \text{ is non-trivial}\} \subset \mathcal{M}_2(\mathbf{C})$$

is defined and called the symmetry locus.

For each $\mu \in \mathbf{C}$, let $\mathrm{Per}_1(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps f having a fixed point with multiplier μ. Each of $\mathrm{Per}_1(\mu)$ forms a straight line as follows:

$$\mathrm{Per}_1(\mu) = \left\{\langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1})\right\}$$

(Lemmas 3.4 and 3.6 in [10]).

Topological partition

For map $f \in \mathrm{Rat}_2(\mathbf{R})$, the two critical points of f are two real numbers or a pair of complex conjugate numbers. If f has a pair of complex conjugate critical points, this map is two-to-one covering map on $S^1 = \mathbf{R} \cup \{\infty\}$. In this case, if $f' > 0$ then f is called the map of degree $+2$, else $f' < 0$ then the map of degree -2.

While a map f with real critical points is called monotone (resp. unimodal, bimodal) if the interval $I = int(f(S^1))$ contains no (resp. one, two) critical points ([10]).

2.1 Real slices of hyperbolic escape locus

A rational map is hyperbolic if and only if the orbit of every critical point converges to some attracting periodic orbit. The hyperbolic maps form an open subset of moduli space, and the connected components of this open set are called hyperbolic components. M. Rees ([19]) shows that the hyperbolic components can be divided into four classes,Type B: Bitransitive, Type C: Capture, Type D: Disjoint attractors, and Type E: Escape. The names are due to J. Milnor ([10]).

Type E: Escape. Both critical orbits converge to the same attracting fixed point. There is just one such hyperbolic component.

In the complex case the escape locus is connected. But the real cut of this component splits into two parts; the upper part and the lower part. The boundary curve of the upper part is given by Milnor (Caption of Figure 16 in [10]).

Now, we specify the lower boundary. Proof is given in [13] and [7]. This boundary curve will play a key role in our later discussions of section 3.

Theorem 1 *Escape loci on the real moduli space is the union of the following sets;*

$$\{\sigma_2 > -2\sigma_1 + 1, \quad \sigma_2 > 2\sigma_1 - 3\}, \ \{\sigma_2 < 2\sigma_1 - 3, \quad \sigma_1 < -1\},$$
$$\{\sigma_2 < \tfrac{-2\sigma_1^2 - 7\sigma_1 - 10}{2+\sigma_1}, \quad \sigma_1 \geq -1\}.$$

3 Bifurcations

Let $\{f_\lambda\}_\Lambda$ be a one-parameter family of discrete dynamical systems on $\mathbf{R}$ where Λ is an interval of $\mathbf{R}$. As the parameter increases, a parameter value λ_0 is called orbit creating if, at λ_0, new periodic orbits are created and no periodic orbits are annihilated; λ_0 is called orbit annihilating if periodic orbits are annihilated and no new periodic orbits are created; λ_0 is called neutral if no periodic orbits are annihilated and no periodic orbits are created.

A family $\{f_\lambda\}_\Lambda$ is said to be monotone increasing (resp. decreasing) if every parameter value in Λ is neutral or orbit creating (resp. annihilating). A family $\{f_\lambda\}_\Lambda$ is called non-monotone if Λ contains both orbit creating and orbit annihilating parameter values. A family $\{f_\lambda\}_\Lambda$ is called antimonotone if any neighborhood of a suitable parameter λ_0 in Λ contains both infinitely many orbit creating and orbit annihilating parameter values.

4 Counter examples

4.1 Counter example to the Monotonicity conjecture

In this section we shall present a counter example, which is a one parameter family of quadratic rational maps, to the monotonicity conjecture enounced in the paper [18].

4.1.1 Monotone and non-monotone bifurcations of quadratic rational families

Now, we investigate the dynamics of a certain real 2-parameter family given by M. Bier and T. C. Bountis [1] and rewritten by H. E. Nusse and J. A. Yorke ([18]):

$$\left\{ f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}.$$

Here the map $f_{0,r}(x)$ should be thought of as an ideal limit map, in the natural compactification of $\mathcal{M}_2(\mathbf{C})$ (cf. [9]), of quadratic rational maps which degenerate towards the constant zero map. Then it makes sense to discuss the bifurcations of this family including the parameter value $m=0$, though in the real moduli space $\mathcal{M}_2(\mathbf{R})$ the maps diverge to infinity according as $m \to \pm 0$. Since the maps $f_{m,r}$ and $f_{m,-r}$ are conjugate to each other for any r, it suffices to consider the case $r \geq 0$.

Theorem 2 *In $\mathcal{M}_2(\mathbf{R})$, the one-parameter family $\{f_{m,r}(x)\}_m$ for each fixed r ($r \geq 0$) lies exactly on an irreducible algebraic curve $\mathcal{H}_r$:*
For $r \neq \frac{1}{2}, 0$, the curve $\mathcal{H}_r$ is of degree 4 defined by the equation

$$\begin{aligned} H_r(\sigma_1,\sigma_2) \;=\;& -r^2\sigma_1^4 + (8r^2-2)\sigma_1^3 + ((8r^2-1)\sigma_2 - 128r^4 + 8r^2 + 1)\sigma_1^2 + ((-32r^2+8)\sigma_2 \\ & +512r^4 - 96r^2 - 12)\sigma_1 + (-16r^2+4)\sigma_2^2 + (512r^4 - 96r^2 - 12)\sigma_2 \\ & -4096r^6 + 1536r^4 - 144r^2 + 36 = 0. \end{aligned} \tag{1}$$

For $r = \frac{1}{2}$ or $r = 0$, the curve $\mathcal{H}_r$ is of degree 3.

The proof is given in our paper [6].
Example (Antimonotone) Consider the one-parameter family defined on a suitable interval I_m, $F_m(x) = m\frac{x^2+ax+b}{1+x^2}$, where constant a is the positive root of the following equation $49a^2 - 32 = 0$, and b is the unique positive root of the following equation

$$117649b^7 + 684285b^6 + 1721517b^5 + 2358566b^4 + 1670655b^3 + 991301b^2 - 257125b = 0.$$

It is clear that this family satisfies the conditions of monotonicity conjecture (M), namely, (1) each F_m is concave, (2) the set of periodic points in I_1 of F_1 consists of two fixed points, and (3) F_m has a negative schwarzian derivative.

In this moduli space, a defining equation of the algebraic curve defined by $\{F_m\}_m$ is given as follows;

$$\begin{aligned} S_{a,b} = {}& (2\sigma_1^3+(\sigma_2-1)\sigma_1^2+(-8\sigma_2+12)\sigma_1-4\sigma_2^2+12\sigma_2-36)a^6+((2\sigma_1^3+(\sigma_2+24)\sigma_1^2+(12\sigma_2+72)\sigma_1+ \\ & 36\sigma_2)b^2+(-14\sigma_1^3+(-6\sigma_2-20)\sigma_1^2+(32\sigma_2+24)\sigma_1+16\sigma_2^2+24\sigma_2+144)b+\sigma_1^4+4\sigma_1^3+(-3\sigma_2- \\ & 12)\sigma_1^2-12\sigma_2\sigma_1+36\sigma_2)a^4+((-10\sigma_1^3+(-4\sigma_2-132)\sigma_1^2+(-48\sigma_2-504)\sigma_1-144\sigma_2-432)b^3+ \\ & (2\sigma_1^4+46\sigma_1^3+(4\sigma_2+188)\sigma_1^2+(-16\sigma_2-216)\sigma_1-240\sigma_2-720)b^2+(-4\sigma_1^4-30\sigma_1^3+(4\sigma_2+ \\ & 84)\sigma_1^2+(48\sigma_2+152)\sigma_1-112\sigma_2-336)b+2\sigma_1^4-6\sigma_1^3+(-4\sigma_2-12)\sigma_1^2+(16\sigma_2+56)\sigma_1-16\sigma_2- \\ & 48)a^2+(\sigma_1^4+24\sigma_1^3+216\sigma_1^2+864\sigma_1+1296)b^4+(-4\sigma_1^4-64\sigma_1^3-288\sigma_1^2+1728)b^3+(6\sigma_1^4+48\sigma_1^3- \\ & 48\sigma_1^2-576\sigma_1+864)b^2+(-4\sigma_1^4+96\sigma_1^2-256\sigma_1+192)b+\sigma_1^4-8\sigma_1^3+24\sigma_1^2-32\sigma_1+16=0. \end{aligned}$$

We remark that this curve tangent to a boundary curve of the lower locus of escape. Then we see this family is antimonotone at this tangent point [3].

4.2 Antimonotonicity conjecture

In this section we shall present an example, which is a one parameter family of cubic polynomials, to the antimonotonicity conjecture enounced in the paper [2]. The one-parameter family $f_\lambda(x) = -x^3 + 1.2675x - \lambda$, defined in [2], is antimonotone under (Anti). It turns out that this family exactly on a half line $\sigma_1 = -3.8025$ in the moduli space. On the other hand, we can present a set BC1: $\sigma_3 = -\frac{8}{3}(\sigma_1 - 6)^2$, of classes of the maps one of whose two critical points maps to another one (see [16], [17]). The set BC1 corresponds to the one parameter family: $BC1$: $g_a(x) = -x^3 + ax + (1 + \frac{2}{3}a)\sqrt{\frac{a}{3}}$. We can show with computer experiments that this family is monotone (naturally not antimonotone) under (Anti).

Recently we know that J. Milnor and Ch. Tresser also treat of this problem and they said in [12] that

The analogue of the Antimonotonicity Conjecture for the stunted sawtooth families is certainly false, since by 5.8, it is very easy to find smooth curves along which there are only orbit creations. Thus, if the conjecture is true for the cubic family, then any complexity preserving correspondence between the stunted sawtooth and cubic parameter triangles must be very wild indeed.

We remark that the entropy of the family $\{f_\lambda\}_\lambda$ is not monotone but one of our family $\{g_a\}_a$ is monotone.

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Chaotic bifurcation of one-parameter family

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Abstract

We give a defining equation of the boundary of the real cut of escape locus in the real moduli space for the real quadratic rational maps. And we discuss problems of non-monotone bifurcations for two cases of one-parameter families: quadratic rational maps and cubic polynomials. We present counter examples by computer experiments to the monotonicity conjecture and the antimonotonicity conjecture.

1 Introduction

In the paper [11], we obtaine a defining equation for a real cut of hyperbolic region, called escape locus, of real moduli space for the quadratic rational maps And, by using this moduli space and properties of escape locus, we develope arguments for bifurcation phenomenas. While in the paper [12], we discussed some one-parameter family with monotone or non-monotone bifurcations and presented two counter examples for conjectures described in [2] and [16].

Now combining these two papers [11] and [12], we presents some results in this paper. Therefor we give notice that this paper does not include new results.

This paper has two parts. First one is concerned with geometry of the space $\mathrm{Rat}_2(\mathbf{C})$ of the quadratic rational maps, namely its moduli space $\mathcal{M}_2(\mathbf{C})$, consisting of all holomorphic conjugacy classes of maps, which can be described as an orbifold whose underlying space is isomorphic to $\mathbf{C}^2$, and having a natural compactification, isomorphic to the projective plane $\mathbf{CP}^2$. Maps which are hyperbolic on their Julia set give rise to hyperbolic components in the moduli space. M. Rees shows in ([17]) that the hyperbolic components can be divided into four classes: type B,C,D and E. There is just one hyperbolic component of type E in $\mathcal{M}_2(\mathbf{C})$, so-called (hyperbolic) escape component, consisting of maps with totally disconnected Julia set. This component has a more complicated topology. If we work in the real space, there are just two escape components in the real moduli space $\mathcal{M}_2(\mathbf{R})$, namely two real slices. We call these loci upper escape locus and lower escape locus. For a map in the upper escape locus with two real critical points, its real dynamics is completely trivial: the compactified real line converges to the real fixed point under iterations. Milnor gives in ([8]) a defining equation of this boundary. On the other hand, a map in the lower escape locus has complicated real dynamics. We will give a defining equation of the boundary of this lower escape locus, as Theorem 1 in section 2.1.2.

The second one is concerned with some topics from the bifurcation problems for a one parameter real family of quadratic rational maps or of cubic polynomials.

System of iterated maps, viewed as real dynamical systems is considered as an important model for the chaotic behavior in certain parameterized systems. Creation and annihilation of periodic orbits is one of the most fundamental bifurcation processes, often illustrated by the pitchforks oriented either one-way or both-ways. J. Milnor and W. Thurston ([9]) proved by using Teichmüller theory that the logistic family $\{\lambda x(1-x)\ ;\ \lambda \in [1,4]\}$, which is a family of simple maps with extremely complicated dynamics, has only orbit-creation parameter values and no orbit-annihilation values as the parameter increases. Unlike monotonicity of the logistic family, however, there exist many one-parameter families exhibiting a non-monotone orbit-bifurcation structure, namely the pitchforks oriented both-ways.

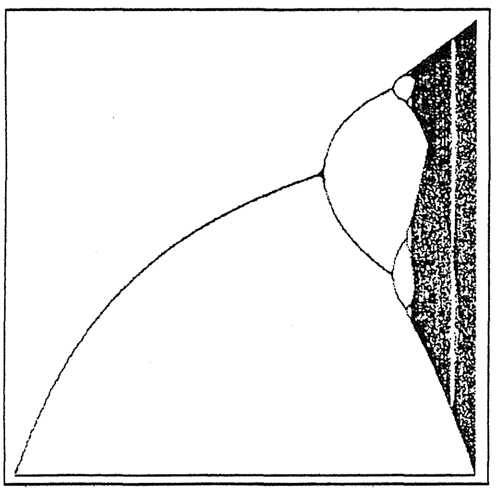

Figure 1: One-way pitchfork in the logistic family. $\{\lambda x(1-x)\}, (1 \le \lambda \le 4)$.

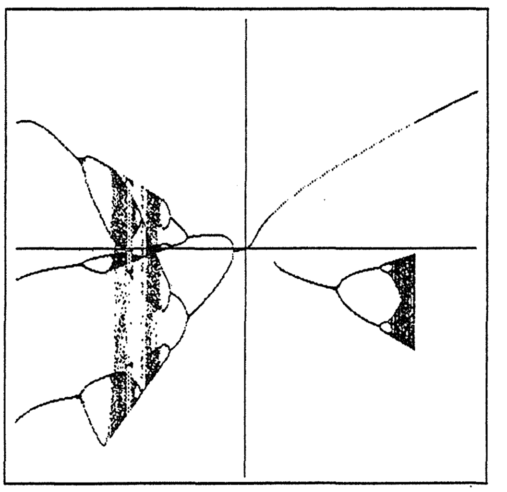

Figure 2: Both-ways pitchforks in the quadratic rational family. $\{m(\frac{1}{5} + \frac{x}{1+x^2})\}$, $(-20 \le m \le 20)$.

We discuss monotonicity conjecture (M) indicated in several papers, now reformulated in [16] as follows:

> (M) *Let $f_m(x) = mf(x)$ be a one-parameter family of differential maps from closed interval I_m into itself which satisfies the following properties:* (1) *f_m is concave on I_m,* (2) *the set of periodic points of f_1 consists of two fixed points,* (3) *f_m has a negative schwarzian derivative. As the parameter m is increased, this one-parameter family is monotone.*

We consider a family $\{mf(x)\}$, where $f(x) = r + \frac{x}{1+x^2}$. The bifurcation diagram of this family can be monotone, non-monotone, or antimonotone according to the choice of the function f, namely the choice of r (cf. [5]).

Our method of approach to a bifurcation problem is to analyze an algebraic curve, defined by one-parameter family in the moduli space associated of a family, e.g., we examine "which hyperbolic locus does the curve lie in?" or "which dynamical curves does the curve intersect with?"

To the monotonicity conjecture, we will give a counter example using the defining equation of the lower escape locus, obtained in the section 2.1.2.

Next, we present a counter example to the antimonotonicity conjecture (A), enounced in the paper ([2]) with their heuristic argument and numerical evidence:

> (A) *A smooth one-dimensional map depending on one parameter has an antimonotone parameter value whenever at least two independent critical points are contained in the interior of a chaotic attractor.*

Hereafter we call the part *"at least two independent critical points are contained in the interior of a chaotic attractor"*, anti-condition:(Anti). To construct a one-parameter family under (Anti), having no antimonotone parameter value, we use an algebraic curve, so-called center curve defined in our papers ([14], [4]), in the moduli space of the cubic maps with the multiplier-coordinates system.

2 Moduli spaces

2.1 Moduli space of quadratic rational maps

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\mathrm{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\mathrm{Rat}_2(\mathbf{C})$ by conjugation, $g \circ f \circ g^{-1} \in \mathrm{Rat}_2(\mathbf{C})$ for $g \in \mathrm{PSL}_2(\mathbf{C})$, $f \in \mathrm{Rat}_2(\mathbf{C})$. The quotient space of $\mathrm{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the moduli space of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f. The multipliers coordinates are introduced in $\mathcal{M}_2(\mathbf{C})$. For each $f \in \mathrm{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be the fixed points of f and μ_i the multipliers of z_i; $\mu_i = f'(z_i)$ $(1 \leq i \leq 3)$. Consider the elementary symmetric functions of the three multipliers, $\sigma_1 = \mu_1 + \mu_2 + \mu_3$, $\sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1$, $\sigma_3 = \mu_1\mu_2\mu_3$, which are subject only to the restriction that $\sigma_3 = \sigma_1 - 2$. Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ (Lemma 3.1 in [8]). Let $\mathrm{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. We remark that the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\mathrm{Rat}_2(\mathbf{R})$ is the real cut of $\mathcal{M}_2(\mathbf{C})$ (see [5]).

By an automorphism of a quadratic rational map f, we will mean $g \in \mathrm{PSL}_2(\mathbf{C})$ which commutes with f. The collection $\mathrm{Aut}(f)$ of all automorphisms of f forms a finite group. Since $\mathrm{Aut}(\tilde{f})$ is isomorphic to $\mathrm{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$, the set

$$\mathcal{S} = \{\langle f \rangle \, ; \ \mathrm{Aut}(f) \text{ is non-trivial}\} \subset \mathcal{M}_2(\mathbf{C})$$

is defined and called the symmetry locus.

For each $\mu \in \mathbf{C}$, let $\mathrm{Per}_1(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps f having a fixed point with multiplier μ. Each of $\mathrm{Per}_1(\mu)$ forms a straight line as follows:

$$\mathrm{Per}_1(\mu) = \left\{\langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1})\right\}$$

(Lemmas 3.4 and 3.6 in [8]).

2.1.1 Topological partition

For map $f \in \mathrm{Rat}_2(\mathbf{R})$, the two critical points of f are two real numbers or a pair of complex conjugate numbers. If f has a pair of complex conjugate critical points, this map is two-to-one covering map on $S^1 = \mathbf{R} \cup \{\infty\}$. In this case, if $f' > 0$ then f is called the map of degree $+2$, else $f' < 0$ then the map of degree -2.

While a map f with real critical points is called monotone (resp. unimodal, bimodal) if the interval $I = int(f(S^1))$ contains no (resp. one, two) critical points ([8]).

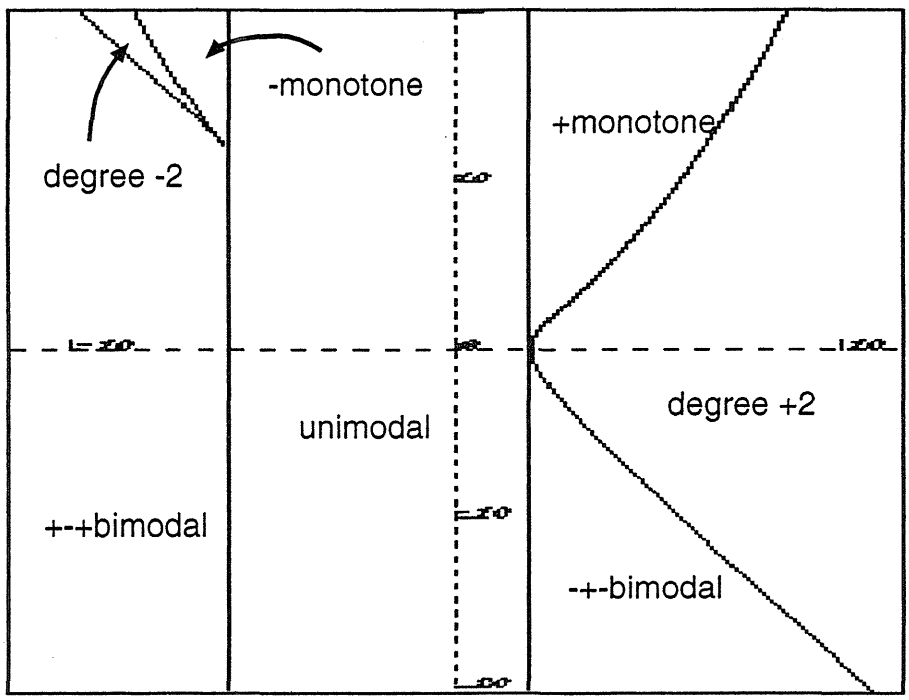

Figure 3: The topological partition of the moduli space of real quadratic rational maps. These regions are bounded by the real cut of the symmetry locus and two lines $\sigma_1 = 2, 6$.

2.1.2 Real slices of hyperbolic escape locus

A rational map is hyperbolic if and only if the orbit of every critical point converges to some attracting periodic orbit. The hyperbolic maps form an open subset of moduli space, and the connected components of this open set are called hyperbolic components. M. Rees ([17]) shows that the hyperbolic components can be divided into four classes as follows. The names are due to J. Milnor ([8]).

Type B: Bitransitive. Each of the two critical points belongs to the immediate basin of some attracting periodic point, where these two periodic points are distinct but belong to the same orbit. Evidently the period must be two or more.

Type C: Capture. Only one critical point belongs to the immediate basin on a periodic point, but the orbit of the other critical point eventually falls into this immediate basin. Again the period must be two or more.

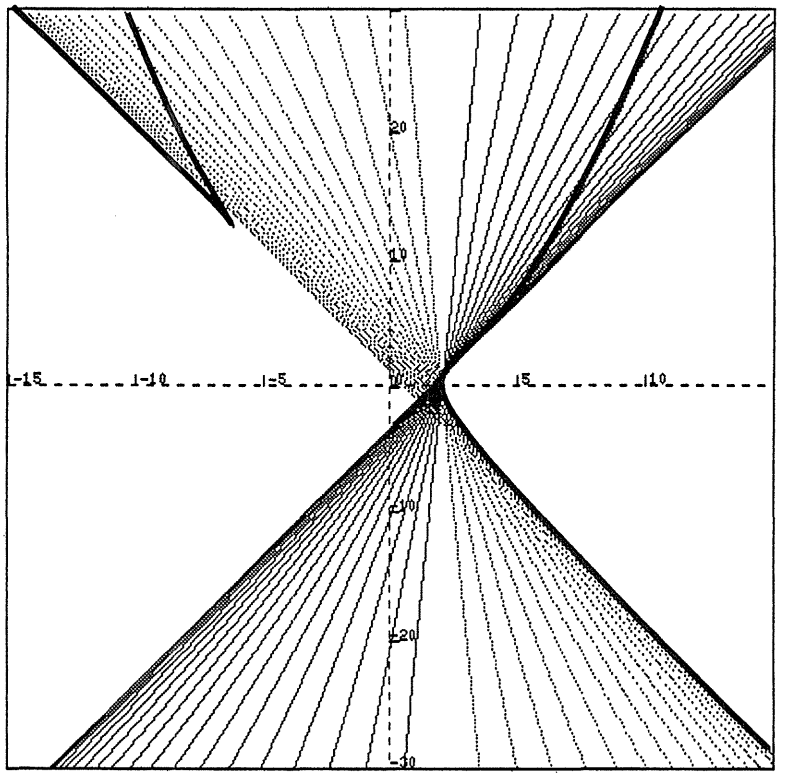

Figure 4: Lines $\mathrm{Per}_1(\mu)$: gray lines show $-1 < \mu < 0$, black lines show $0 < \mu < 1$ and thick curve shows the symmetry locus.

Type D: Disjoint attractors. The two critical points belong to the attracting basins for two disjoint attracting periodic orbits.

Type E: Escape. Both critical orbits converge to the same attracting fixed point. There is just one such hyperbolic component.

In the complex case the escape locus is connected. But the real cut of this component splits into two parts; the upper part and the lower part. The boundary curve of the upper part is given by Milnor (Caption of Figure 16 in [8]).

Now, we specify the lower boundary. This boundary curve will play a key role in our later discussions of section 3.

Theorem 1 *Escape loci on the real moduli space is the union of the following sets;*

$$\{\sigma_2 > -2\sigma_1 + 1, \quad \sigma_2 > 2\sigma_1 - 3\}, \ \{\sigma_2 < 2\sigma_1 - 3, \quad \sigma_1 < -1\},$$
$$\{\sigma_2 < \tfrac{-2\sigma_1^2 - 7\sigma_1 - 10}{2+\sigma_1}, \quad \sigma_1 \geq -1\}.$$

Proof. Here, we use the following real two-parameter family of quadratic rational maps induced by M. Bier and T. C. Bountis ([1]) and rewritten by H. E. Nusse and J. A. Yorke ([16]);

$$\left\{ f_{m,r}(x) = m\left(r + \frac{x}{1+x^2} \right) \right\}_{(m,r)\in\mathbf{R}^2}.$$

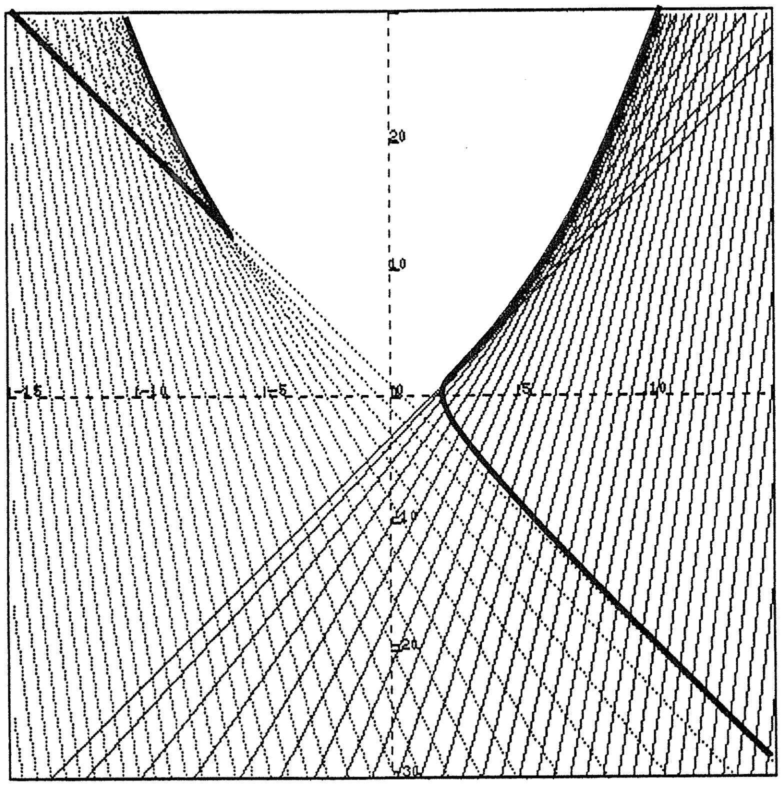

Figure 5: Lines $\mathrm{Per}_1(\mu)$: gray lines show $\mu < -1$, black lines show $\mu > 1$ and thick curve shows the symmetry locus.

This family covers the real moduli space $\{(\sigma_1, \sigma_2)\}$ expect for the degree ± 2 regions and the half line $\{(\sigma_1, \sigma_2)\,;\, \sigma_1 = 2, \sigma_2 < -1\}$ of the quadratic polynomial region (See [13]). Since the maps $f_{m,r}$ and $f_{m,-r}$ are conjugate to each other for any r, it suffices to consider the case $r \geq 0$. We note that the algebraic curve defined by one-parameter family $\{f_{m,0}\}$ coincides with the boundary curve of degree ± 2 regions.

The fixed pints of $f_{m,r}$ are the three roots of equation,

$$x^3 - mrx^2 - (m-1)x - mr = 0.$$

Two critical points of this map are ± 1 and its critical values are $mr \pm \frac{m}{2}$. Therefore this family can not cover the degree ± 2 regions.

The denominator of map $f_{m,r}$ is always positive, and

$$\lim_{x \to \pm\infty} f_{m,r} = mr,$$

$$f_{m,r}(0) = mr.$$

Hence the horizontal line $y = mr$ is unique asymptotic line of this map. For the case of $m > 0$ (resp. $m < 0$), graph is $(- + -)$ (resp. $(+ - +)$), and it is sufficient to consider the dynamics on the closed interval $[mr - \frac{m}{2}, mr + \frac{m}{2}]$ (resp. $[mr + \frac{m}{2}, mr - \frac{m}{2}]$).

From the graphical analysis it is clear that a map $f_{m,r}$ belongs to the escape if and only if $f_{m,r}$ satisfying one of the following condition.

1. In $\pm$ monotone- regions, $f_{m,r}$ has only one attracting fixed point.
2. In unimodal- region,
   - $f_{m,r}$ do not have real fixed point except for an attracting fixed point with multiplier $-1 < \mu < 0$, or
   - $f_{m,r}$ has an attracting fixed point with multiplier $0 < \mu < 1$ and two repelling fixed points, and its two minimal intervals containing each critical orbits has intersection.
3. In $(-+-)$-bimodal- region, $f_{m,r}$ has an attracting fixed point and two repelling fixed points, and its two minimal intervals containing each critical orbits has non-trivial intersection.
4. In $(+-+)$-bimodal- region, $f_{m,r}$ has an attracting fixed point and two repelling fixed points.

In the parameter space $\{(r,m)\}$, after specifying "escape"- regions in each cases, we map these regions to the real moduli space by using transformation formula:

$$\begin{cases} ST1 : 4m^2r^2 - m^2 + (\sigma_1 + 2)m - 4 = 0, \\ ST2 : -4m^4r^4 + (m^4 - 12m^3 - 8m^2)r^2 + 2m^3 + (\sigma_2 - 5)m^2 + 4m - 4 = 0. \end{cases}$$

For example, the escape region corresponding to the above condition 3 ($(-+-)$-bimodal case) is given as the condition $f^2_{m,r}(-1) < f^3_{m,r}(-1)$, translated into an inequality $N_{m,r}/D_{m,r} > 0$:

$$\begin{aligned} N_{m,r} = {} & m^2(-2mr + m - 2)^3(4m^3r^3 - 4(m+1)m^2r^2 + (m^2 + 6m + 4)mr - 2m^2 - 2m - 4) \\ D_{m,r} = {} & (m^2(2r-1)^2 + 4)(16m^6r^6 - 32m^6r^5 + 8(3m^2 + 4m + 6)m^4r^4 \\ & -8(m^2 + 6m + 8)m^4r^3 + (m^4 + 24m^3 + 48m^2 + 32m + 48)m^2r^2 \\ & -4(m^3 + 6m^2 + 4m + 8)m^2r + 5m^4 + 8m^2 + 16). \end{aligned}$$

From calculation we can see the factor $(-2mr + m - 2)$ of $N_{m,r}$ and the second factor of $D_{m,r}$ are always positive under the condition 3. Therefore the condition 3 is finally reduced to an inequality:

$$4m^3r^3 - 4(m+1)m^2r^2 + (m^2 + 6m + 4)mr - 2m^2 - 2m - 4 > 0.$$

Mapping this parameter region to the moduli space by transformation formula, we have

$$2\sigma_1^2 + (\sigma_2 + 7)\sigma_1 + 2\sigma_2 + 10 < 0, \quad \sigma_1 > 2.$$

For the other cases, we can get region of "escape" in the same way.

We conclude this proof by treating the region where the family $\{f_{m,r}\}$ do not cover.

For the case of degree ± 2 regions, a map belongs to the escape if and only if the map has unique attracting fixed point (See Caption of Figure 16 of [8]).

A quadratic polynomial family $\{x^2 + \sigma_2/4\}_{\sigma_2}$ coincides with the line $\{(2,\sigma_2)\}$ on the moduli space. Hence two half line $\{(\sigma_1,\sigma_2)\,;\,\sigma_1 = 2, \sigma_2 < -8\}$, $\{(\sigma_1,\sigma_2)\,;\,\sigma_1 = 2, \sigma_2 > 1\}$ belong to escape loci. ∎

2.2 Moduli space of cubic polynomials

Let $\mathrm{Poly}_3(\mathbf{C})$ be the space of all cubic polynomials from $\mathbf{C}$ to itself. The group $\mathrm{Poly}_1(\mathbf{C})$ of affine transformations acts on the space $\mathrm{Poly}_3(\mathbf{C})$, by conjugation, $g \circ p \circ g^{-1} \in \mathrm{Poly}_3(\mathbf{C})$ for $g \in \mathrm{Poly}_1(\mathbf{C})$, $p \in \mathrm{Poly}_3(\mathbf{C})$. Two maps $p_1, p_2 \in \mathrm{Poly}_3(\mathbf{C})$ are holomorphically conjugate, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \mathrm{Poly}_1(\mathbf{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\mathrm{Poly}_3(\mathbf{C})$ under this action will be denoted by $\mathrm{M}_3(\mathbf{C})$, and called the moduli space of holomorphic conjugacy classes $\langle p \rangle$ of cubic polynomials p. For each $p \in \mathrm{Poly}_3(\mathbf{C})$, let $z_1,\ z_2,\ z_3,\ z_4(=\infty)$ be the fixed points of p and μ_i the multipliers of z_i; $\mu_i = p'(z_i)$ $(1 \le i \le 3)$, and $\mu_4 = 0$. Consider the elementary symmetric functions of the four multipliers,

$$\begin{aligned}
\sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 = \mu_1 + \mu_2 + \mu_3 \\
\sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\
\sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 = \mu_1\mu_2\mu_3 \\
\sigma_4 &= \mu_1\mu_2\mu_3\mu_4 = 0.
\end{aligned}$$

These multipliers determine uniquely p up to holomorphic conjugacy, and are subject only to the restriction that $3 - 2\sigma_1 + \sigma_2 = 0$. Now an affine structure is imposed on $\mathrm{M}_3(\mathbf{C})$ by this multipliers coordinate system (σ_1, σ_3). We remark that the singular part of this moduli space is given the following algebraic variety:

$$S_3(\sigma_1, \sigma_3) = 4\sigma_1^3 - 36\sigma_1^2 + 81\sigma_1 + 27\sigma_3 - 54 = 0. \tag{1}$$

A map in $\mathrm{Poly}_3(\mathbf{C})$ is always conjugate to a map of the normal form $z^3 + az + b$, and its parameters (a, b^2) is used as a coordinate system of $\mathrm{M}_3(\mathbf{C})$ which is isomorphic to $\mathbf{C}^2$ ([6]). These coordinates relate to (σ_1, σ_3) as follows:

$$\begin{aligned}
\sigma_1 &= -3a + 6, \\
\sigma_3 &= 27b^2 + a(2a-3)^2,
\end{aligned} \tag{2}$$

Let $\mathrm{Poly}_3(\mathbf{R})$ be the set of real cubic polynomials. We simply define the real moduli space $\mathrm{M}_3(\mathbf{R})$ for $\mathrm{Poly}_3(\mathbf{R})$ as the real (σ_1, σ_3)-plane.

3 Bifurcations

Let $\{f_\lambda\}_\Lambda$ be a one-parameter family of discrete dynamical systems on $\mathbf{R}$ where Λ is an interval of $\mathbf{R}$. As the parameter increases, a parameter value λ_0 is called orbit creating if, at λ_0, new periodic orbits are created and no periodic orbits are annihilated; λ_0 is called orbit annihilating if periodic orbits are annihilated and no new periodic orbits are created; λ_0 is called neutral if no periodic orbits are annihilated and no periodic orbits are created.

A family $\{f_\lambda\}_\Lambda$ is said to be monotone increasing (resp. decreasing) if every parameter value in Λ is neutral or orbit creating (resp. annihilating). A family $\{f_\lambda\}_\Lambda$ is called non-monotone if Λ contains both orbit creating and orbit annihilating parameter values. A family $\{f_\lambda\}_\Lambda$ is called antimonotone if any neighborhood of a suitable parameter λ_0 in Λ contains both infinitely many orbit creating and orbit annihilating parameter values.

3.1 Counter example to the Monotonicity conjecture

In this section we shall present a counter example, which is a one parameter family of quadratic rational maps, to the monotonicity conjecture enounced in the paper [16].

3.1.1 Monotone and non-monotone bifurcations of quadratic rational families

Now, we investigate the dynamics of a certain real 2-parameter family given by M. Bier and T. C. Bountis [1] and rewritten by H. E. Nusse and J. A. Yorke ([16]):

$$\left\{ f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}.$$

Here the map $f_{0,r}(x)$ should be thought of as an ideal limit map, in the natural compactification of $\mathcal{M}_2(\mathbf{C})$ (cf. [7]), of quadratic rational maps which degenerate towards the constant zero map. Then it makes sense to discuss the bifurcations of this family including the parameter value $m = 0$, though in the real moduli space $\mathcal{M}_2(\mathbf{R})$ the maps diverge to infinity according as $m \to \pm 0$. Since the maps $f_{m,r}$ and $f_{m,-r}$ are conjugate to each other for any r, it suffices to consider the case $r \geq 0$.

Theorem 2 *In $\mathcal{M}_2(\mathbf{R})$, the one-parameter family $\{f_{m,r}(x)\}_m$ for each fixed r $(r \geq 0)$ lies exactly on an irreducible algebraic curve $\mathcal{H}_r$:*
For $r \neq \frac{1}{2}, 0$, the curve $\mathcal{H}_r$ is of degree 4 defined by the equation

$$\begin{aligned} H_r(\sigma_1,\sigma_2) &= -r^2\sigma_1^4 + (8r^2-2)\sigma_1^3 + ((8r^2-1)\sigma_2 - 128r^4 + 8r^2 + 1)\sigma_1^2 \\ &\quad +((-32r^2+8)\sigma_2 + 512r^4 - 96r^2 - 12)\sigma_1 + (-16r^2+4)\sigma_2^2 \\ &\quad +(512r^4 - 96r^2 - 12)\sigma_2 - 4096r^6 + 1536r^4 - 144r^2 + 36 \\ &= 0. \end{aligned}$$

For $r = \frac{1}{2}$ or $r = 0$, the curve $\mathcal{H}_r$ is of degree 3.

The proof is given in our paper [5].

Example (Primary bubbling)
For $r = 0.58$, the one parameter family $\{f_{m,0.58}\}_m$ is non-monotone. More precisely, this bifurcation diagram is so-called primary bubbling ([5]).
Example (Antimonotone) Consider the one-parameter family defined on a suitable interval I_m,

$$F_m(x) = m\frac{x^2+ax+b}{1+x^2},$$

where constant a is the positive root of the following equation

$$49a^2 - 32 = 0,$$

and b is the unique positive root of the following equation

$$117649b^7 + 684285b^6 + 1721517b^5 + 2358566b^4 + 1670655b^3 + 991301b^2 - 257125b = 0.$$

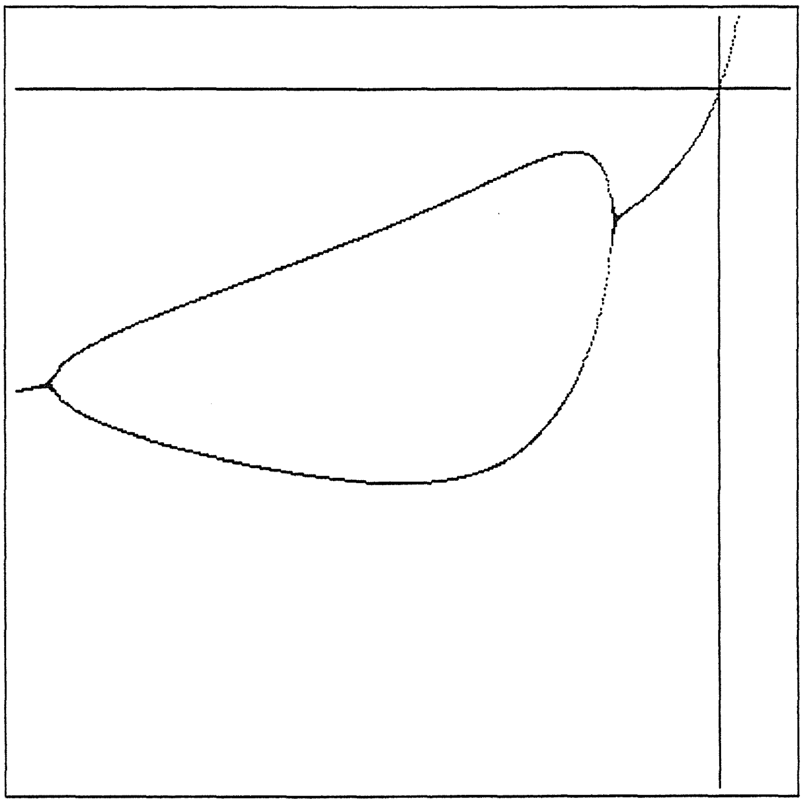

Figure 6: Primary bubbling bifurcation for $\{f_{m,0.58}(x)\}_m$: $-10 \leq m \leq 1$, $-2 \leq x \leq 0.2$.

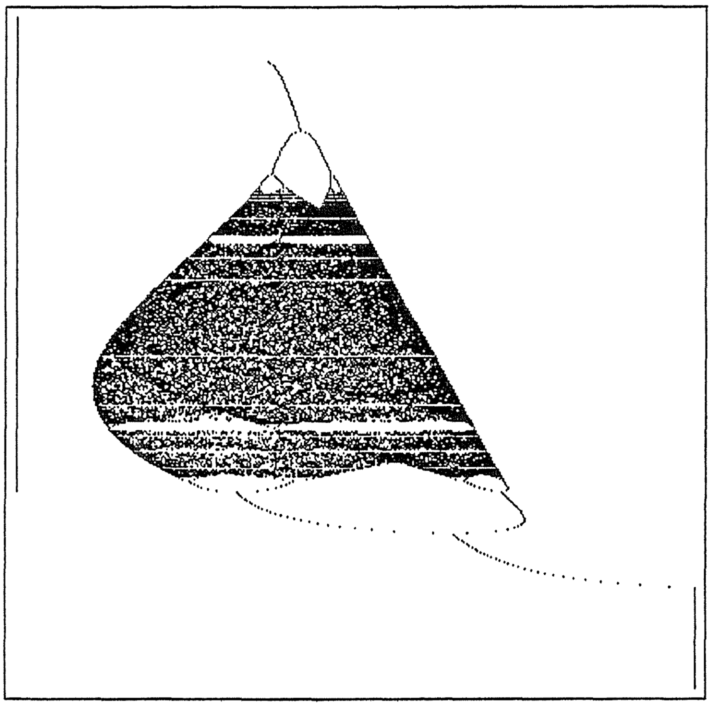

Figure 7: Bifurcation diagram of family $\{F_m(x)\}_m$: $-0.8 < x < 0.2$, $-25 < m < 5$

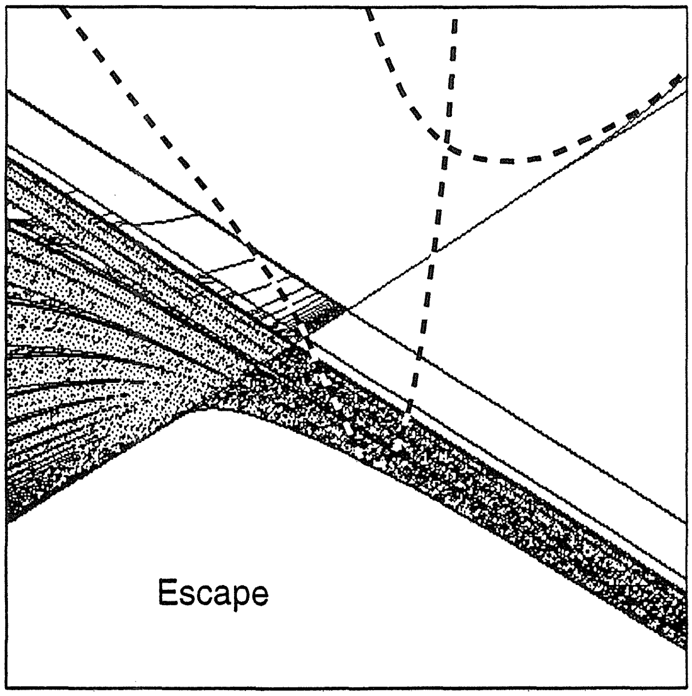

Figure 8: Real moduli space with an algebraic curve defined by $\{F_m\}_m$: $-3 < \sigma_1 < 5,\ -15 < \sigma_2 < 10$

It is clear that this family satisfies the conditions of monotonicity conjecture (M), namely, (1) each F_m is concave, (2) the set of periodic points in I_1 of F_1 consists of two fixed points, and (3) F_m has a negative schwarzian derivative.

In this moduli space, a defining equation of the algebraic curve defined by $\{F_m\}_m$ is given as follows;

$$\begin{aligned} S_{a,b} =& (2\sigma_1^3 + (\sigma_2 - 1)\sigma_1^2 + (-8\sigma_2 + 12)\sigma_1 - 4\sigma_2^2 + 12\sigma_2 - 36)a^6 + ((2\sigma_1^3 + (\sigma_2 + 24)\sigma_1^2 + \\ & (12\sigma_2 + 72)\sigma_1 + 36\sigma_2)b^2 + (-14\sigma_1^3 + (-6\sigma_2 - 20)\sigma_1^2 + (32\sigma_2 + 24)\sigma_1 + 16\sigma_2^2 + 24\sigma_2 + \\ & 144)b + \sigma_1^4 + 4\sigma_1^3 + (-3\sigma_2 - 12)\sigma_1^2 - 12\sigma_2\sigma_1 + 36\sigma_2)a^4 + ((-10\sigma_1^3 + (-4\sigma_2 - 132)\sigma_1^2 + \\ & (-48\sigma_2 - 504)\sigma_1 - 144\sigma_2 - 432)b^3 + (2\sigma_1^4 + 46\sigma_1^3 + (4\sigma_2 + 188)\sigma_1^2 + (-16\sigma_2 - 216)\sigma_1 - \\ & 240\sigma_2 - 720)b^2 + (-4\sigma_1^4 - 30\sigma_1^3 + (4\sigma_2 + 84)\sigma_1^2 + (48\sigma_2 + 152)\sigma_1 - 112\sigma_2 - 336)b + \\ & 2\sigma_1^4 - 6\sigma_1^3 + (-4\sigma_2 - 12)\sigma_1^2 + (16\sigma_2 + 56)\sigma_1 - 16\sigma_2 - 48)a^2 + (\sigma_1^4 + 24\sigma_1^3 + 216\sigma_1^2 + \\ & 864\sigma_1 + 1296)b^4 + (-4\sigma_1^4 - 64\sigma_1^3 - 288\sigma_1^2 + 1728)b^3 + (6\sigma_1^4 + 48\sigma_1^3 - 48\sigma_1^2 - 576\sigma_1 + \\ & 864)b^2 + (-4\sigma_1^4 + 96\sigma_1^2 - 256\sigma_1 + 192)b + \sigma_1^4 - 8\sigma_1^3 + 24\sigma_1^2 - 32\sigma_1 + 16 = 0. \end{aligned}$$

We remark that this curve tangent to a boundary curve of the lower locus of escape. Then we see this family is antimonotone at this tangent point [3].

3.2 Example to the Antimonotonicity conjecture

In this section we shall present a counter example, which is a one parameter family of cubic polynomials, to the antimonotonicity conjecture enounced in the paper [2].

3.2.1 One parameter cuvic polynomials with Monotone bifurcations

The one-parameter family $f_\lambda(x) = -x^3 + 1.2675x - \lambda$, defined in [2], is antimonotone under (Anti). It turns out that this family exactly on a half line $\sigma_1 = -3.8025$ in the moduli space.

On the other hand, we can present a set BC1: $\sigma_3 = -\frac{8}{3}(\sigma_1 - 6)^2$, of classes of the maps one of whose two critical points maps to another one (see [14], [15]). The set BC1 corresponds to the one parameter family:

$$BC1: \quad g_a(x) = -x^3 + ax + (1 + \frac{2}{3}a)\sqrt{\frac{a}{3}}.$$

We can show with computer experiments that this family is monotone (naturally not antimonotone) under (Anti).

Recently we know that J. Milnor and Ch. Tresser also treat of this problem and they said in [10] that

> *The analogue of the Antimonotonicity Conjecture for the stunted sawtooth families is certainly false, since by 5.8, it is very easy to find smooth curves along which there are only orbit creations. Thus, if the conjecture is true for the cubic family, then any complexity preserving correspondence between the stunted sawtooth and cubic parameter triangles must be very wild indeed.*

We remark that the entropy of the family $\{f_\lambda\}_\lambda$ is not monotone but one of our family $\{g_a\}_a$ is monotone.

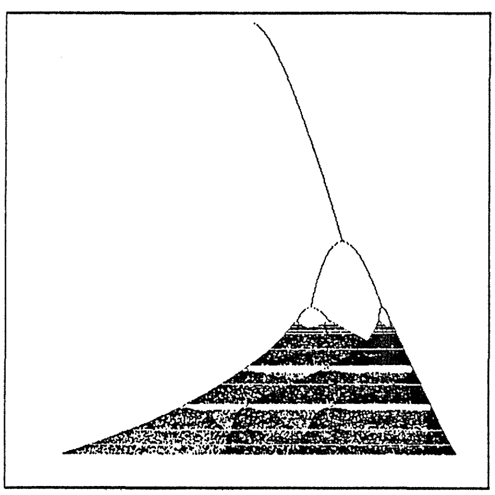

Figure 9: Bifurcation diagram of family $\{g_a(x)\}_a$: $-2 < x < 2$, $0 < a < 2$

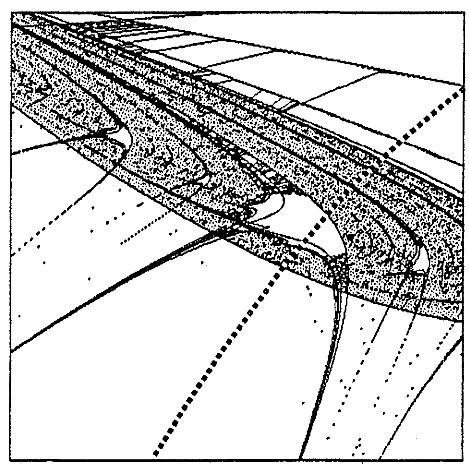

Figure 10: Real moduli space with a center curve BC1: $0 < \sigma_1 < 4$, $-60 < \sigma_3 < 0$

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NACA98

INTERNATIONAL CONFERENCE ON

NONLINEAR ANALYSIS
AND
CONVEX ANALYSIS

Niigata Washington Hotel, Niigata, Japan

July 28–31, 1998

CONFERENCE PROGRAM

Niigata University, Tokyo Institute of Technology, Hirosaki University

❑ Welcome

On behalf of the organizing committee of NACA98, it gives us much pleasure to welcome you to the International Conference on Nonlinear Analysis and Convex Analysis held at Niigata Washington Hotel in Niigata, Japan. We hope and believe that each participant in this meeting will have good opportunities not only to exchange scientific information on generic and applied areas of the conference but also to establish and deepen our friendship.

This conference has several unique features and to held this meeting is very important for development of mathematical science and its related area all over the world. During the last three decades, the study of Nonlinear Analysis has been devoted vigorously and such activity had great influence on other areas of science as much as mathematics. At the same time, Convex Analysis has grown in connection with the study of problems of optimization, equilibrium, control, and stability of linear and nonlinear systems. These two mathematical disciplines have no border and they rather have good effects each other. Now, we are requested and asked for new ideas and human technology based on reliable and theoretical methodology. We believe that world-wide experts on nonlinear analysis and convex analysis can answer reasonable and optimal solutions for such problems if they cooperate with each other. Accordingly, we organize this conference.

In organizing this conference we have been very aware of the excellent standard of several recent conferences and symposiums all over the world. When we attended other conferences, we had aspiration to held an international meeting for our topics in Japan. Then, we decided to organize this conference and to locate it in Niigata where is in the middle of Japan and also the entrance the rim of Japan sea.

The program of the conference consists of invited papers and contributed papers in six plenary sessions, eleven organized sessions, and regular parallel sessions. We are very happy to have 85 papers and to arrange 30 sessions in this conference, and also to have 30 people and their families from abroad. We have been lucky to receive support and encouragement from many individuals and it will be literally impossible for us to thank everybody.

We are indebted to all, in particular the members of the organizing committee, who have supported us in planning and implementing this conference. First, we wish to thank Professor Kensuke Tanaka of Niigata University, who has encouraged us to organize this conference. He has also given us precious advice and ideas. Also, we are very grateful to Professor Daishi Kuroiwa of Shimane University, graduate students Yutaka Kimura and Yoichi Sawasaki of Niigata University, who have accepted immediately to take an active part in the organization and have worked hardly for this conference. My gratitude also goes to the various members of the committees for the time and effort they put into the promotion of the conference. Especially, students in Hirosaki University made great efforts for several preparations of the conference, and we thank them.

Finally, we would like to thank the different organizations which support and sponsor this conference. Firstly, we want acknowledge the strong supports of Niigata University, Tokyo Institute of Technology, and Hirosaki University. We also appreciate the financial support that Intelligent Cosmos Academic Foundation has given us, and last but not least Niigata Convention Bureau for its many help.

We hope we would succeed in creating an environment so that all participants can freely exchange their ideas and strike up friendships around the world. We wish you an exciting and fruitful conference.

高橋 渉

Wataru Takahashi
Chair, NACA98 Conference Organizing Committee

田中 環

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Bifurcations and Hyperbolic Components

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ABSTRACT: We give a defining equation of the boundary of the real cut of escape locus in the real moduli space for the real quadratic rational maps. And we discuss problems of non-monotone bifurcations for two cases of one-parameter families: quadratic rational maps and cubic polynomials. We present counter examples by computer experiments to the monotonicity conjecture and the antimonotonicity conjecture.

Key words: chaotic dynamical systems - bifurcations - moduli space - real algebraic varieties - hyperbolic components.

AMS Subject Classification: 30C15, 39B12, 58D19, 58F14

1 Introduction

This paper has two parts. First one is concerned with geometry of the space $\mathrm{Rat}_2(\mathbf{C})$ of the quadratic rational maps, namely its moduli space $\mathcal{M}_2(\mathbf{C})$, consisting of all holomorphic conjugacy classes of maps, which can be described as an orbifold whose underlying space is isomorphic to $\mathbf{C}^2$, and having a natural compactification, isomorphic to the projective plane $\mathbf{CP}^2$. Maps which are hyperbolic on their Julia set give rise to hyperbolic components in the moduli space. M. Rees shows in ([12]) that the hyperbolic components can be divided into four classes: type B,C,D and E. There is just one hyperbolic component of type E in $\mathcal{M}_2(\mathbf{C})$, so-called (hyperbolic) escape component, consisting of maps with totally disconnected Julia set. This component has a more complicated topology. If we work in the real space, there are just two escape components in the real moduli space $\mathcal{M}_2(\mathbf{R})$, namely two real slices. We call these loci upper escape locus and lower escape locus. For a map in the upper escape locus with two real critical points, its real dynamics is completely trivial: the compactified real line converges to the real fixed point under iterations. Milnor gives in ([6]) a defining equation of this boundary. On the other hand, a map in the lower escape locus has complicated real dynamics. We will give a defining equation of the boundary of this lower escape locus, as Theorem 1 in section 2.2.

The second one is concerned with some topics from the bifurcation problems for a one parameter real family of quadratic rational maps or of cubic polynomials.

System of iterated maps, viewed as real dynamical systems is considered as an important model for the chaotic behavior in certain parameterized systems. Creation and annihilation of periodic orbits is one of the most fundamental bifurcation processes, often illustrated by the pitchforks oriented either one-way or both-ways. J. Milnor and W. Thurston ([7]) proved by using Teichmüller theory that the logistic family $\{\lambda x(1-x)\ ;\ \lambda \in [1,4]\}$, which is a family of simple maps with extremely complicated dynamics, has only orbit-creation parameter values and no orbit-annihilation values as the parameter increases. Unlike monotonicity of the logistic family, however, there exist many one-parameter families exhibiting a non-monotone orbit-bifurcation structure, namely the pitchforks oriented both-ways.

We discuss monotonicity conjecture (M) in several papers, now reformulated as follows ([11]): (M) *Let* $f_m(x) = mf(x)$ *be a one-parameter family of differential maps from closed interval* I_m *into itself which satisfies the following properties:* (1) f_m *is concave on* I_m, (2) *the set of periodic points of* f_1 *consists of two fixed points,* (3) f_m *has a negative schwarzian derivative. As the parameter* m *is increased, this one-parameter family is monotone.*

We consider a family $\{mf(x)\}$, where $f(x) = r + \frac{x}{1+x^2}$. The bifurcation diagram of this family can be monotone, non-monotone, or antimonotone according to the choice of the function f, namely the choice of r (cf. [4]).

Our method of approach to a bifurcation problem is to analyze an algebraic curve, defined by one-parameter family in the moduli space associated of a family, e.g., we examine "which hyperbolic locus does the curve lie in?" or "which dynamical curves does the curve intersect with?"

To the monotonicity conjecture, we will give a counter example using the defining equation of the lower escape locus, obtained in the section 2.2.

Next, we present a counter example to the antimonotonicity conjecture (A), enounced in the paper ([2]) with their heuristic argument and numerical evidence: (A) *A smooth one-dimensional map depending on one parameter has an antimonotone parameter value whenever at least two independent critical points are contained in the interior of a chaotic attractor.* Hereafter we call the part *"at least two independent critical points are contained in the interior of a chaotic attractor"*, anti-condition:(Anti). To construct a one-parameter family under (Anti), having no antimonotone parameter value, we use an algebraic curve, so-called center curve defined in our papers ([10], [3]), in the moduli space of the cubic maps with the multiplier-coordinates system.

2 Quadratic rational maps

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\mathrm{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\mathrm{Rat}_2(\mathbf{C})$ by conjugation, $g \circ f \circ g^{-1} \in \mathrm{Rat}_2(\mathbf{C})$ for $g \in \mathrm{PSL}_2(\mathbf{C})$, $f \in \mathrm{Rat}_2(\mathbf{C})$. The quotient space of $\mathrm{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the moduli space of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f. The multipliers coordinates are introduced in $\mathcal{M}_2(\mathbf{C})$. For each $f \in \mathrm{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be the fixed points of f and μ_i the multipliers of z_i; $\mu_i = f'(z_i)$ $(1 \leq i \leq 3)$. Consider the elementary symmetric functions of the three multipliers, $\sigma_1 = \mu_1 + \mu_2 + \mu_3$, $\sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1$, $\sigma_3 = \mu_1\mu_2\mu_3$, which are subject only to the restriction that $\sigma_3 = \sigma_1 - 2$. Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ (Lemma 3.1 in [6]). Let $\mathrm{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. We remark that the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\mathrm{Rat}_2(\mathbf{R})$ is the real cut of $\mathcal{M}_2(\mathbf{C})$ (see [4]).

By an automorphism of a quadratic rational map f, we will mean $g \in \mathrm{PSL}_2(\mathbf{C})$ which commutes with f. The collection $\mathrm{Aut}(f)$ of all automorphisms of f forms a finite group. Since $\mathrm{Aut}(\tilde{f})$ is isomorphic to $\mathrm{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$, the set

$$\mathcal{S} = \{\langle f \rangle\,;\ \mathrm{Aut}(f) \text{ is non-trivial}\} \subset \mathcal{M}_2(\mathbf{C})$$

is defined and called the symmetry locus.

For each $\mu \in \mathbf{C}$, let $\mathrm{Per}_1(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps f having a fixed point with multiplier μ. Each of $\mathrm{Per}_1(\mu)$ forms a straight line as follows:

$$\mathrm{Per}_1(\mu) = \left\{\langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1})\right\}$$

(Lemmas 3.4 and 3.6 in [6]).

2.1 Topological partition

For map $f \in \mathrm{Rat}_2(\mathbf{R})$, the two critical points of f are two real numbers or a pair of complex conjugate numbers. If f has a pair of complex conjugate critical points, this map is two-to-one covering map on $S^1 = \mathbf{R} \cup \{\infty\}$. In this case, if $f' > 0$ then f is called the map of degree $+2$, else $f' < 0$ then the map of degree -2.

While a map f with real critical points is called monotone (resp. unimodal, bimodal) if the interval $I = int(f(S^1))$ contains no (resp. one, two) critical points ([6]).

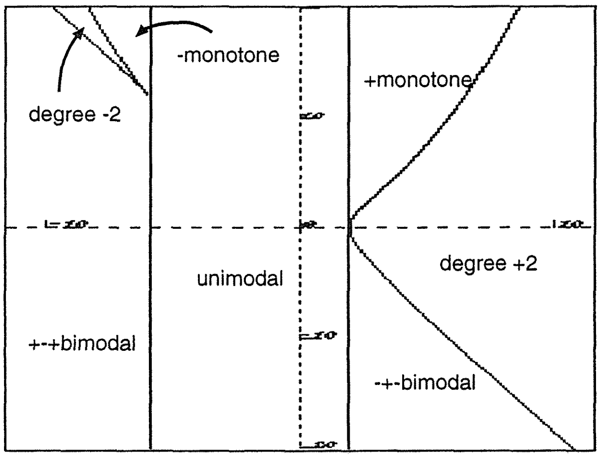

Figure 1: The topological partition of the moduli space of real quadratic rational maps. These regions are bounded by the real cut of the symmetry locus and two lines $\sigma_1 = 2, 6$.

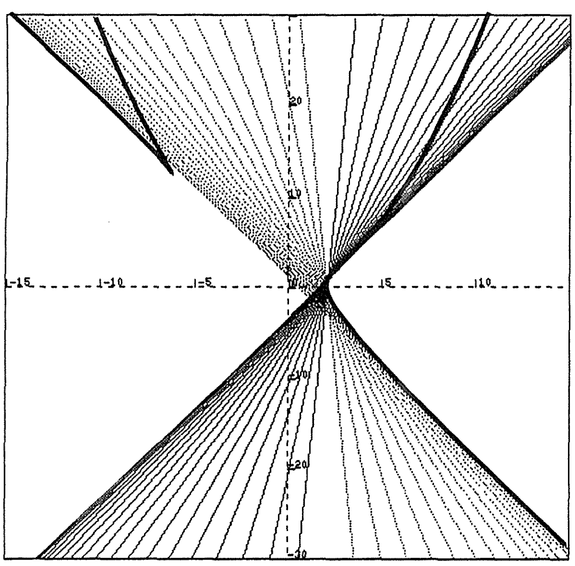

Figure 2: Lines $\mathrm{Per}_1(\mu)$: gray lines show $-1 < \mu < 0$, black lines show $0 < \mu < 1$ and thick curve shows the symmetry locus.

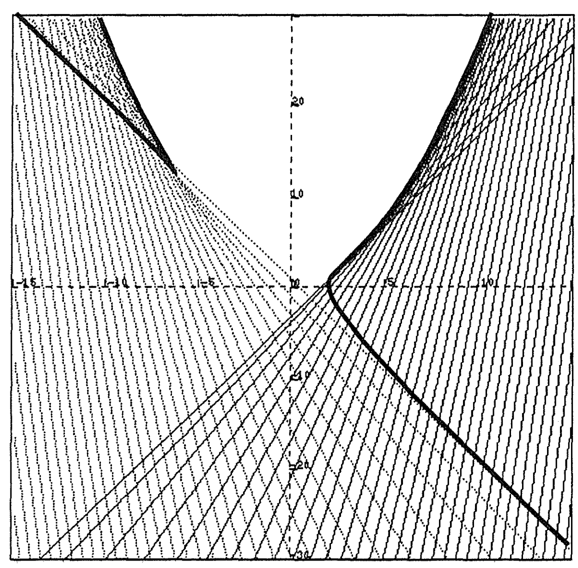

Figure 3: Lines $\mathrm{Per}_1(\mu)$: gray lines show $\mu < -1$, black lines show $\mu > 1$ and thick curve shows the symmetry locus.

2.2 Real slices of hyperbolic escape locus

A rational map is hyperbolic if and only if the orbit of every critical point converges to some attracting periodic orbit. The hyperbolic maps form an open subset of moduli space, and the connected components of this open set are called hyperbolic components. M. Rees ([12]) shows that the hyperbolic components can be divided into four classes as follows. The names are due to J. Milnor ([6]).

Type B: Bitransitive. Each of the two critical points belongs to the immediate basin of some attracting periodic point, where these two periodic points are distinct but belong to the same orbit. Evidently the period must be two or more.

Type C: Capture. Only one critical point belongs to the immediate basin on a periodic point, but the orbit of the other critical point eventually falls into this immediate basin. Again the period must be two or more.

Type D: Disjoint attractors. The two critical points belong to the attracting basins for two disjoint attracting periodic orbits.

Type E: Escape. Both critical orbits converge to the same attracting fixed point. There is just one such hyperbolic component.

In the complex case the escape locus is connected. But the real cut of this component splits into two parts; the upper part and the lower part. The boundary curve of the upper part is given by Milnor (Caption of Figure 16 in [6]).

Now, we specify the lower boundary. This boundary curve will play a key role in our later discussions of section 3.

Theorem 1 *Escape loci on the real moduli space is the union of the following sets;*

$$\begin{aligned}&\{\sigma_2 > -2\sigma_1 + 1, \quad \sigma_2 > 2\sigma_1 - 3\},\\ &\{\sigma_2 < 2\sigma_1 - 3, \quad \sigma_1 < -1\},\\ &\{\sigma_2 < \tfrac{-2\sigma_1^2 - 7\sigma_1 - 10}{2+\sigma_1}, \quad \sigma_1 \geq -1\}.\end{aligned}$$

Proof. Here, we use the following real two-parameter family of quadratic rational maps induced by M. Bier and T. C. Bountis ([1]) and rewritten by H. E. Nusse and J. A. Yorke ([11]);

$$\left\{ f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}.$$

This family covers the real moduli space $\{(\sigma_1, \sigma_2)\}$ expect for the degree ± 2 regions and the half line $\{(\sigma_1, \sigma_2)\,;\, \sigma_1 = 2, \sigma_2 < -1\}$ of the quadratic polynomial region (See [9]). Since the maps $f_{m,r}$ and $f_{m,-r}$ are conjugate to each other for any r, it suffices to consider the case $r \geq 0$. We note that the algebraic curve defined by one-parameter family $\{f_{m,0}\}$ coincides with the boundary curve of degree ± 2 regions.

The fixed pints of $f_{m,r}$ are the three roots of equation,

$$x^3 - mrx^2 - (m-1)x - mr = 0.$$

Two critical points of this map are ± 1 and its critical values are $mr \pm \frac{m}{2}$. Therefore this family can not cover the degree ± 2 regions.

The denominator of map $f_{m,r}$ is always positive, and

$$\lim_{x\to\pm\infty} f_{m,r} = mr,$$

$$f_{m,r}(0) = mr.$$

Hence the horizontal line $y = mr$ is unique asymptotic line of this map. For the case of $m > 0$ (resp. $m < 0$), graph is $(-+-)$ (resp. $(+-+)$), and it is sufficient to consider the dynamics on the closed interval $[mr - \frac{m}{2}, mr + \frac{m}{2}]$ (resp. $[mr + \frac{m}{2}, mr - \frac{m}{2}]$).

From the graphical analysis it is clear that a map $f_{m,r}$ belongs to the escape if and only if $f_{m,r}$ satisfying one of the following condition.

1. In $\pm$ monotone - regions, $f_{m,r}$ has only one attracting fixed point.

2. In unimodal- region,

   - $f_{m,r}$ do not have real fixed point except for an attracting fixed point with multiplier $-1 < \mu < 0$, or
   - $f_{m,r}$ has an attracting fixed point with multiplier $0 < \mu < 1$ and two repelling fixed points, and its two minimal intervals containing each critical orbits has intersection.

3. In $(-+-)$-bimodal- region, $f_{m,r}$ has an attracting fixed point and two repelling fixed points, and its two minimal intervals containing each critical orbits has non-trivial intersection.

4. In $(+-+)$-bimodal - region, $f_{m,r}$ has an attracting fixed point and two repelling fixed points.

In the parameter space $\{(r,m)\}$, after specifying "escape"- regions in each cases, we map these regions to the real moduli space by using transformation formula:

$$\begin{cases} ST1 : 4m^2r^2 - m^2 + (\sigma_1+2)m - 4 = 0, \\ ST2 : -4m^4r^4 + (m^4 - 12m^3 - 8m^2)r^2 + 2m^3 + (\sigma_2 - 5)m^2 + 4m - 4 = 0. \end{cases}$$

For example, the escape region corresponding to the above condition 3 ($(-+-)$-bimodal case) is given as the condition $f^2_{m,r}(-1) < f^3_{m,r}(-1)$, translated into an inequality $N_{m,r}/D_{m,r} > 0$:

$$\begin{aligned} N_{m,r} &= m^2(-2mr+m-2)^3(4m^3r^3 - 4(m+1)m^2r^2 + (m^2+6m+4)mr - 2m^2 - 2m - 4) \\ D_{m,r} &= (m^2(2r-1)^2+4)(16m^6r^6 - 32m^6r^5 + 8(3m^2+4m+6)m^4r^4 \\ &\quad -8(m^2+6m+8)m^4r^3 + (m^4+24m^3+48m^2+32m+48)m^2r^2 \\ &\quad -4(m^3+6m^2+4m+8)m^2r + 5m^4 + 8m^2 + 16). \end{aligned}$$

From calculation we can see the factor $(-2mr+m-2)$ of $N_{m,r}$ and the second factor of $D_{m,r}$ are always positive. Therefore the condition is finally reduced to an inequality:

$$4m^3r^3 - 4(m+1)m^2r^2 + (m^2+6m+4)mr - 2m^2 - 2m - 4 > 0.$$

Mapping this parameter region to the moduli space by transformation formula, we have

$$2\sigma_1^2 + (\sigma_2+7)\sigma_1 + 2\sigma_2 + 10 < 0, \quad \sigma_1 > 2.$$

For the other cases, we can get region of "escape" in the same way.

We conclude this proof by treating the region where the family $\{f_{m,r}\}$ do not cover.

For the case of degree ± 2 regions, a map belongs to the escape if and only if the map has unique attracting fixed point (See Caption of Figure 16 of [6]).

A quadratic polynomial family $\{x^2 + \sigma_2/4\}_{\sigma_2}$ coincides with the line $\{(2,\sigma_2)\}$ on the moduli space. Hence two half line $\{(\sigma_1,\sigma_2)\,;\,\sigma_1 = 2, \sigma_2 < -8\}$, $\{(\sigma_1,\sigma_2)\,;\,\sigma_1 = 2, \sigma_2 > 1\}$ belong to escape loci. ∎

3 Bifurcations of one parameter family

3.1 Bifurcations

Let $\{f_\lambda\}_\Lambda$ be a one-parameter family of discrete dynamical systems on $\mathbf{R}$ where Λ is an interval of $\mathbf{R}$. As the parameter increases, a parameter value λ_0 is called orbit creating if, at λ_0, new periodic orbits are created and no periodic orbits are annihilated; λ_0 is called orbit annihilating if periodic orbits are annihilated and no new periodic orbits are created; λ_0 is called neutral if no periodic orbits are annihilated and no periodic orbits are created.

A family $\{f_\lambda\}_\Lambda$ is said to be monotone increasing (resp. decreasing) if every parameter value in Λ is neutral or orbit creating (resp. annihilating). A family $\{f_\lambda\}_\Lambda$ is called non-monotone if Λ contains both orbit creating and orbit annihilating parameter values. A family $\{f_\lambda\}_\Lambda$ is called antimonotone if any neighborhood of a suitable parameter λ_0 in Λ contains both infinitely many orbit creating and orbit annihilating parameter values.

3.2 Counter example to monotonicity conjecture

Now, we investigate the dynamics of the following real two-parameter again:

$$\left\{ f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}.$$

Here the map $f_{0,r}(x)$ should be thought of as an ideal limit map, in the natural compactification of $\mathcal{M}_2(\mathbf{C})$ (cf. [6]), of quadratic rational maps which degenerate towards the constant zero map. Then it makes sense to discuss the bifurcations of this family including the parameter value $m = 0$, though in the real moduli space $\mathcal{M}_2(\mathbf{R})$ the maps diverge to infinity according as $m \to \pm 0$.

Proposition 1 *In $\mathcal{M}_2(\mathbf{R})$, the one-parameter family $\{f_{m,r}(x)\}_m$ for each fixed r ($r \geq 0$) lies exactly on an irreducible algebraic curve $\mathcal{H}_r$:*

For $r \neq \frac{1}{2}, 0$, the curve $\mathcal{H}_r$ is of degree 4 defined by the equation

$$\begin{aligned} H_r(\sigma_1,\sigma_2) \;=\; & -r^2\sigma_1^4 + (8r^2-2)\sigma_1^3 + ((8r^2-1)\sigma_2 - 128r^4 + 8r^2 + 1)\sigma_1^2 \\ & +((-32r^2+8)\sigma_2 + 512r^4 - 96r^2 - 12)\sigma_1 + (-16r^2+4)\sigma_2^2 \\ & +(512r^4 - 96r^2 - 12)\sigma_2 - 4096r^6 + 1536r^4 - 144r^2 + 36 = 0. \end{aligned} \tag{1}$$

For $r = \frac{1}{2}$ or $r = 0$, the curve $\mathcal{H}_r$ is of degree 3.

The proof is given in our paper ([4]).

Example For $r = 0.58$, the one parameter family $\{f_{m,0.58}\}_m$ is non-monotone. More precisely, this bifurcation diagram is so-called primary bubbling ([4]).

Example Counter example to the monotonicity conjecture
Consider the one-parameter family

$$F_m(x) = m\frac{x^2+ax+b}{1+x^2},$$

where constant a is the positive root of the following equation

$$49a^2 - 32 = 0,$$

and b is the unique positive root of the following equation

$$\begin{gathered} 117649b^7 + 684285b^6 + 1721517b^5 + 2358566b^4 \\ +1670655b^3 + 991301b^2 - 257125b = 0, \end{gathered}$$

define on a suitable interval I_m. Then (1) each F_m is concave, (2) the set of periodic points in I_1 of F_1 consists of two fixed points, and (3) F_m has a negative schwarzian derivative.

A defining equation of algebraic curve by this family is given as follows. We remark that this algebraic curve tangent to the lower boundary of escape loci and antimonotone bifurcation occurs at this tangent point.

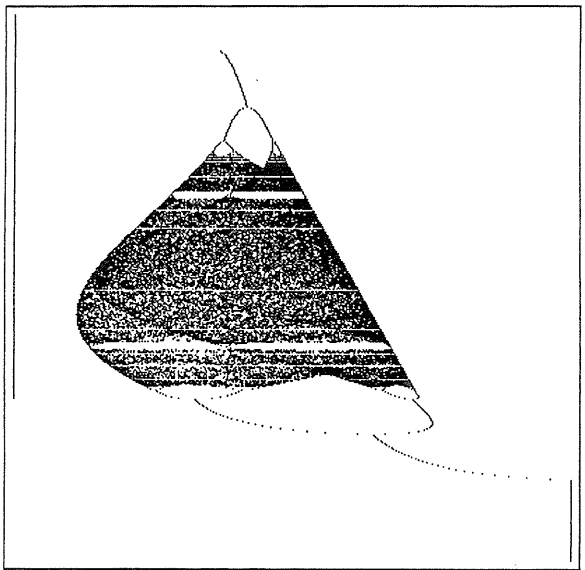

Figure 4: Bifurcation diagram of family $\{F_m(x)\}_m$: $-0.8 < x < 0.2$, $-25 < m < 5$

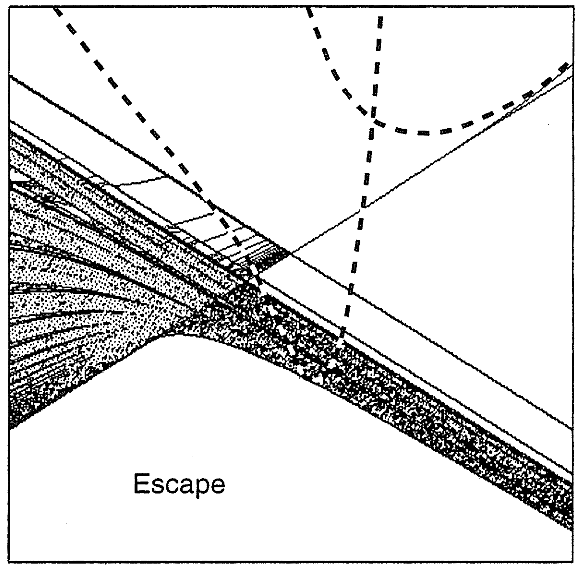

Figure 5: Real moduli space with an algebraic curve defined by $\{F_m\}_m$: $-3 < \sigma_1 < 5$, $-15 < \sigma_2 < 10$

$$\begin{aligned}S_{a,b} = {} & (2\sigma_1^3 + (\sigma_2 - 1)\sigma_1^2 + (-8\sigma_2 + 12)\sigma_1 - 4\sigma_2^2 + 12\sigma_2 - 36)a^6 + ((2\sigma_1^3 + (\sigma_2 + 24)\sigma_1^2 + \\ & (12\sigma_2 + 72)\sigma_1 + 36\sigma_2)b^2 + (-14\sigma_1^3 + (-6\sigma_2 - 20)\sigma_1^2 + (32\sigma_2 + 24)\sigma_1 + 16\sigma_2^2 + 24\sigma_2 + \\ & 144)b + \sigma_1^4 + 4\sigma_1^3 + (-3\sigma_2 - 12)\sigma_1^2 - 12\sigma_2\sigma_1 + 36\sigma_2)a^4 + ((-10\sigma_1^3 + (-4\sigma_2 - 132)\sigma_1^2 + \\ & (-48\sigma_2 - 504)\sigma_1 - 144\sigma_2 - 432)b^3 + (2\sigma_1^4 + 46\sigma_1^3 + (4\sigma_2 + 188)\sigma_1^2 + (-16\sigma_2 - 216)\sigma_1 - \\ & 240\sigma_2 - 720)b^2 + (-4\sigma_1^4 - 30\sigma_1^3 + (4\sigma_2 + 84)\sigma_1^2 + (48\sigma_2 + 152)\sigma_1 - 112\sigma_2 - 336)b + \\ & 2\sigma_1^4 - 6\sigma_1^3 + (-4\sigma_2 - 12)\sigma_1^2 + (16\sigma_2 + 56)\sigma_1 - 16\sigma_2 - 48)a^2 + (\sigma_1^4 + 24\sigma_1^3 + 216\sigma_1^2 + \\ & 864\sigma_1 + 1296)b^4 + (-4\sigma_1^4 - 64\sigma_1^3 - 288\sigma_1^2 + 1728)b^3 + (6\sigma_1^4 + 48\sigma_1^3 - 48\sigma_1^2 - 576\sigma_1 + \\ & 864)b^2 + (-4\sigma_1^4 + 96\sigma_1^2 - 256\sigma_1 + 192)b + \sigma_1^4 - 8\sigma_1^3 + 24\sigma_1^2 - 32\sigma_1 + 16 = 0.\end{aligned}$$

3.3 Counter example to monotonicity conjecture: cubic polynomial family

Let $\mathrm{Poly}_3(\mathbf{C})$ be the space of all cubic polynomials from $\mathbf{C}$ to itself. The quotient space of $\mathrm{Poly}_3(\mathbf{C})$ will be denoted by $\mathrm{M}_3(\mathbf{C})$, and called the moduli space of holomorphic conjugacy classes $\langle p\rangle$ of cubic polynomials p. For each $p \in \mathrm{Poly}_3(\mathbf{C})$, let $z_1,\ z_2,\ z_3,\ z_4(=\infty)$ be the fixed points of p and μ_i the multipliers of z_i; $\mu_i = p'(z_i)$ $(1 \le i \le 3)$, and $\mu_4 = 0$. Consider the elementary symmetric functions of the four multipliers,

$$\begin{aligned}\sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 = \mu_1 + \mu_2 + \mu_3\\ \sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3\\ \sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 = \mu_1\mu_2\mu_3\\ \sigma_4 &= \mu_1\mu_2\mu_3\mu_4 = 0.\end{aligned}$$

These multipliers determine uniquely p up to holomorphic conjugacy, and are subject only to the restriction that $3 - 2\sigma_1 + \sigma_2 = 0$. Now an affine structure is imposed on $\mathrm{M}_3(\mathbf{C})$ by this multipliers coordinate system (σ_1, σ_3).

A map in $\mathrm{Poly}_3(\mathbf{C})$ is always conjugate to a map of the normal form $z^3 + az + b$, and its parameters (a, b^2) is used as a coordinate system of $\mathrm{M}_3(\mathbf{C})$ which is isomorphic to $\mathbf{C}^2$ ([5]). These coordinates relate to (σ_1, σ_3) as follows:

$$\begin{aligned}\sigma_1 &= -3a + 6,\\ \sigma_3 &= 27b^2 + a(2a-3)^2,\end{aligned} \tag{2}$$

Let $\mathrm{Poly}_3(\mathbf{R})$ be the set of real cubic polynomials. We simply define the real moduli space $\mathrm{M}_3(\mathbf{R})$ for $\mathrm{Poly}_3(\mathbf{R})$ as the real (σ_1, σ_3)-plane.

3.3.1 Monotone bifurcations of cubic polynomials

The one-parameter family $f_\lambda(x) = -x^3+1.2675x-\lambda$, defined in ([2]), is antimonotone under (Anti), and lies exactly on a half line $\sigma_1 = -3.8025$ in $M_3(\mathbf{R})$. On the other hand, we can present a set BC1: $\sigma_3 = -\frac{8}{3}(\sigma_1 - 6)^2$, of classes of the maps one of the whose two critical points maps to another one (see [10]). The set BC1 corresponds to the one parameter family $g_a(x) = -x^3+ax+(1+\frac{2}{3}a)\sqrt{\frac{a}{3}}$. We can show via Thurston's rigidity theorem that this family is monotone (naturally not antimonotone) under (Anti). We conclude that this family is a counter example to the conjecture (A) in ([2]).

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FOURTH INTERNATIONAL CONFERENCE ON DIFFERENCE EQUATIONS AND APPLICATIONS
AUGUST 27 – 31, 1998,
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Poznań, 1998.03.31.

Professor
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Dear Professor Kiyoko Nishizawa,

It is my pleasure to inform you that the paper

"Chaotic bifurcations along algebraic curves."

has been accepted for oral presentation in the 4th International Conference on Difference Equations and Applications which will be held on August 27 - 31, 1998, in Poznań, Poland.

The exact time and place of your presentation will be given when the Third Announcement reaches you (by July the 1st).

We are happy to say that many applications have been sent to us. At the same time we are sorry to say that it limited the time we can offer for presentations. That's why at the moment we leave 20 minutes only for your presentation and the discussion on your talk.

We are kindly inviting you to send us (by July the 1st, 1998) the extended abstract (up to 4 pages) of your presentation. It has to be prepared according to the demands stated in the **Guidelines for Preparation of Manuscript** which are attached to this letter. This extended abstract will be included into the conference materials offered to every participant on his/her arrival.

Prof. Jerzy Popenda
Chairman of the Scientific Committee

{kargajda.doc}

Preface

One of the main questions in nowadays science is the following : are the Earth, the Universe discrete or continuous, finite or infinite. The answer to these questions yields several consequences. In fact, each artificial world created by computers is discrete and finite. Therefore there is finite number of functions which can describe the behaviour of each cell, each creature of this world. Consequently mathematics says: the behaviour of this world is periodical. People has been discretizing their world since the beginning. That is impossible to observe continuous area of the space. Supposing we want to describe some real phenomena we are observing, we have to omit several parameters, relations and even unknowns, and we concentrate only on some (finite) of them. Then we build relations, functions, next phase state, examine this system and check its validity in the nature, in the real life. We say we have built a model. Difference equations are found one of the very useful tools in this creation process. They appeared essentially earlier than their „great brother" (G. Ladas) - differential equations. It is only recently that they have started gaining the attention they deserve, although they are applied in probability theory, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, graph theory, electrical networks, physics, genetics in biology, automata theory, economics, psychology, sociology, etc. Perhaps the new time for our equations has been opened together with the appearance of the first computers and numerical methods where recurrence algorithms, discretizations of differential equations and other discrete methods are widely used. Large spectrum of the applications mentioned above, of the methods and pure difference equations problems is announced in this book. It contains selected abstracts of the lectures presented on the ICDEA98, The Fourth International Conference on Difference Equations and Applications, which took place in Poznań, 27-31 August 1998. An idea of this conference was born in 1994 during The First International Conference on Difference Equations which was held at Trinity University in San Antonio, Texas, USA. The next two took place in Veszprem, Hungary (1995), and Taipei, Taiwan (1997). We hope that this one will be at least such successful as previous, and the idea will move to the future.

J.Popenda

CHAOTIC BIFURCATIONS ALONG ALGEBRAIC CURVES

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ABSTRACT We discuss problems of non-monotone bifurcations for two cases of one-parameter families: rational maps and cubic maps. At first, we present a counter example to the conjecture (A) in [2]. And then we discuss a non-monotonicity problem presented in several papers, by analyzing an algebraic curve defined by the given family in these moduli spaces.

1. INTRODUCTION

System of iterated maps, viewed as real dynamical systems, is considered as an important model for the chaotic behavior in certain physical, chemical and biological systems. Creation and annihilation of periodic orbits is one of the most fundamental bifurcation processes in a parameterized family of maps, often illustrated by the pitchforks oriented either one-way or both-ways. For example, the logistic family $\{\lambda x(1-x) \; ; \; \lambda \in [1,4]\}$, which is a family of simple maps with extremely complicated dynamics, serves as a population growth model in theoretical population dynamics. J. Milnor and W. Thurston ([8]) proved by using Teichmüller theory that the logistic family has only orbit-creation parameter values and no orbit-annihilation values as the parameter increases. Unlike monotonicity of the logistic family, however, there exist many one-parameter families exhibiting a non-monotone orbit-bifurcation structure.

We discuss in this paper problems of non-monotone bifurcations for two cases of one-parameter families: cubic maps and rational maps.

First, we present a counter example to the antimonotonicity conjecture (A), enounced in the paper [2] with their heuristic argument and numerical evidence:

> (A) *A smooth one-dimensional map depending on one parameter has an antimonotone parameter value whenever at least two independent critical points are contained in the interior of a chaotic attractor.*

Hereafter we call the part *"at least two independent critical points are contained in the interior of a chaotic attractor"*, anti-condition: (Anti).

To construct a one-parameter family under (Anti), having no antimonotone parameter value, we use an algebraic curve, so-called center curve defined in our papers ([11], [12], [3]), in the moduli space of the cubic maps with the multiplier-coordinates system.

Second, we discuss a monotonicity problem presented as monotonicity conjecture (M) in several papers, now reformulated as follows [13]:

> (M)*Let* $f_m(x) = mf(x)$ *be a one-parameter family of differential maps from closed interval* I_m *into itself which satisfies the following properties:*
> *(1)* f_m *is concave on* I_m*,*
> *(2) the set of periodic points of* f_1 *consists of two fixed points,*
> *(3)* f_m *has a negative schwarzian derivative.*
> *As the parameter* m *is increased, this one-parameter family is monotone.*

We consider a family $\{mf(x)\}$, where $f(x) = r + \frac{x}{1+x^2}$. The bifurcation diagram of this family can be monotone, non-monotone, or antimonotone according to the choice of the function f, namely the choice of r (cf. [4]). We explain these variations of bifurcation processes. Our method of approach to the bifurcation problems of these cases is to analyze an algebraic curve, defined by one-parameter family in its moduli space.

2. BIFURCATIONS

Let $\{f_\lambda\}_\Lambda$ be a one-parameter family of discrete dynamical systems on $\mathbf{R}$ where Λ is an interval of $\mathbf{R}$. As the parameter increases, a parameter value λ_0 is called orbit creating if, at λ_0, new periodic orbits are created and no periodic orbits are annihilated; λ_0 is called orbit annihilating if periodic orbits are annihilated and no new periodic orbits are created; λ_0 is called neutral if no periodic orbits are annihilated and no periodic orbits are created.

A family $\{f_\lambda\}_\Lambda$ is said to be monotone increasing (resp. decreasing) if every parameter value in Λ is neutral or orbit creating (resp. annihilating). A family $\{f_\lambda\}_\Lambda$ is called non-monotone if Λ contains both orbit creating and orbit annihilating parameter values. A family $\{f_\lambda\}_\Lambda$ is called antimonotone if any neighborhood of a suitable parameter λ_0 in Λ contains both infinitely many orbit creating and orbit annihilating parameter values.

3. COUNTER EXAMPLE TO THE ANTIMONOTONICITY CONJECTURE

In this section we shall present a counter example, which is a one parameter family of cubic polynomials, to the antimonotonicity conjecture enounced in the paper [2].

3.1 Moduli space of cubic polynomials

Let $\mathrm{Poly}_3(\mathbf{C})$ be the space of all cubic polynomials from $\mathbf{C}$ to itself. The group $\mathrm{Poly}_1(\mathbf{C})$ of affine transformations acts on the space $\mathrm{Poly}_3(\mathbf{C})$, by conjugation, $g \circ p \circ g^{-1} \in \mathrm{Poly}_3(\mathbf{C})$ for $g \in \mathrm{Poly}_1(\mathbf{C})$, $p \in \mathrm{Poly}_3(\mathbf{C})$. Two maps $p_1, p_2 \in \mathrm{Poly}_3(\mathbf{C})$ are holomorphically conjugate, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \mathrm{Poly}_1(\mathbf{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\mathrm{Poly}_3(\mathbf{C})$ under this action will be denoted by $\mathrm{M}_3(\mathbf{C})$, and called the moduli space of holomorphic conjugacy classes $\langle p \rangle$ of cubic polynomials p. For each $p \in \mathrm{Poly}_3(\mathbf{C})$, let $z_1,\ z_2,\ z_3,\ z_4 (= \infty)$ be the fixed points of p and μ_i the multipliers of z_i; $\mu_i = p'(z_i)$ $(1 \le i \le 3)$, and $\mu_4 = 0$. Consider the elementary symmetric functions of the four multipliers,

$$
\begin{aligned}
\sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 = \mu_1 + \mu_2 + \mu_3 \\
\sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 \\
&= \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\
\sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 = \mu_1\mu_2\mu_3 \\
\sigma_4 &= \mu_1\mu_2\mu_3\mu_4 = 0.
\end{aligned}
$$

These multipliers determine uniquely p up to holomorphic conjugacy, and are subject only to the restriction that $3 - 2\sigma_1 + \sigma_2 = 0$. Now an affine structure is imposed on $M_3(\mathbf{C})$ by this multipliers coordinate system (σ_1, σ_3). We remark that the singular part of this moduli space is given the following algebraic variety:

$$S_3(\sigma_1, \sigma_3) = 4\sigma_1^3 - 36\sigma_1^2 + 81\sigma_1 + 27\sigma_3 - 54 = 0. \tag{1}$$

A map in $\mathrm{Poly}_3(\mathbf{C})$ is always conjugate to a map of the normal form $z^3 + az + b$, and its parameters (a, b^2) is used as a coordinate system of $M_3(\mathbf{C})$ which is isomorphic to $\mathbf{C}^2$ ([6]). These coordinates relate to (σ_1, σ_3) as follows:

$$\begin{aligned} \sigma_1 &= -3a + 6, \\ \sigma_3 &= 27b^2 + a(2a-3)^2, \end{aligned} \tag{2}$$

Let $\mathrm{Poly}_3(\mathbf{R})$ be the set of real cubic polynomials. We simply define the real moduli space $M_3(\mathbf{R})$ for $\mathrm{Poly}_3(\mathbf{R})$ as the real (σ_1, σ_3)-plane.

3.2 One parameter cubic polynomials with Monotone bifurcations

The one-parameter family $f_\lambda(x) = -x^3 + 1.2675x - \lambda$, defined in [2], is antimonotone under (Anti). It turns out that this family exactly on a half line $\sigma_1 = -3.8025$ in the moduli space.

On the other hand, we can present a set BC1: $\sigma_3 = -\frac{8}{3}(\sigma_1 - 6)^2$, of classes of the maps one of whose two critical points maps to another one (see [11], [12]). The set BC1 corresponds to the one parameter family:

$$BC1: \quad g_a(x) = -x^3 + ax + (1 + \frac{2}{3}a)\sqrt{\frac{a}{3}}.$$

We can show with computer experiments that this family is monotone (naturally not antimonotone) under (Anti).

Recently we know that J. Milnor and Ch. Tresser also treat of this problem and they said in [9] that

> *The analogue of the Antimonotonicity Conjecture for the stunted sawtooth families is certainly false, since by 5.8, it is very easy to find smooth curves along which there are only orbit creations. Thus, if the conjecture is true for the cubic family, then any complexity preserving correspondence between the stunted sawtooth and cubic parameter triangles must be very wild indeed.*

We remark that the entropy of the family $\{f_\lambda\}_\lambda$ is not monotone but one of our family $\{g_a\}_a$ is monotone.

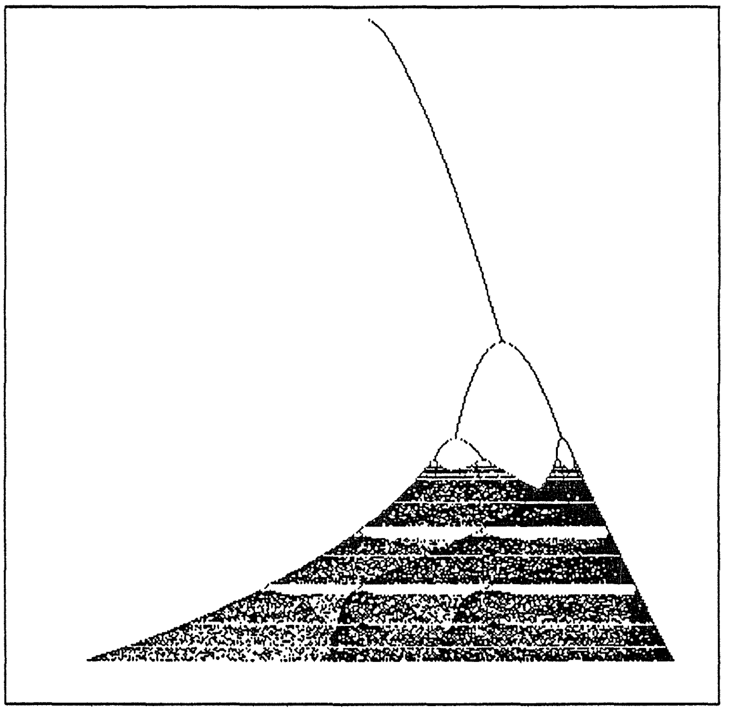

FIGURE 1. Bifurcation diagram of family $\{g_a(x)\}_a$: $-2 < x < 2$, $0 < a < 2$

4. COUNTER EXAMPLE TO THE MONOTONICITY CONJECTURE

In this section we shall present a counter example, which is a one parameter family of quadratic rational maps, to the monotonicity conjecture enounced in the paper [13].

4.1 Moduli space of quadratic rational functions

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\mathrm{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\mathrm{Rat}_2(\mathbf{C})$ by conjugation, $g \circ f \circ g^{-1} \in \mathrm{Rat}_2(\mathbf{C})$ for $g \in \mathrm{PSL}_2(\mathbf{C})$, $f \in \mathrm{Rat}_2(\mathbf{C})$. The quotient space of $\mathrm{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the moduli space of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f. The Multipliers coordinates are introduced in $\mathcal{M}_2(\mathbf{C})$ as in the case of cubic maps. For each $f \in \mathrm{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be the fixed points of f and μ_i the multipliers of z_i; $\mu_i = f'(z_i)$ $(1 \leq i \leq 3)$. Consider the elementary symmetric functions of the three multipliers, $\sigma_1 = \mu_1 + \mu_2 + \mu_3$, $\sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1$, $\sigma_3 = \mu_1\mu_2\mu_3$, which

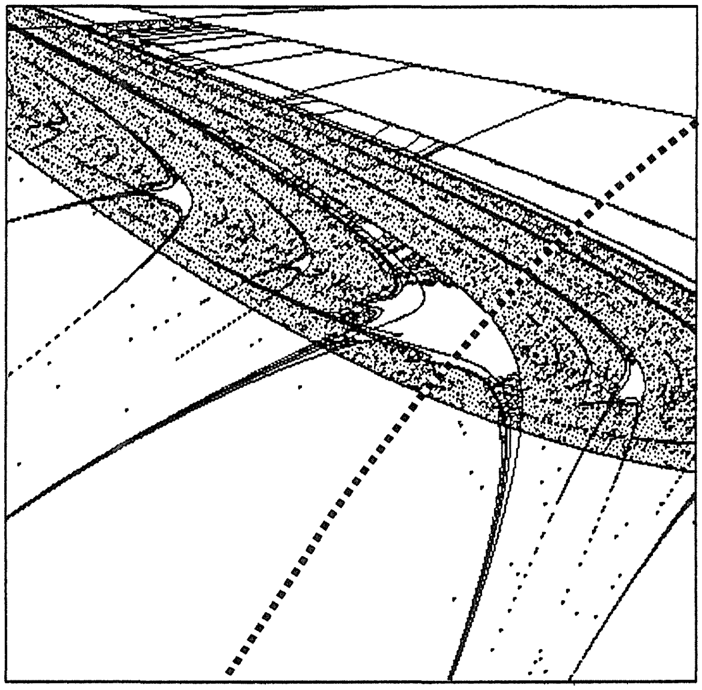

FIGURE 2. Real moduli space with a center curve BC1: $0 < \sigma_1 < 4$, $-60 < \sigma_3 < 0$

are subject only to the restriction that $\sigma_3 = \sigma_1 - 2$. Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ (Lemma 3.1 in [7]). Let $\mathrm{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. We remark that the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\mathrm{Rat}_2(\mathbf{R})$ is the real cut of $\mathcal{M}_2(\mathbf{C})$.

4.2 Hyperbolic components

A rational map is hyperbolic if and only if the orbit of every critical point converges to some attracting periodic orbit. The hyperbolic maps form an open subset of moduli space, and the connected components of this open set are called hyperbolic components. M. Rees ([14]) shows that the hyperbolic components can be divided into four classes: type B (Bitransitive), type C (Capture), type D (Disjoint), and type E (Escape). The names are due to J. Milnor ([7]).

The region escape consists of classes of maps which both critical orbits converge to the same attracting fixed point. This "escape" play an important role in bifurcation theory, because any bifurcation don't occurs in this region.

It is known that there is just one such hyperbolic component. But the real cut of this component splits into two parts. The boundary equation of upper part are given by Milnor. We specify in [10] the defining equation of lower boundary as follows;

$$\begin{aligned}\sigma_2 &= 2\sigma_1 - 3, & \sigma_1 < -1,\\ \sigma_2 &= \frac{-2\sigma_1^2 - 7\sigma_1 - 10}{2+\sigma_1}, & \sigma_1 \geq -1.\end{aligned}$$

4.3 Monotone and non-monotone bifurcations of quadratic rational families

Now, we investigate the dynamics of a certain real 2-parameter family given by M. Bier and T. C. Bountis [1] and rewritten by H. E. Nusse and J. A. Yorke ([13]):

$$\left\{ f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}.$$

Here the map $f_{0,r}(x)$ should be thought of as an ideal limit map, in the natural compactification of $\mathcal{M}_2(\mathbf{C})$ (cf. [7]), of quadratic rational maps which degenerate towards the constant zero map. Then it makes sense to discuss the bifurcations of this family including the parameter value $m=0$, though in the real moduli space $\mathcal{M}_2(\mathbf{R})$ the maps diverge to infinity according as $m \to \pm 0$. Since the maps $f_{m,r}$ and $f_{m,-r}$ are conjugate to each other for any r, it suffices to consider the case $r \geq 0$.

Theorem 1 *In $\mathcal{M}_2(\mathbf{R})$, the one-parameter family $\{f_{m,r}(x)\}_m$ for each fixed r $(r \geq 0)$ lies exactly on an irreducible algebraic curve $\mathcal{H}_r$:*
For $r \neq \frac{1}{2}, 0$, the curve $\mathcal{H}_r$ is of degree 4 defined by the equation

$$\begin{aligned}H_r(\sigma_1,\sigma_2) = {} & -r^2\sigma_1^4 + (8r^2-2)\sigma_1^3 + ((8r^2-1)\sigma_2 - 128r^4\\ & +8r^2+1)\sigma_1^2 + ((-32r^2+8)\sigma_2 + 512r^4 - 96r^2 - 12)\sigma_1\\ & +(-16r^2+4)\sigma_2^2 + (512r^4 - 96r^2 - 12)\sigma_2 - 4096r^6\\ & +1536r^4 - 144r^2 + 36 = 0.\end{aligned} \tag{3}$$

For $r = \frac{1}{2}$ or $r=0$, the curve $\mathcal{H}_r$ is of degree 3.

The proof is given in our paper [4].

Example (Primary bubbling)
For $r = 0.58$, the one parameter family $\{f_{m,0.58}\}_m$ is non-monotone. More precisely, this bifurcation diagram is so-called primary bubbling ([4]).

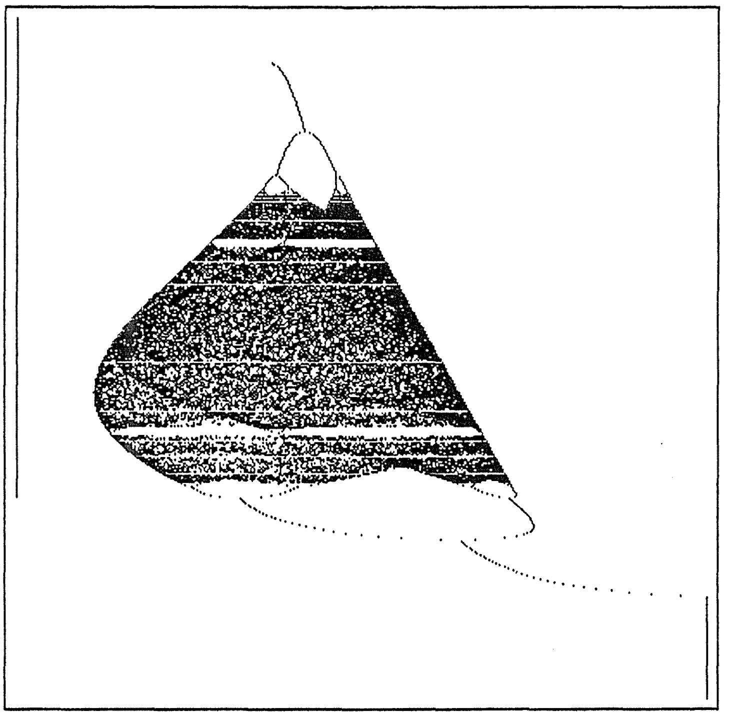

FIGURE 3. Bifurcation diagram of family $\{F_m(x)\}_m$: $-0.8 < x < 0.2$, $-25 < m < 5$

Example (Antimonotone)
Consider the one-parameter family defined on a suitable interval I_m,

$$F_m(x) = m\frac{x^2 + ax + b}{1 + x^2},$$

where constant a is the positive root of the following equation

$$49a^2 - 32 = 0,$$

and b is the unique positive root of the following equation

$$\begin{gathered} 117649b^7 + 684285b^6 + 1721517b^5 + 2358566b^4 \\ +1670655b^3 + 991301b^2 - 257125b = 0. \end{gathered}$$

It is clear that this family satisfies the conditions of monotonicity conjecture (M), namely, (1) each F_m is concave, (2) the set of periodic points in I_1 of F_1 consists of two fixed points, and (3) F_m has a negative schwarzian derivative.

In this moduli space, a defining equation of the algebraic curve defined by $\{F_m\}_m$ is given as follows;

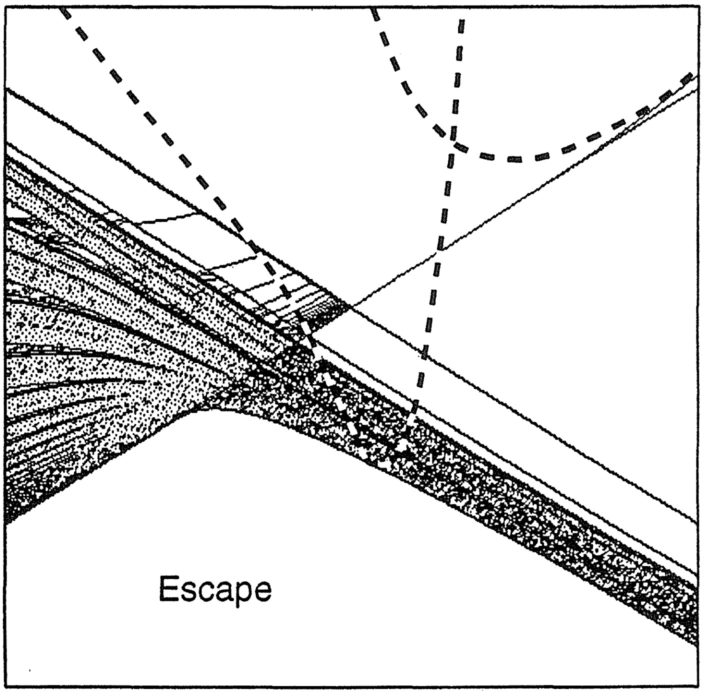

FIGURE 4. Real moduli space with an algebraic curve defined by $\{F_m\}_m$: $-3 < \sigma_1 < 5,\ -15 < \sigma_2 < 10$

$$S_{a,b} = (2\sigma_1^3 + (\sigma_2 - 1)\sigma_1^2 + (-8\sigma_2 + 12)\sigma_1 - 4\sigma_2^2 + 12\sigma_2 - 36)a^6 + ((2\sigma_1^3 + (\sigma_2 + 24)\sigma_1^2 + (12\sigma_2 + 72)\sigma_1 + 36\sigma_2)b^2 + (-14\sigma_1^3 + (-6\sigma_2 - 20)\sigma_1^2 + (32\sigma_2 + 24)\sigma_1 + 16\sigma_2^2 + 24\sigma_2 + 144)b + \sigma_1^4 + 4\sigma_1^3 + (-3\sigma_2 - 12)\sigma_1^2 - 12\sigma_2\sigma_1 + 36\sigma_2)a^4 + ((-10\sigma_1^3 + (-4\sigma_2 - 132)\sigma_1^2 + (-48\sigma_2 - 504)\sigma_1 - 144\sigma_2 - 432)b^3 + (2\sigma_1^4 + 46\sigma_1^3 + (4\sigma_2 + 188)\sigma_1^2 + (-16\sigma_2 - 216)\sigma_1 - 240\sigma_2 - 720)b^2 + (-4\sigma_1^4 - 30\sigma_1^3 + (4\sigma_2 + 84)\sigma_1^2 + (48\sigma_2 + 152)\sigma_1 - 112\sigma_2 - 336)b + 2\sigma_1^4 - 6\sigma_1^3 + (-4\sigma_2 - 12)\sigma_1^2 + (16\sigma_2 + 56)\sigma_1 - 16\sigma_2 - 48)a^2 + (\sigma_1^4 + 24\sigma_1^3 + 216\sigma_1^2 + 864\sigma_1 + 1296)b^4 + (-4\sigma_1^4 - 64\sigma_1^3 - 288\sigma_1^2 + 1728)b^3 + (6\sigma_1^4 + 48\sigma_1^3 - 48\sigma_1^2 - 576\sigma_1 + 864)b^2 + (-4\sigma_1^4 + 96\sigma_1^2 - 256\sigma_1 + 192)b + \sigma_1^4 - 8\sigma_1^3 + 24\sigma_1^2 - 32\sigma_1 + 16 = 0.$$

We remark that this curve tangent to a boundary curve of the lower locus of escape. Then we see this family is antimonotone at this tangent point [5].

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