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# SOME KINDS OF UNCERTAINTY RELATIONS REPRESENTED BY TRACIAL OR NON-TRACIAL POSITIVE LINEAR MAPS

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Abstract. Uncertainty relations are inequalities representing the impossibility of simultaneous measurement in quantum mechanics. The most well-known uncertainty relations were presented by Heisenberg and Schrödinger. We aim to replace the usual trace Tr by positive linear map from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$ and to replace $x^\alpha$ and $x^{1-\alpha}$ by functions $f$ and $g$ under certain conditions. We obtain some extensions of uncertainty relations represented by tracial or non-tracial positive linear map.

## 1. Introduction

In quantum information theory, when the obsearvable$A$ (self-adjoint operator) is measured under the quantum state $\rho$ (density operator) , it is known that the expectation is given by $\mathrm{Tr}(\rho A)$ and the variance $\mathrm{Var}_\rho(A) := \mathrm{Tr}(\rho A^2) - (\mathrm{Tr}(\rho A))^2$. Under the quantum state $\rho$, the fact that two obsearvables $A, B$ can't be measured simultaneously is represented by Heisenberg uncertainty relation

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2,$$

where $[A,B] = AB - BA$ is a commutator. The stronger inequality is given by Schrödinger as follows:

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) - |\mathrm{Re}(\mathrm{Cov}_\rho(A,B))|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2,$$

where $\mathrm{Cov}_\rho(A) := \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)$ is a covariance. Yanagi-Furuichi-Kuriyama [12] defined the one parameter correlation and one parameter Wigner-Yanase skew information as follows: for $\alpha \in [0,1]$

$$\mathrm{Corr}_\rho^\alpha(A,B) = \mathrm{Tr}(\rho A^* B) - \mathrm{Tr}(\rho^{1-\alpha} A^* \rho^\alpha B), \; I_\rho^\alpha(A) = \mathrm{Corr}_\rho^\alpha(A,A).$$

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And they give the trace inequality

$$I_\rho^\alpha(A) I_\rho^\alpha(B) \geq \left| \mathrm{Re}(\mathrm{Corr}_\rho^\alpha(A,B)) \right|^2.$$

When $\alpha = \frac{1}{2}$, it is clear that $\mathrm{Corr}_\rho^{1/2}(A,B) = \mathrm{Corr}_\rho(A,B)$ is said correlation and $I_\rho(A)$ is said Wigner-Yanase skew information. Luo [10] defined the quality representing the uncertainty

$$U_\rho(A) = \sqrt{\mathrm{Var}_\rho(A)^2 - (\mathrm{Var}_\rho(A) - I_\rho(A))^2}.$$

and gave the extension of Heisenberg type inequality

$$U_\rho(A) U_\rho(B) \geq \frac{1}{4} |\mathrm{Tr}(\rho[A,B])|^2.$$

We introduce some notations. $M_n(\mathbb{C})$ is the set of all $n \times n$ complex matrices, $M_{n,sa}(\mathbb{C})$ is the set of all $n \times n$ self-adjoint matrices, $M_{n,+}(\mathbb{C})$ is the set of all $n \times n$ positive semi-definite matrices and $M_{n,+,1}(\mathbb{C})$ is the set of all $n \times n$ density matrices. We remark that the usual trace $\mathrm{Tr}(A)$ has the following properties:

**(1):** $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$
**(2):** $A \geq 0 \Longrightarrow \mathrm{Tr}(A) \geq 0$
**(3):** $\mathrm{Tr}(A^*) = \overline{\mathrm{Tr}(A)}$
**(4):** $\mathrm{Tr}(\alpha A + \beta B) = \alpha \mathrm{Tr}(A) + \beta \mathrm{Tr}(B)$ for all $\alpha, \beta \in \mathbb{C}$
**(5):** $\mathrm{Tr}(A)\mathrm{Tr}(B) = \mathrm{Tr}(B)\mathrm{Tr}(A)$

## 2. UR for Non-Tracial Positive Linear Maps

We consider the mapping $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ which is an extension of usual trace $\mathrm{Tr}(A)$ for $A \in M_n(\mathbb{C})$. We assume that $\Phi$ satisfies the following conditions:

**(1):** $\Phi(AB) = \Phi(BA)$ (tracial)
**(2):** $A \geq 0 \Longrightarrow \Phi(A) \geq 0$ (positive)
**(3):** $\Phi(A^*) = \Phi(A)^*$ ($*$-invariant)
**(4):** $\Phi(\alpha A + \beta B) = \alpha \Phi(A) + \beta \Phi(B)$ for all $\alpha, \beta \in \mathbb{C}$ (linear).

When $\Phi$ doesn't satisfy the condition (1), $\Phi$ is said to be non-tracial. In this section we deal with non-tracial positive linear maps. We use the followng lemma.

**Lemma 2.1.** *Let $A, B, X \in M_n(\mathbb{C})$ satisfy $A > 0, B > 0$. Then* (1)$\sim$(3) *are equivalent.*

$$(1) \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0,\ (2)\ A \geq X B^{-1} X^*,\ (3)\ B \geq X^* A^{-1} X$$

**Definition 2.2.** *Let $A, B \in M_n(\mathbb{C})$, $\rho \in M_{n,sa}(\mathbb{C})$ and $h$ be a real-valued function defined on the interval which includes the spectrum of $\rho$. Then we define*

$$Var^h_{\rho,\Phi}(A) = \Phi(h^{1/2}A^*Ah^{1/2}) - \Phi(h^{\frac{1}{2}}A^*h^{\frac{1}{2}})\Phi^{-1}\Phi(h^{\frac{1}{2}}Ah^{\frac{1}{2}}),$$

$$Cov^h_{\rho,\Phi}(A,B) = \Phi(h^{\frac{1}{2}}A^*Bh^{\frac{1}{2}}) - \Phi(h^{\frac{1}{2}}A^*h^{\frac{1}{2}})\Phi^{-1}\Phi(h^{\frac{1}{2}}Bh^{\frac{1}{2}}),$$

*where $h(\rho)^{\frac{1}{2}}$ is denoted by $h^{\frac{1}{2}}$ and $\Phi(h(\rho))^{-1}$ is denoted by $\Phi^{-1}$.*

**Theorem 2.3.** *For $A, B \in M_n(\mathbb{C})$, $\rho \in M_{n,sa}(\mathbb{C})$,*

$$\begin{pmatrix} Var^h_{\rho,\Phi}(A) & Cov^h_{\rho,\Phi}(A,B) \\ Cov^h_{\rho,\Phi}(B,A) & Var^h_{\rho,\Phi}(B) \end{pmatrix} \geq 0.$$

*Proof.*

$$0 \leq \begin{pmatrix} h^{\frac{1}{2}}A^* & 0 & 0 \\ h^{\frac{1}{2}}B^* & 0 & 0 \\ h^{\frac{1}{2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} Ah^{\frac{1}{2}} & Bh^{\frac{1}{2}} & h^{\frac{1}{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} h^{\frac{1}{2}}A^*Ah^{\frac{1}{2}} & h^{\frac{1}{2}}A^*Bh^{\frac{1}{2}} & h^{\frac{1}{2}}A^*h^{\frac{1}{2}} \\ h^{\frac{1}{2}}B^*Ah^{\frac{1}{2}} & h^{\frac{1}{2}}B^*Bh^{\frac{1}{2}} & h^{\frac{1}{2}}B^*h^{\frac{1}{2}} \\ h^{\frac{1}{2}}Ah^{\frac{1}{2}} & h^{\frac{1}{2}}Bh^{\frac{1}{2}} & h(\rho) \end{pmatrix}.$$

Since $\Phi$ has three-positivity property,

$$\begin{pmatrix} \Phi(h^{\frac{1}{2}}A^*Ah^{\frac{1}{2}}) & \Phi(h^{\frac{1}{2}}A^*Bh^{\frac{1}{2}}) & \Phi(h^{\frac{1}{2}}A^*h^{\frac{1}{2}}) \\ \Phi(h^{\frac{1}{2}}B^*Ah^{\frac{1}{2}}) & \Phi(h^{\frac{1}{2}}B^*Bh^{\frac{1}{2}}) & \Phi(h^{\frac{1}{2}}B^*h^{\frac{1}{2}}) \\ \Phi(h^{\frac{1}{2}}Ah^{\frac{1}{2}}) & \Phi(h^{\frac{1}{2}}Bh^{\frac{1}{2}}) & \Phi(h(\rho)) \end{pmatrix} \geq 0.$$

Then

$$\begin{aligned} & \begin{pmatrix} \Phi(h^{\frac{1}{2}}A^*Ah^{\frac{1}{2}}) & \Phi(h^{\frac{1}{2}}A^*Bh^{\frac{1}{2}}) \\ \Phi(h^{\frac{1}{2}}B^*Ah^{\frac{1}{2}}) & \Phi(h^{\frac{1}{2}}B^*Bh^{\frac{1}{2}}) \end{pmatrix} \\ \geq & \begin{pmatrix} \Phi(h^{\frac{1}{2}}A^*h^{\frac{1}{2}}) \\ \Phi(h^{\frac{1}{2}}B^*h^{\frac{1}{2}}) \end{pmatrix} \Phi^{-1} \begin{pmatrix} \Phi(h^{\frac{1}{2}}Ah^{\frac{1}{2}}) & \Phi(h^{\frac{1}{2}}Bh^{\frac{1}{2}}) \end{pmatrix} \\ = & \begin{pmatrix} \Phi(h^{\frac{1}{2}}A^*h^{\frac{1}{2}})\Phi^{-1}\Phi(h^{\frac{1}{2}}Ah^{\frac{1}{2}}) & \Phi(h^{\frac{1}{2}}A^*h^{\frac{1}{2}})\Phi^{-1}\Phi(h^{\frac{1}{2}}Bh^{\frac{1}{2}}) \\ \Phi(h^{\frac{1}{2}}B^*h^{\frac{1}{2}})\Phi^{-1}\Phi(h^{\frac{1}{2}}Ah^{\frac{1}{2}}) & \Phi(h^{\frac{1}{2}}B^*h^{\frac{1}{2}})\Phi^{-1}\Phi(h^{\frac{1}{2}}Bh^{1/2}) \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{pmatrix} Var^h_{\rho,\Phi}(A) & Cov^h_{\rho,\Phi}(A,B) \\ Cov^h_{\rho,\Phi}(B,A) & Var^h_{\rho,\Phi}(B) \end{pmatrix} \geq 0.$$

□

**Theorem 2.4** (Heisenberg Type). *We assume that $\Phi(M_n(\mathbb{C}))$ is commutative, that is, $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)$ for any $A, B \in M_n(\mathbb{C})$. Then for $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,sa}(\mathbb{C})$ and positive real-valued continuous function $h$ defined on the interval which includes the spectrum of $\rho$, we have*

$$\begin{pmatrix} Var^h_{\rho,\Phi}(A) & \frac{1}{2}\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}}) \\ -\frac{1}{2}\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}}) & Var^h_{\rho,\Phi}(B) \end{pmatrix} \geq 0.$$

**Theorem 2.5** (Schrödinger Type). *Under the same condition in Theorem 2.4, we have*

$$\begin{pmatrix} Var^h_{\rho,\Phi}(A) & \mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\} + \frac{1}{2}\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}}) \\ \mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\} - \frac{1}{2}\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}}) & Var^h_{\rho,\Phi}(B) \end{pmatrix}$$
$$\geq 0.$$

*Proofs of Theorem 2.4 and 2.5.* By using the notations for $A \in M_n(\mathbb{C})$

$$\mathrm{Re}(A) = \frac{A + A^*}{2},\ \mathrm{Im}(A) = \frac{A - A^*}{2i},$$

we have

$$\begin{aligned} & Cov^h_{\rho,\Phi}(A,B) - Cov^h_{\rho,\Phi}(B,A) \\ = \; & \Phi(h^{\frac{1}{2}}ABh^{\frac{1}{2}}) - \Phi(h^{\frac{1}{2}}Ah^{\frac{1}{2}})\Phi^{-1}\Phi(h^{\frac{1}{2}}Bh^{\frac{1}{2}}) \\ & -\Phi(h^{\frac{1}{2}}BAh^{\frac{1}{2}}) + \Phi(h^{\frac{1}{2}}Bh^{\frac{1}{2}})\Phi^{-1}\Phi(h^{\frac{1}{2}}Ah^{\frac{1}{2}}) \\ = \; & \Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}}) \text{ (by using the commutativity of } \Phi(M_n(\mathbb{C}))) \\ = \; & 2i\mathrm{Im}\{Cov^h_{\rho,\Phi}(A,B)\}. \end{aligned}$$

$$Cov^h_{\rho,\Phi}(A,B) + Cov^h_{\rho,\Phi}(B,A) = 2\mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\}$$

Then

$$2Cov^h_{\rho,\Phi}(A,B) = \Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}}) + 2\mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\}.$$

By the commutativity of $\Phi(M_n(\mathbb{C}))$, we have

$$|Cov^h_{\rho,\Phi}(A,B)|^2 = |\mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\}|^2 + \frac{1}{4}|\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}})|^2.$$

By Theorem 2.3

$$Var^h_{\rho,\Phi}(A) \geq Cov^h_{\rho,\Phi}(A,B)Var^h_{\rho,\Phi}(B)^{-1}Cov^h_{\rho,\Phi}(B,A).$$

Since $\Phi(M_n(\mathbb{C}))$ is commutative,

$$\begin{aligned}Var^h_{\rho,\Phi}(A)\cdot Var^h_{\rho,\Phi}(B) &\geq |Cov^h_{\rho,\Phi}(A,B)|^2\\ &= |\mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\}|^2+\frac{1}{4}|\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}})|^2.\end{aligned}$$

Therefore

$$\begin{aligned}
&Var^h_{\rho,\Phi}(A)\\
\geq\ & \left(|\mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\}|^2+\frac{1}{4}|\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}})|^2\right)(Var^h_{\rho,\Phi}(B))^{-1}\\
=\ & \left((\mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\})^2-\frac{1}{4}(\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}}))^2\right)(Var^h_{\rho,\Phi}(B))^{-1}\\
& (\textit{by using } \Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}})^*=-\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}}))\\
=\ & \left(\mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\}+\frac{1}{2}\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}})\right)^*(Var^h_{\rho,\Phi}(B))^{-1}\\
& \left(\mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\}-\frac{1}{2}\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}})\right).(\textit{by using } \Phi(M_n(\mathbb{C})))
\end{aligned}$$

Then

$$\begin{pmatrix} Var^h_{\rho,\Phi}(A) & \mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\}+\frac{1}{2}\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}})\\ \mathrm{Re}\{Cov^h_{\rho,\Phi}(A,B)\}-\frac{1}{2}\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}}) & Var^h_{\rho,\Phi}(B)\end{pmatrix}\geq 0.$$

Then we get Theorem 2.5. By Theorem 2.3,

$$M=\begin{pmatrix} Var^h_{\rho,\Phi}(A) & -Cov^h_{\rho,\Phi}(B,A)\\ -Cov^h_{\rho,\Phi}(A,B) & Var^h_{\rho,\Phi}(B)\end{pmatrix}\geq 0.$$

By the commutativity of $\Phi(M_n(\mathbb{C}))$,

$$N=\begin{pmatrix} Var^h_{\rho,\Phi}(A) & Cov^h_{\rho,\Phi}(A,B)\\ Cov^h_{\rho,\Phi}(B,A) & Var^h_{\rho,\Phi}(B)\end{pmatrix}\geq 0.$$

Therefore

$$0\leq M+N=\begin{pmatrix} 2Var^h_{\rho,\Phi}(A) & \Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}})\\ -\Phi(h^{\frac{1}{2}}[A,B]h^{\frac{1}{2}}) & 2Var^h_{\rho,\Phi}(B)\end{pmatrix}.$$

Then we get Theorem 2.4. □

## 3. UR for Tracial positive Linear Maps

Let $f, g$ be real valued functions defined on the interval which include the spectral of $\rho \in M_{n,sa}(\mathbb{C})$. $f, g$ are said to be monotone pair, if $(f(x)-f(y))(g(x)-g(y)) \geq 0$ for all $x, y \in D$. (see [2, 7])

**Definition 3.1.** *For $A, B \in M_n(\mathbb{C}),\ \rho \in M_{n,sa}(\mathbb{C})$, we difine*

$$I^{f,g}_{\rho,\Phi}(A) = \Phi(f(\rho)g(\rho)A^*A) - \Phi(f(\rho)A^*g(\rho)A),$$

$$Corr^{f,g}_{\rho,\Phi}(A,B) = \Phi(f(\rho)g(\rho)A^*B) - \Phi(f(\rho)A^*g(\rho)B).$$

**Theorem 3.2.** *For $A \in M_n(\mathbb{C})$, we have $I^{f,g}_{\rho,\Phi}(A)+I^{f,g}_{\rho,\Phi}(A^*) \geq 0$. In particular, $I^{f,g}_{\rho,\Phi}(A) \geq 0$ for $A \in M_{n,sa}(\mathbb{C})$.*

*Proof.* Let $\rho = \sum_{i=1}^{n} \lambda_i E_i$ be the spectral decomposition of $\rho$. Since

$$\begin{aligned}
f(\rho)g(\rho)A^2 &= \sum_i f(\lambda_i)g(\lambda_i)E_iA^2 \\
&= \sum_{i,j} f(\lambda_i)g(\lambda_i)E_iAE_jA \\
&= \sum_i f(\lambda_i)g(\lambda_i)E_iAE_iA + \sum_{i<j} f(\lambda_i)g(\lambda_i)E_iAE_jA \\
&\quad + \sum_{i>j} f(\lambda_i)g(\lambda_i)E_iAE_jA \\
&= \sum_i f(\lambda_i)g(\lambda_i)E_iAE_iA + \sum_{i<j} f(\lambda_i)g(\lambda_i)E_iAE_jA \\
&\quad + \sum_{i<j} f(\lambda_j)g(\lambda_j)E_jAE_iA,
\end{aligned}$$

we have

$$\begin{aligned}
\Phi(f(\rho)g(\rho)A^2) &= \Phi(\sum_i f(\lambda_i)g(\lambda_i)E_iA^2) \\
&= \Phi(\sum_i f(\lambda_i)g(\lambda_i)E_iAE_iA) + \Phi(\sum_{i<j} f(\lambda_i)g(\lambda_i)E_iAE_jA \\
&\quad + \sum_{i<j} f(\lambda_j)g(\lambda_j)E_jAE_iA)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_i f(\lambda_i)g(\lambda_i)\Phi(E_iAE_iA) + \sum_{i<j}\Phi(\{f(\lambda_i)g(\lambda_i) \\
&\quad + f(\lambda_j)g(\lambda_j)\}E_iAE_jA) \\
&\geq \sum_i f(\lambda_i)g(\lambda_i)\Phi(E_iAE_iA) \\
&\quad + \sum_{i<j}\Phi(\{f(\lambda_i)g(\lambda_j) + f(\lambda_j)g(\lambda_i)\}E_iAE_jA) \\
&= \Phi(f(\rho)Ag(\rho)A).
\end{aligned}
$$

□

**Definition 3.3.** *For* $A, B \in M_n(\mathbb{C}),\ \rho \in M_{n,sa}(\mathbb{C})$, *we define*

$$
Corr_{\rho,\Phi}^{\prime f,g}(A,B) = \frac{1}{2}\{Corr_{\rho,\Phi}^{f,g}(A,B) + Corr_{\rho,\Phi}^{f,g}(B^*,A^*)\},
$$

$$
I_{\rho,\Phi}^{\prime f,g}(A) = \frac{1}{2}\{I_{\rho,\Phi}^{f,g}(A) + I_{\rho,\Phi}^{f,g}(A^*)\}.
$$

**Theorem 3.4.** *For* $A, B \in M_{n,sa}(\mathbb{C})$, *we have*

$$
\begin{aligned}
&\begin{pmatrix} I_{\rho,\Phi}^{f,g}(A) & Corr_{\rho,\Phi}^{\prime f,g}(A,B) \\ Corr_{\rho,\Phi}^{\prime f,g}(B,A) & I_{\rho,\Phi}^{f,g}(B) \end{pmatrix} \\
= &\begin{pmatrix} I_{\rho,\Phi}^{f,g}(A) & \mathrm{Re}\{Corr_{\rho,\Phi}^{f,g}(A,B)\} \\ \mathrm{Re}\{Corr_{\rho,\Phi}^{f,g}(B,A)\} & I_{\rho,\Phi}^{f,g}(B) \end{pmatrix} \geq 0.
\end{aligned}
$$

*Proof.* $\Psi : M_{2n}(\mathbb{C}) \to M_{2m}(\mathbb{C})$ is defined by

$$
\Psi(\begin{pmatrix} A & B \\ C & D \end{pmatrix}) = \begin{pmatrix} \frac{1}{2}(\Phi(A) + \Phi(D)) & 0 \\ 0 & 0 \end{pmatrix}.
$$

It is clear that $\Psi$ is tracial positive linear map. For $A \in M_n(\mathbb{C}), \rho \in M_{n,sa}(\mathbb{C})$

$$
\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix},\ \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} \in M_{2n,sa}(\mathbb{C}).
$$

Then

$$\begin{aligned}
&\begin{pmatrix} \Phi(f(\rho)g(\rho)A^*A)+\Phi(f(\rho)g(\rho)AA^*) & 0 \\ 0 & 0 \end{pmatrix} \\
= \;& 2\Psi(\begin{pmatrix} f(\rho)g(\rho) & 0 \\ 0 & f(\rho)g(\rho) \end{pmatrix}\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}) \\
\geq \;& 2\Psi(\begin{pmatrix} f(\rho) & 0 \\ 0 & f(\rho) \end{pmatrix}\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}\begin{pmatrix} g(\rho) & 0 \\ 0 & g(\rho) \end{pmatrix}\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}) \\
= \;& \begin{pmatrix} \Phi(f(\rho)A^*g(\rho)A)+\Phi(f(\rho)Ag(\rho)A^*) & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Therefore we have

$$\Phi(f(\rho)g(\rho)A^*A)+\Phi(f(\rho)g(\rho)AA^*) \geq \Phi(f(\rho)A^*g(\rho)A)+\Phi(f(\rho)Ag(\rho)A^*).$$

Hence we get

$$I^{f,g}_{\rho,\Phi}(A)+I^{f,g}_{\rho,\Phi}(A^*) \geq 0.$$

□

**Corollary 3.5.** *If $\Phi(M_n(\mathbb{C}))$ is commutative, then for $A,B \in M_{n,sa}(\mathbb{C})$ we have*

$$|\mathrm{Re}\{Corr^{f,g}_{\rho,\Phi}(A,B)\}|^2 \leq I^{f,g}_{\rho,\Phi}(A) I^{f,g}_{\rho,\Phi}(B).$$

**Theorem 3.6.** *For $A \in M_n(\mathbb{C})$, we have*

$$\begin{aligned}
&\frac{1}{2}\{\Phi(f(\rho)A^*g(\rho)A)+\Phi(f(\rho)Ag(\rho)A^*)\} \\
&\qquad \geq \Phi(f(\rho)^{1/2}g(\rho)^{1/2}A^*f(\rho)^{1/2}g(\rho)^{1/2}A) \\
&\qquad \geq \Phi(f(\rho)A^*g(\rho))\Phi(f(\rho)g(\rho))^{-1}\Phi(f(\rho)Ag(\rho)).
\end{aligned}$$

*Then for $A \in M_{n,sa}(\mathbb{C})$ we have*

$$I^{f,g}_{\rho,\Phi}(A) \leq I^{\sqrt{fg},\sqrt{fg}}_{\rho,\Phi}(A) \leq Var^{fg}_{\rho,\Phi}(A).$$

*Proof.* Let $\rho = \sum_{i=1}^n \lambda_i E_i$ be spectral decomposition of $\rho$. Let $\lambda_i$ $(i = 1,2,\ldots,n)$ be real numbers and $E_i,:\ i = 1,2,\ldots,n$ be ortogonal projections satisfying

$\sum_{i=1}^n E_i = I$. We denote $f(\lambda_i),\ g(\lambda_j)$ by $f_i,\ g_j$. Then we have

$$
\begin{aligned}
&\begin{pmatrix} \Phi(f(\rho)g(\rho)A^2) & W \\ W & \Phi(f(\rho)g(\rho)B^2) \end{pmatrix} \\
&\textit{where } W = \Phi(f(\rho)g(\rho)AB) + \Phi(f(\rho)g(\rho)BA), \\
= &\begin{pmatrix} \sum_{i=1}^n f_i g_i \Phi(E_i A^2) & \sum_{i=1}^n f_i g_i \Phi(E_i AB) \\ \sum_{i=1}^n f_i g_i \Phi(E_i BA) & \sum_{i=1}^n f_i g_i \Phi(E_i B^2) \end{pmatrix} \\
&+ \begin{pmatrix} \sum_{i=1}^n f_i g_i \Phi(E_i A^2) & \sum_{i=1}^n f_i g_i \Phi(E_i AB) \\ \sum_{i=1}^n f_i g_i \Phi(E_i AB) & \sum_{i=1}^n f_i g_i \Phi(E_i B^2) \end{pmatrix} \\
= &\begin{pmatrix} \sum_{i=1}^n f_i g_i \Phi(E_i A(\sum_{j=1}^n E_j)A) & \sum_{i=1}^n f_i g_i \Phi(E_i A(\sum_{j=1}^n E_j)B) \\ \sum_{i=1}^n f_i g_i \Phi(E_i B(\sum_{j=1}^n E_i)A) & \sum_{i=1}^n f_i g_i \Phi(E_i B(\sum_{j=1}^n E_j)B) \end{pmatrix} \\
&+ \begin{pmatrix} \sum_{i=1}^n f_i g_i \Phi(E_i A(\sum_{j=1}^n E_i)A) & \sum_{i=1}^n f_i g_i \Phi(E_i B(\sum_{j=1}^n E_i)A) \\ \sum_{i=1}^n f_i g_i \Phi(E_i A(\sum_{j=1}^n E_j)B) & \sum_{i=1}^n f_i g_i \Phi(E_i B(\sum_{j=1}^n E_j)B) \end{pmatrix} \\
&(\textit{by using } \sum_{j=1}^n E_j = I) \\
= \ &2\begin{pmatrix} \sum_{i=1}^n f_i g_i \Phi(E_i A E_i A) & \sum_{i=1} f_i g_i \Phi(E_i B E_i A) \\ \sum_{i=1}^n f_i g_i \Phi(E_i A E_i B) & \sum_{i=1}^n f_i g_i \Phi(E_i B E_i B) \end{pmatrix} \\
&+ \begin{pmatrix} 2\sum_{i<j} f_i g_i \Phi(E_i A E_j A) & \sum_{i=1}^n (f_i g_i + f_j g_j)\Phi(E_i A E_j B) \\ \sum_{i<j} f_j g_j \Phi(E_j B E_i A) & 2\sum_{i<j} f_i g_i \Phi(E_i B E_j B) \end{pmatrix} \\
&+ \begin{pmatrix} 2\sum_{i<j} f_j g_j \Phi(E_j A E_i A) & \sum_{i<j} (f_i g_i + f_j g_j)\Phi(E_i B E_j A) \\ \sum_{i<j} f_i g_i \Phi(E_i A E_j B) & 2\sum_{i<j} f_j g_j \Phi(E_j B E_i B) \end{pmatrix} \\
= \ &2\begin{pmatrix} \sum_{i=1}^n f_i g_i \Phi(E_i A E_i A) & \sum_{i<j} f_i g_i \Phi(E_i B E_i A) \\ \sum_{i=1}^n f_i g_i \Phi(E_i A E_i B) & \sum_{i=1}^n f_i g_i \Phi(E_i B E_i B) \end{pmatrix} \\
&+ \begin{pmatrix} 2\sum_{i<j} \delta_{ij} \Phi(E_i A E_j A) & \sum_{i<j} \delta_{ij}\Delta_{ij} \\ \sum_{i<j} \delta_{ij}\Delta_{ij} & 2\sum_{i<j} \delta_{ij} \Phi(E_i B E_j B) \end{pmatrix} \\
&(\textit{by using thetracial condition of } \Phi), \\
&\textit{where } \delta_{ij} = f_i g_i + f_j g_j,\ \Delta_{ij} = \Phi(E_i A E_j B) + \Phi(E_j A E_i B) \\
= \ &2\begin{pmatrix} \sum_{i=1}^n f_i g_i \Phi(E_i A E_i A) & \sum_{i=1}^n f_i g_i \Phi(E_i B E_i A) \\ \sum_{i=1}^n f_i g_i \Phi(E_i A E_i B) & \sum_{i=1}^n f_i g_i \Phi(E_i B E_i B) \end{pmatrix} \\
&+ \sum_{i<j} \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta_{ij} \end{pmatrix} \begin{pmatrix} 2\Phi(E_i A E_j A) & \Delta_{ij} \\ \Delta_{ij} & 2\Phi(E_i B E_j B) \end{pmatrix} \\
\geq \ &2\begin{pmatrix} \sum_{i=1}^n f_i g_i \Phi(E_i A E_i A) & \sum_{i=1}^n f_i g_i \Phi(E_i B E_i A) \\ \sum_{i=1}^n f_i g_i \Phi(E_i A E_i B) & \sum_{i=1}^n f_i g_i \Phi(E_i B E_i B) \end{pmatrix}
\end{aligned}
$$

$$+\sum_{i<j}\begin{pmatrix}\xi_{ij} & 0\\ 0 & \xi_{ij}\end{pmatrix}\begin{pmatrix}2\Phi(E_iAE_jA) & \Delta_{ij}\\ \Delta_{ij} & 2\Phi(E_iBE_jB)\end{pmatrix}$$

*(by using the monotone pairness of $f,\ g$),*
*where $\xi_{ij}=f_ig_j+f_jg_i$*

$$= \begin{pmatrix}\sum_{i=1}^n\sum_{j=1}^n f_ig_j\Phi(E_iAE_jA) & \sum_{i=1}^n\sum_{j=1}^n f_ig_j\Phi(E_iBE_jA)\\ \sum_{i=1}^n\sum_{j=1}^n f_ig_j\Phi(E_iAE_jB) & \sum_{i=1}^n\sum_{j=1}^n f_ig_j\Phi(E_iBE_jB)\end{pmatrix}$$

$$+\begin{pmatrix}\sum_{i=1}^n\sum_{j=1}^n f_ig_j\Phi(E_iAE_jA) & \sum_{i=1}^n\sum_{j=1}^n f_ig_j\Phi(E_iAE_jB)\\ \sum_{i=1}^n\sum_{j=1}^n f_ig_j\Phi(E_iBE_jA) & \sum_{i=1}^n\sum_{j=1}^n f_ig_j\Phi(E_iBE_jB)\end{pmatrix}$$

$$= \begin{pmatrix}2\Phi(f(\rho)g(\rho)A) & V\\ V & 2\Phi(f(\rho)Bg(\rho)A)\end{pmatrix},$$

*where $V=\Phi(f(\rho)Ag(\rho)B)+\Phi(f(\rho)Bg(\rho)A)$.*

Then

$$\begin{pmatrix}\Phi(f(\rho)g(\rho)A^2) & \frac{1}{2}W\\ \frac{1}{2}W & \Phi(f(\rho)g(\rho)B^2)\end{pmatrix}\geq\begin{pmatrix}\Phi(f(\rho)Ag(\rho)A) & \frac{1}{2}V\\ \frac{1}{2}V & \Phi(f(\rho)Bg(\rho)B)\end{pmatrix},$$

Therefore we get

$$\begin{pmatrix}I_{\rho,\Phi}^{f,g}(A) & \mathrm{Re}\{Corr_{\rho,\Phi}^{f,g}(A,B)\}\\ \mathrm{Re}\{Corr_{\rho,\Phi}^{f,g}(A,B)\} & I_{\rho,\Phi}^{f,g}(B)\end{pmatrix}\geq 0.$$

□

**Corollary 3.7.** *For $\rho\in M_{n,+}(\mathbb{C}),\ A\in M_{n,sa}(\mathbb{C})$, we have*

$$I_{\rho,\Phi}^{\alpha}(A)\leq I_{\rho,\Phi}^{1/2}(A)\leq Var_{\rho,\Phi}(A).$$

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## References

[1] T. Ando, *Convexity of certain maps on positive definite matrices and applications to Hadamard products*, Linear Algebra Appl. **26** (1979), 203–241.

[2] J. C. Bourin, *Some inequalities for norms on matrices and operators*, Linear Algebra Appl. **292** (1999), 139–154.

[3] M. D. Choi and S. K. Tsui,*Tracial positive linear maps of $C^*$-algebaras*, Proc. Amer. Math. Soc. **87** (1983), 57–61.

[4] A. Dadkhah and M. S. Moslehian, *Quantum information inequalities via tracial positive linear maps*, J. Math. Anal. Appl. **447** (2017), 666–680.

[5] A. Dadkhan, M. S. Moslehian and K. Yanagi, *Noncommutative versions of inequalities in quantum information theory*, Anal. Math. Phys. **9**(2019), 2151–2169.

[6] S. Friedland, V. Gheorghiu, and G. Gour, *Universal uncertainty relations*, Phys. Rev. Lett. **111** (2013), 230401.

[7] J. I. Fujii, *A trace inequality arising from quantum information theory*, Linear Algebra Appl. **400** (2004), 141–146.

[8] T. Furuta, J. Mićić Hot, J. Pečarić, and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Self-adjoint Operators on a Hilbert Space*, Element, Zagreb, Croatia, 2005.

[9] E. H. Lieb, *Convex trace functions and the Wigner–Yanase–Dyson conjecture*, Adv. Math. **11** (1973), 267–288.

[10] S. Luo, *Heisenberg uncertainty relation for mixed states*, Phys. Rev. A **72** (2005), 042110, pp.3.

[11] E. Schrödinger, *About Heisenberg uncertainty relation*, Proc. Prussian Acad. Sci. Phys. Math. Section **XIX** (1930), p.293.

[12] K. Yanagi, S. Furuichi, and K. Kuriyama, *A generalized skew information and uncertainty relation*, IEEE Trans. Info. Theory **51** (2005), 4401–4404.



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